Autoresonant (nonstationary) excitation of pendulums, Plutinos, plasmas, and other nonlinear oscillators

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A weakly driven pendulum cannot be strongly excited by a fixed frequency drive. The only way to strongly excite the pendulum is to use a drive whose frequency decreases with time. Feedback is often used to control the rate at which the frequency decreases. Feedback need not be employed, however; the drive frequency can simply be swept downwards. With this method, the drive strength must exceed a threshold proportional to the sweep rate raised to the 3/4 power. This threshold has been discovered only recently, and holds for a very broad class of driven nonlinear oscillators. The threshold may explain the abundance of 3:2 resonances and dearth of 2:1 resonances observed between the orbital periods of Neptune and the Plutinos (Pluto and many of the Kuiper Belt objects), and has been extensively investigated in the Diocotron system in pure-electron plasmas.

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I. INTRODUCTION

How can you excite a pendulum to high amplitude with a weak drive? The usual answer is to drive the pendulum at its linear resonant frequency, f_0. However, as the pendulum amplitude increases, its oscillation frequency decreases and the pendulum goes out of phase with its drive. As shown in Fig. 1, the amplitude only attains a modest value before beating all the way back down to zero.[1]

It is not surprising that driving at f_0 will not strongly excite the pendulum. As shown by the solutions of the pendulum equation,

$$\ddot{\theta}+(2\pi f_0)^2 \sin\theta=\overline{\epsilon}\cos(2\pi f t), \tag{1}$$

graphed in Fig. 2,[2–4] the equilibrium response of the pendulum is large only when the drive frequency f is lower than the linear frequency f_0. Even this is not sufficient, however, because there exist two stable branches to the response, one with a much higher amplitude than the other. Suddenly applying a low frequency drive to an initially quiescent pendulum results in oscillations around the low amplitude branch, not the high amplitude branch. Here θ is the angle of the pendulum from vertical, and $\overline{\epsilon}$ is the drive strength.

The only way to strongly excite the pendulum from rest is to sweep the frequency downward and hope that the pendulum's response follows the response function's upper branch to high amplitude. Feedback can be used to appropriately sweep the frequency; this is how we excite a child's swing. But there are circumstances in which using feedback is impractical or undesirable. For instance, you might not be able to sense the phase of the pendulum well enough to apply feedback properly, or you might want to simultaneously excite several pendulums with the same drive. What can you do then? Simply starting the sweep at a frequency well above f_0, and sweeping downwards through the resonance at a sufficiently slow rate can strongly excite the pendulum.[1] We call this phenomenon autoresonance[5] because, at each instant in time, the pendulum automatically adjusts its amplitude so that its instantaneous nonlinear frequency matches the drive frequency. Autoresonant effects were first observed in particle accelerators,[6] and have since been noted in atomic physics,[5,7] fluid dynamics,[8] plasmas,[9,10] nonlinear waves,[11,12] and planetary dynamics.[13,14]

Autoresonance in pendulums and other systems has been used implicitly in many papers.[15–23] However, these papers do not address a very important question: How fast can the frequency be swept and still strongly excite the pendulum? Specifically, given a pendulum described by

$$\ddot{\theta}+\omega_0^2 \sin\theta=\overline{\epsilon}\cos(\omega_0 t-\alpha t^2/2), \tag{2}$$

how large can the sweep rate α be and still stay in autoresonance? Here $\omega_0/2\pi=f_0$ is the pendulum's linear frequency, and the drive frequency is $\omega_0-\alpha t$.

Recently, we have addressed this problem in a series of papers[9,10,24–26] focused on a similar nonlinear oscillator, the Diocotron mode[27] in non-neutral plasmas. Here we apply these results to the pendulum and find that there is a very sharp threshold for autoresonance; if the drive amplitude $\overline{\epsilon}$ exceeds a threshold proportional to the sweep rate α raised to the three-quarters power, the pendulum will follow the drive to high amplitude (see Figs. 1 and 3). If the drive amplitude is below this threshold, the pendulum amplitude will stay very low. This $\alpha^{3/4}$ scaling applies to a very broad class of nonlinear oscillators, not just to the pendulum and to the Diocotron. Moreover, the scaling can be extended to drives at subharmonics of f_0 (Refs. 24 and 28) and survives in the presence of damping.[26]

We begin this paper with the mathematical analysis of the threshold. Then we describe some of the Diocotron experiments in plasmas, and discuss how the threshold explains the Plutino resonances. We end with some concluding remarks.

II. ANALYSIS

The complete threshold analysis is given in Refs. 8 and 10, so we will only give a précis here. The analysis divides into three regimes: the linear, phase-trapping regime where the pendulum amplitude is so low that its frequency does not

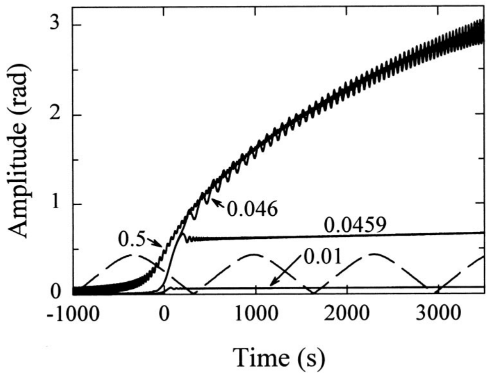

Fig. 1. The response of a pendulum to a drive sweeping at rate $\alpha=0.001$ for four different values of the drive strength $\bar{\epsilon}$ (solid lines) and to a fixed frequency drive of strength $\bar{\epsilon}=0.1$ (dashed line). The response to the $\bar{\epsilon}=0.01$ drive is very small, and the response to the $\bar{\epsilon}=0.0459$ drive is only somewhat greater. However, the response to the $\bar{\epsilon}=0.0460$ drive, which is only 0.2% gerater than the $\bar{\epsilon}=0.0459$ drive, is large and results in the pendulum swinging over its top. The response to the $\bar{\epsilon}=0.5$ drive is qualitatively similar to the response to the $\bar{\epsilon}=0.0460$ drive. Clearly there is a sharp threshold near $\bar{\epsilon}=0.0460$. The oscillations in the $\bar{\epsilon}=0.0460$ curve are the manifestations of the pseudoparticle oscillations in the pseudopotential wells. These curves come from numerically simulating Eq. (2) with $\omega_0=2\pi$. For clarity, the time scale for the fixed frequency curve (the dashed line) has been expanded by a factor of 10.

significantly deviate from f_0, the weakly nonlinear regime where we take only the lowest order amplitude correction to the frequency, and the strongly nonlinear regime where we must use the complete expression for the pendulum frequency. We will consider each regime in turn.

A. Linear, phase-trapping regime

The pendulum will be in the linear regime from the time at which the drive is first applied, $t=t_s<0$, to near the time at which the drive frequency crosses f_0, namely $t\approx 0$. Since the

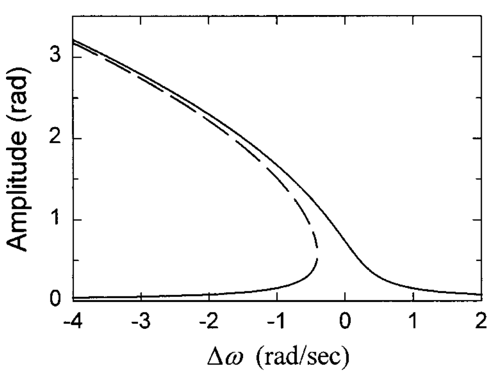

Fig. 2. Equilibrium pendulum amplitude as a function of the drive frequency detuning $\Delta\omega$, defined by $\omega=\omega_0+\Delta\omega$ where ω is the drive frequency. (For the swept drive used in this paper, $\Delta\omega=-\alpha t$.) Only the first-order correction to the frequency is retained in this graph, so the equilibria come from solving for the steady amplitude solutions to the equation $\ddot{\theta}+\omega_0^2(\theta-\theta^3/6)=\bar{\epsilon}\cos[(\omega_0+\Delta\omega)t]$. The two solid lines are stable equilibria, and the dashed line is an unstable equilibrium. Far from resonance, the low amplitude solutions reduce to the standard solutions of the linearized pendulum equation, $\ddot{\theta}+\omega_0^2\theta=\bar{\epsilon}\cos[(\omega_0+\Delta\omega)t]$. The high amplitude solutions comes from balancing the nonlinearities in the next order equation. The θ^3 term introduces the possibility, realized for large negative $\Delta\omega$, of three equilibria at a given $\Delta\omega$. For this figure, the drive strength is $\bar{\epsilon}=2$ and $\omega_0=2\pi$.

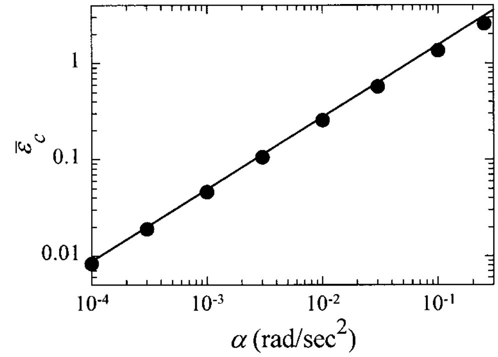

Fig. 3. The numerically determined critical drive strength as a function of the sweep rate α (dots) and the theoretical scaling from Eq. (24) (line) for $\omega_0=2\pi$. The numeric data are in excellent agreement with the theory. Experiments (Refs. 9 and 10) with the Diocotron mode in non-neutral plasmas have verified this scaling law over five orders of magnitude in α.

system is effectively linear here, it reduces to a simple harmonic oscillator (SHO) subject to a swept frequency drive,

$$\ddot{\theta}+\omega_0^2\theta=\bar{\epsilon}\cos(\omega_0 t-\alpha t^2/2). \tag{3}$$

This system was solved exactly in terms of complicated Fresnel sine and cosine functions by Lewis.[29] However, the behavior of the solution can be understood quite readily. Like any SHO, the sudden application of the drive excites a driven mode at the drive frequency and an undriven, homogeneous mode at the linear frequency. Assuming that the pendulum is initially quiescent, the two modes have equal and opposite amplitude, are proportional to the drive, and inversely proportional to the deviation $\Delta\omega(t_s)=\omega(t_s)-\omega_0=-\alpha t_s$ from the linear resonant frequency. Thereafter, just as in a steady state SHO, the amplitude of the driven mode continues to be approximately inversely proportional to $\Delta\omega(t)$. Consequently, the driven mode grows in proportion to $1/t$ as $t\rightarrow 0$. Remarkably, the homogeneous mode is not further excited by the sweeping frequency.

The net motion of the pendulum is the sum of the phase-locked driven mode and the free-running homogeneous mode. Initially, the net motion will not be phase locked because the amplitudes of the two modes are comparable. But since the driven mode amplitude grows while the homogeneous mode amplitude is constant, the driven mode will soon dominate, and the net pendulum motion phase locks to the drive.

B. Weakly nonlinear regime

In the weakly nonlinear regime, we need to retain the lowest order amplitude correction to the pendulum frequency. The set of weakly nonlinear equations describing our adiabatically driven pendulum can be derived as follows. We begin by replacing $\sin\theta$ by its truncated expansion in Eq. (2):

$$\ddot{\theta}+\omega_0^2\left(1-\frac{\theta^2}{6}\right)\theta=\bar{\epsilon}\cos(\psi)=\bar{\epsilon}\,\mathrm{Re}(e^{i\psi}), \tag{4}$$

where $\psi=\omega_0 t-\alpha t^2/2$ is the driving phase. Then we seek solutions of Eq. (4) of the form

$$\theta=a\cos\varphi=\mathrm{Re}(ae^{i\varphi}), \tag{5}$$

where $a(t)$ is the real amplitude, and $\varphi(t)$ is the phase of the pendulum, and we have neglected higher harmonics (i.e., the terms with $e^{2i\varphi}$, $e^{3i\varphi}$, etc.) By differentiating Eq. (5), we find

$$\ddot{\theta}\approx \mathrm{Re}[2i\dot{\varphi}\dot{a}+i\ddot{\varphi}a-\dot{\varphi}^2a)e^{i\varphi}], \tag{6}$$

and by cubing Eq. (5)

$$\theta^3\approx \mathrm{Re}(\tfrac{3}{4}a^3e^{i\varphi}), \tag{7}$$

where we have neglected the $\ddot{a}$ term in Eq. (6) (slow amplitude assumption) and kept the first harmonic term only in Eq. (7). With these approximation, Eq. (4) becomes

$$2i\dot{\varphi}\dot{a}+i\ddot{\varphi}a-\dot{\varphi}^2a+\omega_0^2(1-\tfrac{1}{8}a^2)a=\overline{\epsilon}e^{i(\psi-\varphi)}. \tag{8}$$

On separating the real and imaginary parts in the last equation, we obtain

$$\frac{d(a^2\dot{\varphi})}{dt}=a(2\dot{\varphi}\dot{a}+\ddot{\varphi}a)=-\overline{\epsilon}a\,\sin\Phi, \tag{9}$$

and

$$\omega_0^2-\dot{\varphi}^2=\omega_0^2\frac{a^2}{8}+\frac{\overline{\epsilon}}{a}\cos\Phi, \tag{10}$$

where the phase mismatch is defined as $\Phi=\varphi-\psi$. Next we assume that the system is nearly in resonance, i.e., that the pendulum frequency is close to the linear frequency, $\dot{\varphi}\approx\omega_0$. Then $\omega_0^2-\dot{\varphi}^2\approx 2\omega_0(\omega_0-\dot{\varphi})$, and Eqs. (9) and (10) can be rewritten as:

$$\dot{a}=-\frac{\overline{\epsilon}}{2\omega_0}\sin\Phi, \tag{11}$$

$$\dot{\Phi}=\omega_0-\frac{\omega_0}{16}a^2-\dot{\psi}-\frac{\overline{\epsilon}}{2a\omega_0}\cos\Phi. \tag{12}$$

Finally we define the action variable $I=a^2/2$, and the weakly nonlinear frequency of the pendulum $\Omega(I)=\omega_0-\omega_0a^2/16=\omega_0(1-\beta I)$, $(\beta=1/8)$ and substitute the driving frequency $\dot{\psi}=\omega_0-\alpha t$, arriving at[30]

$$\frac{dI}{dt}=-2\epsilon I^{1/2}\sin\Phi, \tag{13}$$

$$\frac{d\Phi}{dt}=\Omega(I)-\omega_0+\alpha t-\epsilon I^{-1/2}\cos\Phi, \tag{14}$$

where $\epsilon=\overline{\epsilon}/(\sqrt{8}\omega_0)$ is the normalized drive strength.

It should be mentioned that the expression $I=a^2/2$ is just the small amplitude limit of the normalized action variable frequently used in describing nonlinear oscillators. More generally, the normalized action, in our case, is defined as

$$I=\frac{1}{2\pi\omega_0}\oint\dot{\theta}\,d\theta, \tag{15}$$

where the integration is over one period of oscillation of the unperturbed pendulum. The action I is a measure of the energy in the system, and, in combination with another variable called the ''angle,'' Θ, is used to replace the phase space variables $(\theta,\dot{\theta})$ in studying nonlinear oscillators. The (I,Θ) representation is very convenient in studying weak perturbations of oscillators because the unperturbed motion in (I,Θ) variables is extremely simple, i.e., $I=\mathrm{const}$ and $\Theta=\Omega(I)t$. Furthermore, the structure of Eqs. (13) and (14) is also preserved in a general nonlinear case, with Φ representing the phase mismatch $\Theta-\psi$, and $2I^{1/2}$ and $I^{-1/2}$ in the interaction terms in these equations replaced by some function $V(I)$ and dV/dI, respectively.

Equations (13) and (14) are easy to interpret. If the drive is turned off ($\epsilon=0$), the action will be constant, and the phase mismatch Φ will advance appropriately. When the drive is on ($\epsilon\neq 0$), the action will increase or decrease depending on whether or not the pendulum is in phase or out of phase with the drive. The rate at which the action changes depends on the action itself, because the work done per cycle depends on the pendulum amplitude. Likewise the drive can ''drag'' the phase mismatch. When the pendulum amplitude is large, it is difficult for the drive to change the pendulum angle quickly, but when the pendulum amplitude is small, the drive can change the pendulum angle readily. Consequently, the rate of change of the phase mismatch is inversely proportional to (the square root of) the action.

The pendulum enters the weakly nonlinear regime from the linear regime at about $t=0$. From the results of Sec. II A, we know that the system is phase locked at this time, so Φ starts out near π. If the system is to stay in autoresonance, it must stay phase locked and Φ must remain near π. Otherwise, the phase between the pendulum and its drive would stray and the drive would not be effectively coupled. If Φ is to stay near π, the right-hand side of Eq. (14) must be close to zero. This requirement will be met if

$$0=\alpha t-\beta\omega_0 I_0+\frac{\epsilon}{I_0^{1/2}}, \tag{16}$$

where we have defined the action to be within a small deviation Δ of the equilibrium action $I_0, I=I_0-\Delta$. Note that I_0 is a slow function of time: The action grows as time increases and the drive frequency gets further and further from the linear resonant frequency. This equation is identical to the standard equation used to find the nonlinear response of a pendulum, and its solution is plotted in Fig. 2.

Since I_0 is a slow function of time, we can expand the system [Eqs. (13) and (14)] around the instantaneous value of I_0, yielding two new equations:

$$\dot{\Delta}=2\epsilon\sqrt{I_0}\,\sin\Phi+\frac{\alpha}{S}, \tag{17}$$

$$\dot{\Phi}=S\Delta, \tag{18}$$

where $S=\beta\omega_0+\epsilon/2I_0^{3/2}$. Note that S is a function of time through I_0. Together, these two equations form a Hamiltonian system with

$$H(\Phi,\Delta)=S\Delta^2/2+V_{\mathrm{pseudo}}(\Phi), \tag{19}$$

where

$$V_{\mathrm{pseudo}}(\Phi)=2\epsilon I_0^{1/2}\cos\Phi-\frac{\alpha}{S}\Phi. \tag{20}$$

Thus, the system reduces to a pseudoparticle of slowly varying effective mass $1/S$ moving in a slowly varying pseudopotential well.

The pseudopotential, V_{pseudo}, looks like a tilted series of potential wells (see Fig. 4). Near $t=0$, we know that the system is phase locked and Φ is near π; for this to remain true, the pseudoparticle must be trapped at the bottom of one of the potential wells. If the pseudopotential and the effective mass change slowly, and the wells continue to exist, the

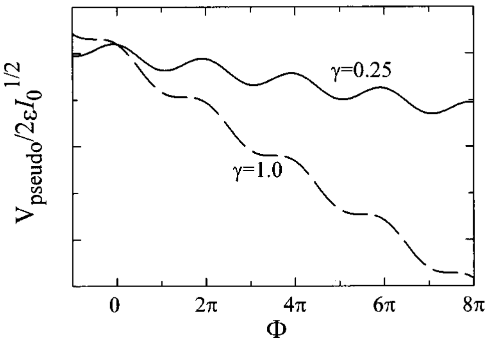

Fig. 4. Critical (dashed line) and typical above-threshold (solid line) normalized pseudopotentials $V_{\mathrm{pseudo}}/2\epsilon I_0^{1/2}=\cos\Phi-\gamma\Phi$ as a function of the phases-lip Φ, where $\gamma=\alpha/2\epsilon I_0^{1/2}S$. The critical pseudopotential, defined by Eq. (21), occurs when $\gamma=1.0$.

pseudoparticle will stay trapped at the bottom of the well as time advances. The phase mismatch Φ will stay near zero, the equilibrium action will continue to satisfy Eq. (16), and the system will stay in autoresonance. However, the pseudopotential wells will only exist if the pseudopotential tilt is less than the well depth, i.e., if

$$2\epsilon I_0^{1/2}>\alpha/S. \tag{21}$$

If this condition is not met, the wells will disappear, the pseudoparticle will escape, phase locking will be lost, and the system will not stay in autoresonance.[1] By differentiating the quantity $SI_0^{1/2}$ (remember that S is itself a function of I_0), it is easy to show that there is a critical action at which condition (21) is most difficult to satisfy and the wells are most likely to disappear,

$$I_{0\mathrm{crt}}=(\epsilon/\beta\omega_0)^{2/3}. \tag{22}$$

If the system makes it past this critical action, the wells will be retained effectively forever, and the system will grow to high amplitude. Thus, replacing I_0 with $I_{0\mathrm{crit}}$ in Eq. (21) yields the minimum drive amplitude for which the system stays in autoresonance:

$$\epsilon_c=\frac{1}{\sqrt{\beta\omega_0}}\left(\frac{\alpha}{3}\right)^{3/4} \tag{23}$$

or

$$\overline{\epsilon_c}=8\sqrt{\omega_0}\left(\frac{\alpha}{3}\right)^{3/4}. \tag{24}$$

This threshold has been verified numerically (see Figs. 1 and 3) and, for the non-neutral plasma Diocotron mode, experimentally.[9,10]

For most system parameters, $I_{0\mathrm{crit}}$ is quite small. For ϵ near the critical value [Eq. (23)], $I_{0\mathrm{crit}}$ is approximately $(1/\beta)(3N)^{-1/2}$, where $N=\omega_0^2/\alpha$ measures the number of cycles in the sweep, so $I_{0\mathrm{crit}}$ is indeed small for any reasonable sweep rate. Consequently, whether or not the system stays in autoresonance is determined at very low pendulum amplitude. Thus, the assumptions made at the beginning of this section, that we need only take the first-order terms in the action-angle equations, and that only the first-order correction to the linear frequency is important, are valid. More importantly, we can generalize our results to *any* driven nonlinear oscillator whose nonlinear frequency dependence reduces to $\omega=\omega_0(1+\beta I)$ for small I, i.e., when the equation of motion reduces to

$$\ddot{\theta}+\omega_0^2(\theta+a\theta^2+b\theta^3)=\overline{\epsilon}\cos(\omega_0 t-\alpha t^2/2). \tag{25}$$

For the pendulum, $\beta=3b/4-5a^2/6$.[31] If $a=0$, Eq. (25) is called the Duffing equation, and is the low amplitude description of many physical systems. Systems that reduce to the Duffing equation are very common because the ''Duffing'' term, $b\omega_0^2\theta^3$, is the first symmetric nonlinearity. Thus any system whose potential goes like $U(\theta)=\overline{a}\theta^2+\overline{b}\theta^4+\cdots$, where $\overline{a}$ and b are constants, will reduce to a Duffing oscillator at low amplitude. The pendulum, the Diocotron (Sec. III),[27] many mechanical systems,[16–20] galvanometers,[15] electronic oscillators,[21] etc., all reduce appropriately. Some two-dimensional systems, like the Neptune–Plutinos systems (Sec. IV), also reduce appropriately.

C. Strongly nonlinear regime

In the strongly nonlinear regime the action becomes large, and the small I assumption used to derive Eqs. (13) and (14) is not strictly valid. Nonetheless, computer simulations of the original equation of motion [Eq. (2)] and analysis of the corresponding action-angle evolution equations[32] confirm that the phase locking in the system continues, and the frequency of the oscillator decreases (i.e., its oscillation amplitude grows) to approximately match that of the drive, until the pendulum swings over the top. This usually happens before the drive frequency reaches zero, as the small oscillations around the equilibrium action will be sufficient to drive the pendulum over.

Intuition might lead one to believe that the drive amplitude would have to increase eventually to excite the pendulum to very high amplitude; perhaps the most remarkable aspect of the strongly nonlinear regime is that the drive amplitude can be *decreased* when the pendulum amplitude is large. This follows from the decreasing pseudopotential tilt once past the critical action, which makes the wells effectively deeper. Thus the drive can be decreased while still maintaining the wells.

III. AUTORESONANCE IN THE DIOCOTRON MODE IN PURE-ELECTRON PLASMAS

Consider a column of electrons that is aligned along a strong magnetic field **B**, and confined within a conducting cylindrical wall. If the column is displaced from the cylindrical wall axis, it will orbit around the axis. This oscillation is called the Diocotron mode,[27] and is found in electron beams in accelerators and in ''pure-electron plasmas'' confined in Malmberg–Penning traps (Fig. 5). Understanding the Diocotron has been crucial to understanding charged plasmas, and dozens of papers have been published on the subject. The Diocotron oscillation comes about because of an interaction between the electron column and its image. At any given instant, the electron column **E×B** drifts azimuthally in the electric field **E** of the image. (See Fig. 6. An electron subjected to crossed **E** and **B** fields will move perpendicular to both fields, hence the name **E×B** drift. The physics of the drift is simple: The electron is accelerated antiparallel to **E**, but the Lorentz force from **B** pushes the electron sideways. The net motion is a cycloid whose center moves at velocity **E×B**.) The image moves with the column,

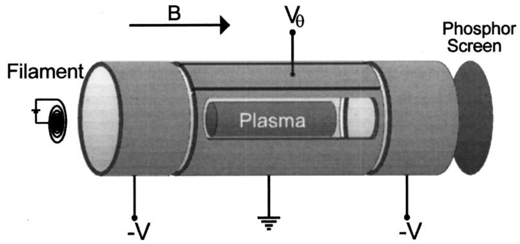

Fig. 5. Malmberg–Penning trap geometry. Electrons are confined axially by negatively biasing the end cylinders, and radially by a strong axial magnetic field. The electrons are emitted from the filament, and loaded into the trap by temporarily grounding the left cylinder. Details of the trap operation can be found in Ref. 37.

so the net column motion is to orbit around the wall axis. The distance off axis is a measure of the amplitude of the oscillation. The image electric field gets stronger nonlinearly as the column approaches the wall, so the oscillation frequency increases with amplitude, following the equation

$$\omega = \frac{\omega_0}{1-r^2}, \tag{26}$$

where ω_0 is the linear frequency of the mode, and r is the suitably normalized oscillation amplitude. The oscillation is very robust, and very lightly damped; the column can orbit the trap hundreds of thousands of times.

At low amplitude (r small) the Diocotron reduces to a Duffing oscillator. It is easy to drive the Diocotron by impressing an oscillating voltage on an azimuthal section of the wall. By starting at a frequency below the linear frequency, and sweeping the frequency upwards, the Diocotron can be grown autoresonantly[9,10,24–26] to so large an amplitude that the electron column hits the wall. As shown in Fig. 7, the system obeys the sweep rate threshold scaling law Eq. (24) over five orders of magnitude. Many other aspects of the autoresonant theory presented here have been tested with the Diocotron, including the behavior in the linear regime, the existence of the pseudopotential wells, and the possibility of lowering the drive strength once in the strongly nonlinear regime.

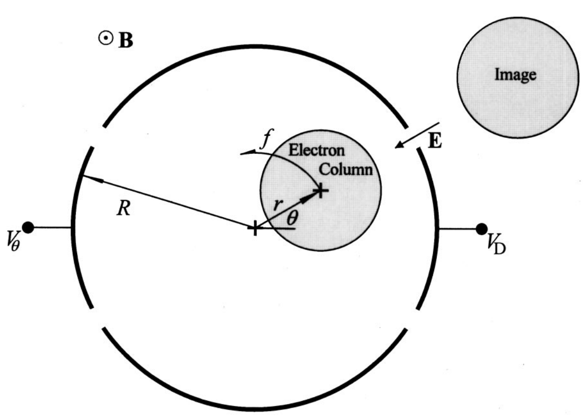

Fig. 6. End view of the Malmberg–Penning trap showing the confining wall at R, the electron column a distance r from the trap center, the electron column image, the image electric field **E**, and the Diocotron drift at frequency f. The mode is detected by monitoring the image charge on the pickup sector V_θ and driven by applying a voltage to the drive sector V_D. Further details are given in Ref. 10.

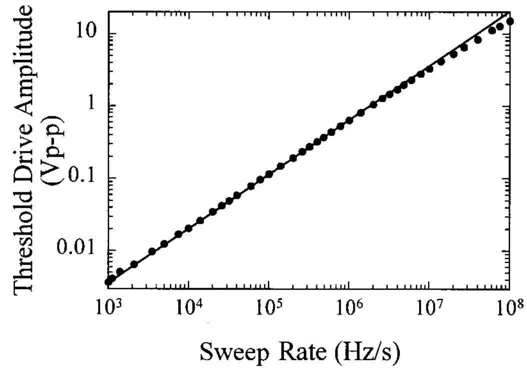

Fig. 7. Critical Diocotron drive amplitude vs sweep rate $\alpha/2\pi$. Measured results (●), and theoretical prediction from Eq. (24) (solid line). For more information, see Ref. 10.

IV. THE PROBLEM OF PLUTINOS

The autoresonance phenomenon and the threshold for capture into resonance described in Sec. II B can be found in more complicated dynamical systems. One of the most remarkable examples is found in Nature in relation to early evolution of the trans-Neptunian region[33] of the solar system. In contrast to the nonlinear pendulum, the Keplerian two-body (Sun–planet) problem has two degrees of freedom and yields elliptical orbits in a plane, given, in polar coordinates (r,φ), by the equation $a/r = 1 + e\cos\varphi$, where e is the eccentricity of the orbit. It is well known that most planets in the solar system move roughly in one plane (the ecliptic) on nearly circular orbits with eccentricities $e < 0.09$. A notable exception is Pluto, the most distant planet in the solar system, which has a very eccentric orbit ($e = 0.25$) and a semi-major axis of about 39 AU. Furthermore, Pluto is observed to be in a 3:2 resonance with Neptune, i.e., Pluto completes two rotations around the Sun during the time Neptune completes three rotations. According to present understanding, this peculiar synchronization has existed since the planetary formation several billion years ago.

Pluto is not the only trans-Neptunian body in the solar system. There exist a large number of smaller masses (estimated number of ~100,000) which comprise the Kuiper Belt, a disk-shaped region at distances roughly between 35 and 100 AU from the Sun (1 AU is the distance between the Earth and the Sun). Remarkably, about one-third of presently observed Kuiper Belt objects (KBOs) are also engaged in the 3:2 resonance with Neptune (these objects resemble Pluto and, thus, are called Plutinos) and also have very eccentric orbits $0.1 < e < 0.35$.[34] How one can explain this anomaly? What is the origin of orbital eccentricity of the resonant trans-Neptunian objects?

It is now widely accepted that the resonances in the Plutino orbits are the result of Neptune's orbit migration during the last stage of planetary formation. Neptune's radius r_N experienced a slow increase (by roughly 30% during some time) due to some migration mechanism, and its orbital frequency decreased concomitantly. The force exerted by the orbiting Neptune acted as a quasi-periodic drive on the Plutinos, passing through, and capturing many of them in resonance,[35] just as the pendulum phase locks to its drive as described in Sec. II A. Beyond the trapping stage, as Neptune's radius continued to increase, the Plutinos entered autoresonance and their initially circular orbits developed growing radial oscillations, i.e., became increasingly eccentric (see the orbit equation above).

The theory of migrating planets explained a number of important features observed today, such as the distribution of eccentricities of Plutinos in the Kuiper belt, for example. Nevertheless, the early theory[36] could not explain the dearth of KBOs in the similar, 2:1 resonance with Neptune. These early calculations yielded similar populations of 3:2 and 2:1 resonances. Astronomical observations, on the other hand, showed almost no 2:1 resonant KBOs. The dearth of 2:1 resonant KBOs was one of the main remaining questions of the resonant KBO capture theory, with a possible explanation given only recently.[14] The explanation is related to the issue of threshold for capture into resonance. We have shown that the governing equations describing orbital evolution of a KBO in the field of slowly migrating Neptune can be described by the reduced system of evolution equations identical to Eqs. (13) and (14), where I is replaced by e^2, while Φ represents the phase mismatch between the rotating Neptune and the KBO. In the $(j+1):j$ autoresonance, $(j+1)\Omega \approx j\omega_N(t)$, where Ω is the Keplerian frequency of the trapped KBO and $\omega_N(t)$ is the slowly varying frequency of migrating Neptune. Each resonance has a different coupling constant ϵ [see Eqs. (3) and (4)] and, thus, yields different Neptune migration time scale $t^{\mathrm{mig}}=r_N(dr_N/dt)^{-1}$ thresholds for trapping into resonance. We have calculated these thresholds[14] and found that $t_{\mathrm{th}}^{\mathrm{mig}}=2\times 10^6$ and 2×10^7 yr for capturing into 3:2 and 2:1 resonances, respectively. The order of magnitude difference between the thresholds is due to the Sun's rotation around the Sun–Neptune center of mass. Inclusion of this rotation required analysis of the associated three-body problem, but affected the time scale for capture into 2:1 resonance *only*, lowering the corresponding value of ϵ significantly, thus increasing the threshold time scale by one order of magnitude. Therefore, if the actual migration time scale of Neptune is between 2×10^6 and 2×10^7 yr, a large fraction of the KBOs would be captured into 3:2 resonance and none into 2:1 resonance, as observed in present solar system. Thus, these arguments allow us to find accurate bounds on the time scales involved in the early stage of the evolution of the solar system on the bases of present astronomical observations, and to discard slower Neptune migration models.

V. CONCLUSIONS

We have shown that a swept-frequency drive can strongly excite a pendulum. The initial drive frequency must be above the pendulum's linear frequency. The pendulum will phase lock to the drive while the drive frequency is still above the linear frequency. If the pendulum remains locked to the drive past a critical drive frequency, the pendulum will be strongly excited as the drive continues to sweep downwards. In this circumstance, the pendulum's amplitude will automatically adjust itself so that the pendulum's nonlinear frequency will match the drive frequency: hence the name autoresonance. The pendulum will remain locked to the drive only if the drive strength exceeds a threshold proportional to the sweep rate α raised to the 3/4 power [Eq. (23)]. The critical point occurs at a frequency which is only slightly lower than the linear frequency, where the pendulum amplitude is still quite low. Because the critical point occurs at such a low amplitude, only the lowest order corrections to the frequency are relevant, and the pendulum can be considered to be a Duffing oscillator [Eq. (25)]. Many if not most nonlinear oscillators reduce to the Duffing oscillator at low amplitude, so the analysis presented here is very general, and the same $\alpha^{3/4}$ drive strength scaling applies to all these systems.

In this paper we assumed that the oscillator is lightly damped, but recent calculations and experiments[26] show that the results still apply in the presence of appreciable damping. We demonstrated elsewhere[24,28] that a very similar scaling law holds if a system is driven near a subharmonic ω_0/n. In this case, the scaling law becomes $\alpha^{3/4n}$. Furthermore, autoresonance phenomena occur when *any* sufficiently slow change is made to the system.[9] Here we have only discussed sweeping the drive frequency, but a pendulum can be autoresonantly excited by a fixed frequency drive if the pendulum bob length is continuously shortened such that the pendulum's linear resonant frequency passes through the drive frequency.[1] A similar scaling law will apply.[10]

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[1]Four movies illustrating the phenomena described in this paper can be found at http://socrates.berkeley.edu/~fajans/Autoresonance/Autoresonance.htm. The movies show: (1) a pendulum driven at its linear resonant frequency, (2) an autoresonant pendulum driven by a swept frequency drive, (3) an autoresonantly driven pendulum with the corresponding pseudopotential wells, and (4) a pendulum with an ever shrinking length driven at a fixed frequency.

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THE SECOND LAW

Dispersal into disorder creates because it need not be uniformly smooth. A flood of chaos there may result in a surge of order here. The purposeless increase in disorder of the world is not a smoothly descending river of energy, but a choppy rapid, that may throw up a structured foam and an elaborate wave as it plunges down. That order may take the form of a protein formed by an enzyme driven ultimately by the energy of the Sun, or the construction of a strip of DNA. It may power the jaws of a cheetah and the emergence on its coat of the stripes of a zebra. Thus the Second Law may erupt into evolution, and stronger cheetahs and better camouflaged zebras may emerge, transitorily, as the universe globally spreads in disorder. Thus the 'Creation'—everything—emerges as chaos spreads.

P. W. Atkins, ''The Limitless Power of Science,'' in *Nature's Imagination—The Frontiers of Scientific Vision*, edited by John Cornwell (Oxford University Press, New York, 1995).