Autoresonant solutions of the nonlinear Schrödinger equation

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Resonant driving of the nonlinear Schrödinger (NLS) equation by small-amplitude oscillations or waves with adiabatically varying frequencies and/or wave vectors is proposed as a method of excitation and control of wave-type solutions of the system. The idea is based on the autoresonance phenomenon, i.e., a continuous nonlinear phase locking between the solutions of the NLS equation and the driving oscillations, despite the space-time variation of the parameters of the driver. We illustrate this phenomenon in the examples of excitation of plane and standing waves in the driven NLS system, where one varies the driver parameters in time or space. The relation of autoresonance in these applications to the corresponding problems in nonlinear dynamics is outlined. One of these dynamical problems comprises a different type of multifrequency autoresonance in a Hamiltonian system with two degrees of freedom. The averaged variational principle is used in studying the problem of autoresonant excitation and stabilization of more general cnoidal solutions of the NLS equation. [S1063-651X(98)07109-8]

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I. INTRODUCTION

The nonlinear Schrödinger equation

$$i\psi_t+\psi_{xx}+|\psi|^2\psi=0 \tag{1}$$

is one of the most important equations of nonlinear physics. It admits a variety of solutions, a well known example being the cnoidal wave [1] $\psi(x,t)=a(x-vt)\exp\{i[k_0x-\omega_0t]\}$, where $v=2k_0$, $\omega_0=k_0^2-\Lambda_0$, and the real function $a(s)$ satisfies

$$a_{ss}-\Lambda_0a+a^3=0. \tag{2}$$

The simplest realizations of the cnoidal waves are plane, constant amplitude waves $\psi=\Lambda_0^{1/2}\exp\{i[k_0x-\omega_0t]\}$ (here and in the following we assume $\Lambda_0>0$) and the standing waves $\psi=a(x)\exp(i\Lambda_0t)$. In the present work we shall study the question of adiabatic excitation and control of the cnoidal solutions by adding an external resonant driving on the right-hand side (RHS) of Eq. (1). We shall find a proper choice of the driving oscillations (waves) for this purpose by using the idea of autoresonance in the system. A similar problem was studied recently for driven systems of a certain general class described by a variational principle [2]. Investigations of particular examples of autoresonant solutions exist for the Korteweg–de Vries [2–4] and sine-Gordon [5] equations as well as for coupled sine-Gordon equations [6].

Autoresonance is a nonlinear phase locking phenomenon, taking place when a resonantly driven nonlinear system remains phase locked with the driving oscillation (or wave) despite the adiabatic variation of the frequency and/or wave vector of the driver. The phase locking is due to the tendency of the driven nonlinear system to preserve the resonance by slowly adjusting its state (energy, frequency, wave vector, etc.) in space-time. Thus variation of the parameters of the driver allows control of the state of the driven system. The autoresonance phenomenon has its roots in nonlinear dynamics and a number of dynamical applications can be found in the literature. These include particle accelerators [7–9], excitation of atoms [10] and dissociation of molecules [11], controlled transition to chaos [12], and autoresonance in higher-order dynamical systems [13,14]. We shall also see the dynamical connection in the present work, which is devoted to studying autoresonance in the driven nonlinear Schrödinger (NLS) equation.

The scope of our presentation is as follows. In Sec. II we shall consider the simplest autoresonant solutions in our system, i.e., autoresonant plane waves. We shall see that these solutions can be excited by driving the system by an external plane wave with slowly varying frequency. In Sec. III we shall study a more complex problem of excitation of standing waves. We shall show that excitation of autoresonant standing waves requires using a superposition of an oscillation and a standing wave in the driver. In Sec. IV we shall further generalize our theory for studying the excitation and stability of more general autoresonant cnoidal waves in the NLS system. We shall approach this problem by constructing an averaged variational principle for the phase locked driven NLS equation. Several applications of the variational theory are given in Sec. V. Finally, Sec. VI presents our conclusions and Table I gives a glossary of symbols and the locations where these symbols appear for the first time.

II. TEMPORAL AUTORESONANCE: AUTORESONANT PLANE WAVES

In this section we illustrate the idea of autoresonant excitation of a nonlinear wave by studying *temporal* autoresonance in the NLS system. Consider the driven NLS equation

$$i\psi_t+\psi_{xx}+|\psi|^2\psi=\varepsilon\ \exp\{i[k_0x-k_0^2t+\phi(t)]\}, \tag{3}$$

where the RHS represents a small, constant amplitude ($\varepsilon\ll1$) plane wave having wave vector k_0 and *slowly* varying frequency $k_0^2-\Lambda(t)$, where $\Lambda(t)\equiv d\phi/dt$ and all variables

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TABLE I. Glossary of symbols.

Symbol	Definition
ψ	a solution of the nonlinear Schrödinger equation (1)
a,k_0,ω_0	the amplitude, wave number, and frequency of a cnoidal wave [below Eq. (1)]
v	the velocity of the filling phase [below Eq. (1)]
ε,ϕ,Λ	the amplitude, phase factor, and associated frequency of the driver [Eq. (3)]
Φ	the phase mismatch [Eq. (4)]
I,θ,H	the canonical action, angle, and Hamiltonian in the dynamical problem associated with temporal autoresonance [below Eq. (5)]
$\Psi,\mathscr{k}$	the phase and wave number of the spatial modulation of the driver [below Eq. (7)]
$V_{\rm eff}$	the effective potential [Eq. (9)]
$J,\Theta;B,\xi$	the radial and azimuthal action-angle pairs associated with the spatial autoresonance [below Eq. (11)]
S	the generating function [below Eq. (12)]
A	the energy parameter [Eq. (13)]
U,V	the radial and azimuthal solutions [Eqs. (15) and (16)]
$\lambda(J,B),\beta(J,B)$	the radial wavelength and azimuthal wave number [below Eq. (16)]
$\alpha_{0,1}$	the amplitudes of the zero and first harmonics of radial oscillations [below Eq. (17)]
μ	the mismatch of the radial oscillations [Eq. (19)]
k,ω	the wave number and frequency associated with the driving phase and amplitude, respectively [below Eq. (25)]
L,L_0	the Lagrangians [Eq. (27)]
Ω,K	the frequency and wave number associated with the radial angle variable [below Eq. (28)]
γ,β	the frequency and wave number associated with the azimuthal angle variable [below Eq. (28)]
$\mathcal{L},\mathcal{L}_{0,1}$	the averaged Lagrangian and its components [Eq. (29)]
h	the velocity parameter [below Eq. (30)]
R	the frequency parameter [below Eq. (33)]
ν,κ	the frequency and wave number of the slow modulations [below Eq. (51)]
d_{1-4}	the coefficients in the dispersion relation (52)
c_{1-3}	the coefficients in the characteristic equation (59)

and parameters are dimensionless. We are interested in finding solutions of Eq. (3) that are phase locked with the driver. Therefore, we write the solution as $\psi=a(x,t)\exp\{i[k_0x-k_0^2t+\theta(x,t)]\}$, where $\mathrm{Im}(a,\theta)=0$. Then $f\equiv a\exp(i\theta)$ satisfies $i(f_t+vf_x)+f_{xx}+|f|^2f=\varepsilon\exp[i\phi(t)]$, where, as in Sec. I, $v=2k_0$. By separating the real and imaginary parts in the last equation and defining the phase mismatch $\Phi\equiv\theta-\phi(t)$, we obtain the equivalent pair of real equations

$$a_t+va_x+a\Phi_{xx}+2a_x\Phi_x=-\varepsilon\sin\Phi,$$
$$a(\Phi_t+v\Phi_x)-a_{xx}+a\Phi_x^2+\Lambda a-a^3=-\varepsilon\cos\Phi. \tag{4}$$

First, we seek the solution of Eq. (3) satisfying a *uniform* initial condition $\psi(x,t=t_0)=\psi_0$, where $|\psi_0|\ll 1$. Consequently, we study solutions of the purely time-dependent limit of Eq. (4):

$$a_t=-\varepsilon\sin\Phi,$$
$$\Phi_t=a^2-\Lambda(t)-(\varepsilon/a)\cos\Phi, \tag{5}$$

subject to the initial conditions $a(t_0)\ll 1$ and $\Phi(t_0)=\Phi_0$ (an arbitrary constant). Equations (5) can be interpreted as Hamilton's equations for a one degree of freedom dynamical problem, with the Hamiltonian $H(I,\theta,t)=I^2/2+2\varepsilon I^{1/2}\cos(\theta-\phi)$, where $I\equiv a^2$ and θ serve as the canonical action-angle pair of the unperturbed Hamiltonian $H_0(I)\equiv I^2/2$. The latter describes an oscillator of frequency $\partial H_0/\partial I=I$, while the perturbed Hamiltonian $H(I,\theta,t)$ has the characteristic form, known as the *single resonance approximation* in the standard theory of nonlinear resonance [15], but for the case of a slowly varying frequency $\phi_t=\Lambda(t)$ of the driving oscillation. A similar adiabatically driven dynamical problem was studied in Ref. [16]. The autoresonant solution is obtained when the driving frequency $\Lambda(t)$ passes linear resonance with the unperturbed oscillator, i.e., the point $\Lambda=0$. We illustrate this solution of Eqs. (5) in Fig. 1, showing the numerical results for the amplitude a [Fig. 1(a)] and the phase mismatch $\Phi(\mathrm{mod}\ 2\pi)$ [Fig. 1(b)] for $\varepsilon=0.05$ and

$$\Lambda(t)=\begin{cases}\Lambda_0\sin(\frac{1}{2}\pi t/T_0), & -T_0>t>T_0\\ \Lambda_0, & t>T_0,\end{cases}$$

where $T_0=200$ and $\Lambda_0=2$. We have used the initial conditions (at $t_0=-T_0$) $a=0.01$ and $\Phi=\pi$ in these calculations. Figure 1 demonstrates the initial phase trapping stage, at $t<0$, where the phase mismatch $\Phi(\mathrm{mod}\ 2\pi)$ settles near zero.

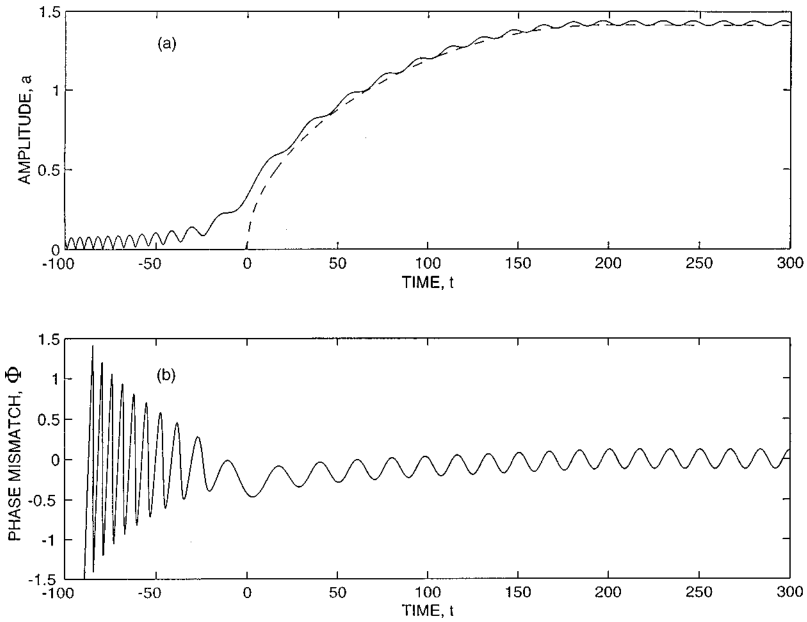

FIG. 1. Autoresonant excitation of a plane wave in the driven NLS system. The solid lines show (a) the amplitude a of the wave and (b) the phase mismatch $\Phi(\mathrm{mod}2\pi)$. The dashed line represents (for $t>0$) the smooth autoresonant solution $[\Lambda(t)]^{1/2}$ for the amplitude.

At later times, after the trapping, one can see a typical autoresonant evolution of the system, as the nonlinear frequency shift a^2 approximately follows the driving frequency $\Lambda(t)$ (we show the values of $[\Lambda(t)]^{1/2}$ by the dashed line in Fig. 1(a)]), despite the time variation of the frequency. One can also see that, in autoresonance, the system performs small (and slow) autoresonant oscillations around the quasi-equilibrium solutions. These oscillations comprise an additional characteristic signature of autoresonance and their frequency scales as $\nu\approx\sqrt{2\varepsilon a}$ [16]. In conclusion, at final evolution times, when Λ approaches Λ_0, one arrives at the simplest, plane cnoidal wave solution $\psi\approx a_0 \exp\{i[k_0x-(k_0^2-\Lambda_0)t]\}$ in the driven NLS system with the amplitude $a_0\approx\Lambda_0^{1/2}$ controlled by the final value of the driving frequency.

At this point we recall that, so far, spatial variations have been suppressed. Now we remove this restriction and consider the question of stability of the temporally autoresonant solution with respect to small spatial perturbations. We shall limit the analysis to studying the stability of the uniform autoresonant solution at times beyond T_0, when the time varying part of the driving frequency arrives at its constant value $\Lambda=\Lambda_0$. We again seek solutions of Eqs. (4), but now a and θ are functions of space-time. The system (4) is satisfied by $\Phi=\bar{\Phi}\equiv 0$ and $a=\bar{a}$ given by $\bar{a}^3-\Lambda_0\bar{a}-\varepsilon=0$, i.e., $\bar{a}\approx\Lambda_0^{1/2}$ (note that $\bar{\Phi}$ and $\bar{a}$ are the time averaged values of Φ and a in Fig. 1 at $t\geq T_0$). By perturbing this steady state, i.e., writing $\Phi(x,t)=\bar{\Phi}+\delta\Phi(x,t)=\delta\Phi(x,t)$ and $a(x,t)=\bar{a}+\delta a(x,t)$, where $\delta\Phi,\delta a\sim\exp[i(\kappa x-\nu t)]$, and linearizing Eqs. (4) around $\bar{\Phi}$ and $\bar{a}$ one finds the dispersion relation

$$(\nu-\kappa v)^2=(\kappa^2-\varepsilon/\bar{a})(\kappa^2-2\Lambda_0-3\varepsilon/\bar{a}). \tag{6}$$

Equation (6) predicts an instability for $\varepsilon/\bar{a}<\kappa^2<2\Lambda_0+3\varepsilon/\bar{a}$. Nevertheless, in contrast to the free ($\varepsilon=0$) plane wave solution, which is always unstable at sufficiently small κ, there exists a *stability window* $\kappa^2<\varepsilon/\bar{a}$ for the autoresonant solution. Thus the autoresonant driving stabilizes the plane wave solutions with respect to sufficiently slow spatial modulations.

III. SPATIAL AUTORESONANCE: AUTORESONANT STANDING WAVES

In this section we consider the boundary value driven NLS problem

$$i\psi_t+\psi_{xx}+|\psi|^2\psi=\varepsilon(x)\exp(i\Lambda_0 t), \tag{7}$$

where, at some position (say, $x=x_0$) and for *all* times, $\psi(x_0,t)$ is the stationary autoresonant plane wave of the preceding section for $k_0=0$ and constant value Λ_0 of the driving frequency. Thus, on the boundary, $\psi(x_0,t)=a_0\exp[i\theta_0(t)]$, where $a_0=\Lambda_0^{1/2}+\delta a_0$, $\theta_0=\Lambda_0 t+\delta\theta_0$ (i.e., $\Phi_0=\theta_0-\Lambda_0 t=\delta\theta_0\equiv\delta\Phi_0$), and $a_{0x}=\Phi_{0x}=0$. Note that we allow small, but constant, perturbations δa_0 and $\delta\Phi_0$ in the boundary conditions, so the following analysis tests the stability of the autoresonant solutions with respect to these perturbations. Also, in contrast to Eq. (3), where the amplitude ε of the driving oscillation was fixed, we add a quasiperiodic *spatial modulation* in the RHS in Eq. (7) and use $\varepsilon(x)\equiv\varepsilon_0+\varepsilon_1\cos[\Psi(x)]$, where $k(x)\equiv\Psi_x$ is a *slowly* varying wave number of the modulation. One could choose a different form of quasiperiodic modulation; the only requirement would be (see below) a sufficiently large average of $\varepsilon(x)$ over one spatial oscillation. Thus we drive our system by the superposition of an oscillation $\varepsilon_0\exp(i\Lambda_0 t)$ and a standing wave $\varepsilon_1\cos\Psi(x)\exp(i\Lambda_0 t)$. We shall see that this *double* driving is essential for stabilizing the resulting autoresonant solution.

Now we write the desired solution of Eq. (7) as $\psi=a(x)\exp\{i[\Lambda_0 t+\Phi(x)]\}$, where $a(x)$ and $\Phi(x)$ satisfy the spatial limit of Eqs. (4) with $\varepsilon=\varepsilon(x)$, $\Lambda=\Lambda_0$, and $v=0$:

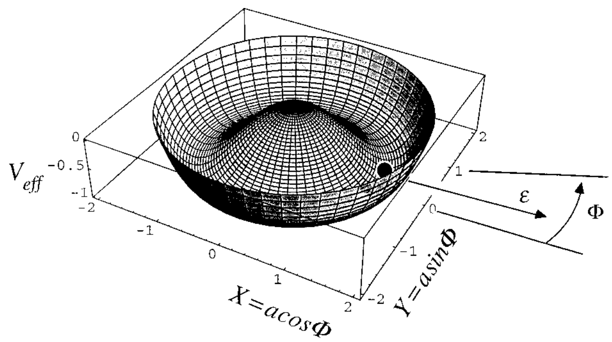

FIG. 2. Quasipotential $V_{\rm eff}(a)$ versus $X=a\cos\Phi$ and $Y=a\sin\Phi$. The initial position of the quasiparticle, i.e., $a_0\approx\Lambda_0^{1/2}$, $\Phi\approx 0$, is shown by a solid dot. The arrow indicates the direction of the uniform field ε acting on the quasiparticle in the dynamical analog.

$$a\Phi_{xx}+2a_x\Phi_x=-\varepsilon(x)\sin\Phi,$$
$$a_{xx}-a\Phi_x^2-\Lambda_0 a+a^3=\varepsilon(x)\cos\Phi. \tag{8}$$

The rest of this section is devoted to studying autoresonant solutions of this purely x-dependent system. The temporal modulations of these solutions will be included in Sec. IV.

We proceed by observing that Eqs. (8) comprise a two degrees of freedom *dynamical* problem, where x plays the role of ''time.'' Indeed Eqs. (8) are Lagrange's equations, when the Lagrangian is $L(a,a_x;\Phi,\Phi_x)=L_0+\varepsilon(x)a\cos\Phi$ and

$$L_0(a,a_x;\Phi_x)\equiv\tfrac{1}{2}(a_x^2+a^2\Phi_x^2)+(\tfrac{1}{2}\Lambda_0 a^2-\tfrac{1}{4}a^4). \tag{9}$$

Now, for convenience, we make the transition from the Lagrangian to the Hamiltonian formulation. To this end, we define the canonical momenta $p^a\equiv\partial L/\partial a_x=a_x$ and $p^\Phi\equiv\partial L/\partial\Phi_x=a^2\Phi_x$ and write the associated Hamiltonian

$$H=a_x p^a+\Phi_x p^\Phi-L=H_0+H_1, \tag{10}$$

where

$$H_0=\tfrac{1}{2}[(p^a)^2+(p^\Phi/a)^2]+V_{\rm eff}(a),$$
$$H_1=-\varepsilon(x)a\cos\Phi, \tag{11}$$

and $V_{\rm eff}(a)\equiv-\frac{1}{2}\Lambda_0 a^2+\frac{1}{4}a^4$. The unperturbed Hamiltonian H_0 describes the *planar* motion of a quasiparticle of unit mass under the action of a *central* force given by the potential $V_{\rm eff}$. The variables a and Φ, in this interpretation, play the roles of polar coordinates of the quasiparticle, while $p^{a,\Phi}$ are the *radial* and *angular* momenta of the quasiparticle, respectively. Then the perturbed part H_1 of the Hamiltonian can be viewed as representing the interaction of the quasiparticle with an external, *uniform* quasiperiodic force of strength $\varepsilon(x)$ acting in the $\Phi=0$ direction in the plane. We illustrate this useful interpretation in Fig. 2, showing the effective potential as a function of the Cartesian coordinates $X=a\cos\Phi$ and $Y=a\sin\Phi$ for the case $\Lambda_0=2$. Note that the *initial* position of the quasiparticle in the X-Y plane, $a_0=\Lambda_0^{1/2}+\delta a_0$, $\Phi_0=\delta\Phi_0$, corresponds to a near equilibrium (minimum) point in the effective potential. In the rest of this section we study autoresonant excitation of the radial oscillations of this *two-dimensional* oscillator. This problem differs from the autoresonance problems studied previously by the presence of the second *resonant* (azimuthal) degree of freedom. The question of autoresonant stabilization of the azimuthal motion will be discussed. The following analysis is also important as a step for studying (in Sec. IV) a more general, space-time autoresonance in the driven NLS system.

The most convenient description of autoresonance in this two degrees of freedom system is obtained by transforming to the action-angle variables of the unperturbed Hamiltonian H_0. This Hamiltonian is independent of Φ (Φ is a *cyclic* variable) and therefore the angular momentum $p^\Phi\equiv B$ remains constant in the unperturbed problem. We use this fact and make the canonical transformation of variables $(a,p^a;\Phi,B)\to(\Theta,J;\xi,B)$ via the generating function

$$F(a,J;\Phi,B)\equiv S(a,J,B)+\Phi B. \tag{12}$$

Here $S(a,J,B)\equiv\int^a p^*[A(J,B),B,a']da'$, where $p^*(A,B,a)$ is defined as the solution of

$$H_0=\tfrac{1}{2}[(p^a)^2+(B/a)^2]+V_{\rm eff}(a)=A \tag{13}$$

for p^a, and the radial action variable J is given by

$$J(A,B)\equiv(2\pi)^{-1}\oint p^*(A,B,a)da. \tag{14}$$

Also, we have rewritten Eq. (14) as $A=A(J,B)$ in the definition of $S(a,J,B)$. The generating function (12) defines the angle variable conjugate to the action J, i.e., $\Theta(a,J,B)\equiv\partial F/\partial J=\partial S/\partial J$ or, by inversion,

$$a=U(\Theta,J,B). \tag{15}$$

Similarly, the canonical angle variable conjugate to B is $\xi\equiv\partial F/\partial B=\Phi-V(a,J,B)$, where $V(a,J,B)\equiv-\partial S/\partial B=-\int^a(\partial p^*/\partial B)da'=V(\Theta,J,B)$, so

$$\Phi=\xi+V(\Theta,J,B). \tag{16}$$

Note that the change of $S(a,J,B)$ during one period of radial oscillations is $\Delta S=2\pi J$, so Θ changes by 2π and thus $U(\Theta,J,B)$ is 2π periodic in Θ. On the other hand, the change of V over an oscillation is $\Delta V=-\partial(\Delta S)/\partial B=0$, so $V(\Theta,J,B)$ is also 2π periodic in Θ.

Since the generating function (12) does not depend on x explicitly (recall that x is our time variable), the transformed unperturbed Hamiltonian is simply $H_0=A(J,B)$. Consequently, in the unperturbed problem, the actions J,B and the *frequencies* $\Theta_x=\partial A/dJ\equiv 2\pi/\lambda(J,B)$ and $\xi_x=\partial A/\partial B\equiv\beta(J,B)$ remain constant. Note that β describes a *constant* angular velocity component characterizing the evolution of the cyclic variable Φ in our central force problem, while $V(\Theta,J,B)$ is the component of Φ oscillating with the period of the radial oscillations. Next we express the perturbed part of the Hamiltonian $H_1=-\varepsilon(x)a\cos\Phi$ in terms of the action-angle variables of the unperturbed problem:

$$H_1=-(\varepsilon_0+\varepsilon_1\cos\Psi)U(\Theta,J,B)\cos[\xi+V(\Theta,J,B)]. \tag{17}$$

Here, because of the periodicity in Θ, we can expand $U\cos V=\Sigma_n\alpha_n(J,B)\exp(in\Theta)$ and $U\sin V=\Sigma_n\beta_n(J,B)\exp(in\Theta)$. Finally, we make the double resonance approximation, i.e., assume a persisting *approximate* double resonance $\xi\approx 0$ and $\Theta-\Psi\approx 0$ in the system and, by neglecting all the nonresonant terms in Eq. (17), replace the perturbed part H_1 of the Hamiltonian (10) by

$$\tilde{H}_1=-\{\varepsilon_0\alpha_0+\mathrm{Re}[\varepsilon_1\alpha_1 e^{i(\Theta-\Psi)}]\}\cos\xi$$
$$+\{\varepsilon_0\beta_0+\mathrm{Re}[\varepsilon_1\beta_1 e^{i(\Theta-\Psi)}]\}\sin\xi.$$

Here we shall also assume that the perturbed motion is such that, in addition to ξ, V remains small for all times (i.e., the cyclic variable $\Phi=\xi+V$ is always small) and, consequently, neglect the part of $\tilde{H}_1$ associated with V. Then our double resonance approximation for the transformed Hamiltonian becomes

$$\tilde{H}(\Theta,\xi;J,B;x)=A(J,B)-\{\varepsilon_0\alpha_0(J,B)+\varepsilon_1\alpha_1(J,B)\times\cos[\Theta-\Psi(x)]\}\cos\xi, \tag{18}$$

where we have chosen $U\cos V$ to be an even function of Θ, so α_1 is real.

At this stage, we proceed to studying the evolution given by Eq. (18). The corresponding Hamilton equations are

$$J_x=\varepsilon_1\alpha_1\sin\mu\cos\xi,$$
$$\mu_x=2\pi/\lambda(J,B)-k(x)-[\varepsilon_0\alpha_{0J}-\varepsilon_1\alpha_{1J}\cos\mu]\cos\xi,$$
$$B_x=-[\varepsilon_0\alpha_0-\varepsilon_1\alpha_1\cos\mu]\sin\xi, \tag{19}$$
$$\xi_x=\beta(J,B)-[\varepsilon_0\alpha_{0B}-\varepsilon_1\alpha_{1B}\cos\mu]\cos\xi,$$

where we have introduced the phase mismatch $\mu\equiv\pi+\Theta-\Psi$ of the radial oscillations and substituted the wave vector of the driving modulation $\Psi_x=k(x)$. The boundary conditions for this system are $J=\delta J_0\ll 1$ (recall that the quasiparticle starts near the minimum of the effective potential well, where $J=0$), $\mu=\pi-\Psi$ (we have set $\Theta=0$ at the boundary), $B=0$, and $\xi=\delta\Phi_0$. Next we observe that Eqs. (19) allow the azimuthal equilibrium with $B=\xi\equiv 0$. Indeed, from Eq. (14), for small B, $A(J,B)=A^0(J)+O(B^2)$ and, similarly, $\alpha_{0.1}=\alpha^0_{0.1}(J)+O(B^2)$. Then β, α_{0B}, and α_{1B} all scale as $O(B)$ and thus vanish at $B=0$. Therefore, the last two equations of Eqs. (19) are satisfied trivially for $B=\xi=0$, while the first two become

$$J_x=\varepsilon_1\alpha_1^0\sin\mu,$$
$$\mu_x=2\pi/\lambda^0(J)-k(x)-(\varepsilon_0\alpha^0_{0J}-\varepsilon_1\alpha^0_{1J}\cos\mu). \tag{20}$$

This system has the same form as the evolution equations for dynamic autoresonance in the one degree of freedom problem described in Sec. II [compare to Eqs. (5)]. Therefore, one can expect to find the phase locked solution of Eq. (20) similarly to that described in Sec. II. In other words, if, at the boundary, J is sufficiently small, the trapping into resonance ($\mu\to 0$) takes place as $k(x)$ passes the linear resonance point x_r, where $k(x_r)=2\pi/\lambda^0(0)$, while beyond this point, under certain conditions, the system enters the autoresonant regime. In autoresonance, due to self-adjustment of the action J, the approximate resonance condition $K^0(J)\equiv 2\pi/\lambda^0(J)\approx k(x)$ is preserved continuously, despite the variation of k.

Finally, we include the azimuthal perturbations and show that they can be stabilized in our system. Indeed, assume that B and ξ do not vanish, but remain $O(\varepsilon^{1/2})$ [ε being $\max(\varepsilon_0,\varepsilon_1)$] continuously (we shall verify this assumption later). Then, for sufficiently small μ, to lowest order in ε, Eqs. (19) become

$$J_x=\varepsilon_1\alpha_1^0\sin\mu,$$
$$\mu_x=K^0(J)-k(x), \tag{21}$$

and

$$B_x=-(\varepsilon_0\alpha_0-\varepsilon_1\alpha_1)\sin\xi,$$
$$\xi_x=\beta(J,B). \tag{22}$$

Equations (21) are independent of B and ξ, so the autoresonant evolution of J and μ is similar to that described previously. On the other hand, by differentiating the second equation of Eqs. (22) and substituting the first equation in the result, one obtains

$$\xi_{xx}\approx-(\varepsilon_0\alpha_0-\varepsilon_1\alpha_1)\beta_B\sin\xi. \tag{23}$$

Therefore, since $\beta_B>0$, we have an oscillating evolution of ξ, as long as $\varepsilon_0\alpha_0>\varepsilon_1\alpha_1$, and the characteristic spatial frequency of these oscillations is $\kappa^{\xi}=[(\varepsilon_0\alpha_0-\varepsilon_1\alpha_1)\beta_B]^{1/2}$. This frequency varies adiabatically as the action J evolves in the autoresonance according to Eqs. (21). Then, if ξ is small at the boundary, it remains small at later times provided the variation of J is sufficiently slow. Also, by integrating the first equation of Eqs. (22), we find the assumed scaling $B\sim(\varepsilon_0\alpha_0-\varepsilon_1\alpha_1)/\kappa^{\xi}\sim O(\varepsilon^{1/2})$. Note that the characteristic frequency κ^{ξ} of the oscillations of ξ differs from that of the oscillations of μ. Indeed, by differentiating the second equation of Eqs. (21) and substituting the first equation in the result, one obtains

$$\mu_{xx}=\varepsilon_1\alpha_1 K^0_J\sin\mu-k_x. \tag{24}$$

Therefore, since $K^0_J<0$, the characteristic frequency of the radial phase mismatch oscillations is $\kappa^{\mu}\approx(\varepsilon_1\alpha_1^0|K^0_J|)^{1/2}$, which differs from κ^{ξ}, but both κ^{ξ} and κ^{μ} scale as $\varepsilon^{1/2}$. Note that, for small values of the radial action J, the Fourier coefficient $\alpha_0\approx a_0$, while $\alpha_1\sim J^{1/2}\ll\alpha_0$. Therefore, the presence of α_0 in Eq. (23) is essential for stabilizing the azimuthal motion in the initial autoresonant excitation stage. Since the presence of the coefficient α_0 in our theory can be traced back to the presence of the nonvanishing spatial average of the modulations of the driver, this average is necessary for the azimuthal stabilization. One can also give a mechanical interpretation of the azimuthal stabilization. Indeed, the presence of the constant force ε_0 in the $\Phi=0$ direction in our driven dynamical problem given by the Hamiltonian (10) can be regarded as due to the additional potential $-\varepsilon_0 a\cos\Phi=-\varepsilon_0 X$, which simply tilts the effective poten-

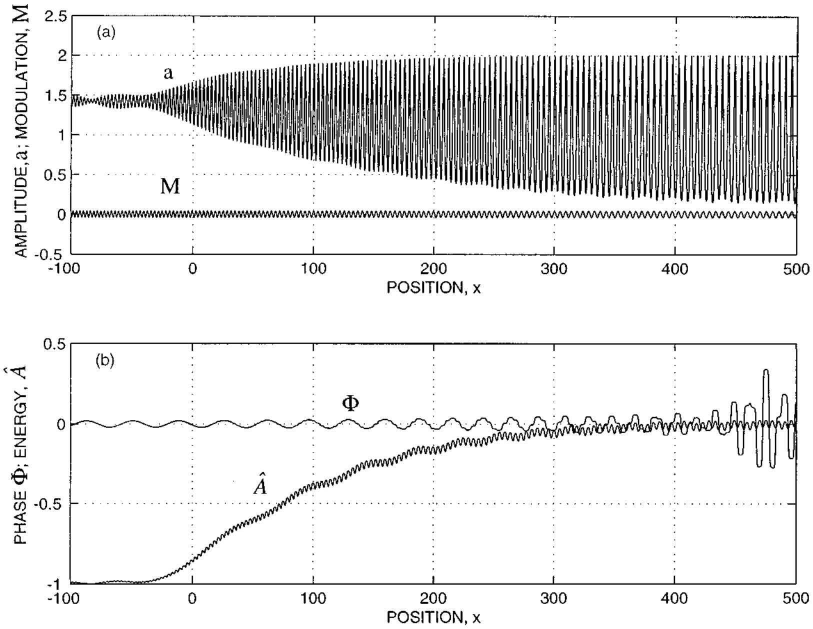

FIG. 3. Autoresonant excitation of standing wave solutions of the NLS equations. (a) The amplitude a and the quasiperiodic modulation of the driver $M=-\varepsilon_1\cos[\Psi(x)]$ versus position. One observes the autoresonant phase locking in the driven system. (b) The phase Φ and the energy $\hat{A}$ versus position. These results demonstrate the azimuthal stability of the autoresonant solutions and the presence of the characteristic autoresonant oscillations of the solutions around their slowly evolving averages.

tial shown in Fig. 2 around the $Y=a\ \sin\Phi$ axis, lowering the $X>0$ part of $V_{\rm eff}$, thus creating a *local minimum* in the potential at $Y=0$, $X\approx\Lambda_0^{1/2}$.

We conclude this section by numerical examples presented in Figs. 3 and 4. We solve Eq. (7) for $\varepsilon(x)=\varepsilon_0+\varepsilon_1\cos[\Psi(x)]$, $\varepsilon_{0,1}=0.04$, $\mathscr{k}(x)=\Psi_x=\mathscr{k}_0+\Delta\mathscr{k}\ \sin[\frac{1}{2}\pi x/X_0]$, $\mathscr{k}_0=(2\Lambda_0)^{1/2}$, $\Delta\mathscr{k}=0.4\mathscr{k}_0$, $\Lambda_0=2$, and $X_0=500$, subject to the boundary conditions (at $x_0=-200$) $\delta a_0=0.005$, $\delta\Phi_0=0.02$, and $a_{0x}=\Phi_{0x}=0$. The results of these calculations for the amplitude a are shown in Fig. 3(a). We observe the efficient excitation of the radial oscillations of a as one passes the linear resonance point $x=0$. We also show the phase-shifted (by π) quasiperiodic part $M=\varepsilon_1\cos[\Psi(x)-\pi]=-\varepsilon_1\cos\Psi$ of the modulation of the driver in Fig. 3(a) and notice the phase locking of the oscillations of a with those of M. The phase locking reflects the predicted spatial autoresonance in the system. We also calculate Φ and the energy $\hat{A}=\frac{1}{2}(a_x^2+a^2\Phi_x^2)+V_{\rm eff}(a)$ and show these results in Fig. 3(b). One can see the autoresonant growth of the averaged energy, as the system self-adjusts its parameters to stay in resonance with the driver. Figure 3 also illustrates the aforementioned slow autoresonant oscillations (around smooth averages) of Φ and $\hat{A}$ with two *different* spatial frequencies κ^{ξ} and κ^{μ}, respectively. Finally, in Fig. 4 we present the solutions Φ and $\hat{A}$ for the same parameters and boundary conditions as in Fig. 3(b), but with $\varepsilon_0=0.01$. One can see the development of the azimuthal instability when one violates (see above) the condition $\varepsilon_0\alpha_0>\varepsilon_1\alpha_1$ at $x\approx200$. The phase locking and the autoresonance discontinue beyond this point. This completes our illustration of the adiabatic resonant excitation of standing waves $\psi=a(x)\exp[i(\Lambda_0t+\Phi)]\approx a(x)\exp(i\Lambda_0t)$ in the driven NLS system and we proceed to the problem of autoresonant excitation and stabilization of more general cnoidal waves.

IV. AVERAGED VARIATIONAL PRINCIPLE: SLOW EVOLUTION EQUATIONS

In this section we study the boundary value problem similar to that described in Sec. III, but use a more general driving term, i.e., consider solutions of

$$i\psi_t+\psi_{xx}+|\psi|^2\psi=\varepsilon(x,t)\exp[i\varphi(x,t)], \tag{25}$$

where $\varphi(x,t)=\phi(x)-\omega_0t$ and $\varepsilon(x,t)=\varepsilon_0+\varepsilon_1\cos[\Psi(x)-\omega t]$. We write $\psi=a\ \exp(i\theta)$ and replace Eq. (25) by the pair of real equations [see the similar system (4) in Sec. II]

$$\begin{aligned} &a_t+va_x+a\Phi_{xx}+2a_x\Phi_x+\tfrac{1}{2}av_x=-\varepsilon(x,t)\sin\Phi,\\ &a(\Phi_t+v\Phi_x)-a_{xx}+a\Phi_x^2+\Lambda a-a^3=-\varepsilon(x,t)\cos\Phi, \end{aligned} \tag{26}$$

where $\Phi(x,t)\equiv\theta(x,t)-\varphi(x,t)$, $v(x)\equiv2k(x)$, $k(x)\equiv d\phi/dx$, and $\Lambda(x)\equiv k^2(x)-\omega_0$. The space and the time dependences of a and Φ are now included in the problem. Time dependence enters via the modulation of the amplitude of the driver and/or time-dependent boundary conditions (at $x=x_0$) $a_0=[\Lambda(x_0)]^{1/2}+\delta a_0(t)$ and $\Phi_0=\delta\Phi_0(t)$, while $a_{0x}=\Phi_{0x}=0$. An important ingredient of the following theory is the assumption of the presence of *slow* and *fast* space-time scales in the problem. Let only the slow time variation be present in the boundary conditions. On the other hand, we assume that both $\phi(x)-\omega_0t$ and $\Psi(x)-\omega t$ in the driving term are varying on the fast space-time scale, but the frequencies ω,ω_0 are constant, while $\mathscr{k}(x)\equiv d\Psi/dx$ and $k(x)\equiv d\phi/dx$ are slowly varying functions of position. The purpose of the theory is to show that a proper choice of $\mathscr{k}(x)$ and $k(x)$ yields a stable, adiabatically varying cnoidal solution with the amplitude a of the form of a quasiperiodic wave moving with the velocity $v(x)\equiv2k(x)$. We shall see that, for exciting this solution, one needs to match $v(x)$ to the

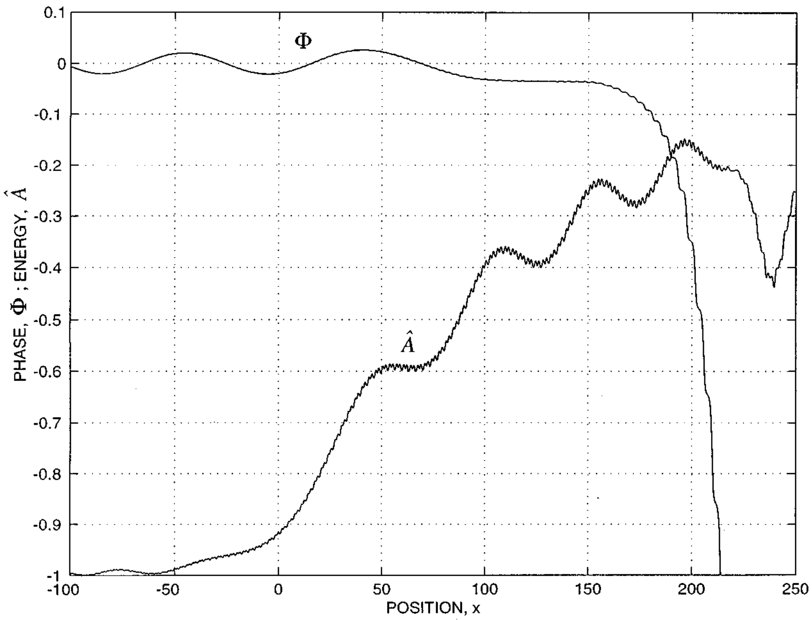

FIG. 4. Phase Φ and energy $\hat{A}$ versus position for the same parameters and boundary conditions as in Fig. 3(b), but with smaller value of $\varepsilon_0=0.01$. The figure shows the azimuthal instability of the autoresonant solution when the condition $\varepsilon_0\alpha_0>\varepsilon_1\alpha_1$ is violated at $x\approx 200$, destroying the phase locking and the autoresonance in the system.

phase velocity $\omega/\mathscr{k}(x)$ of the modulation of the driver, i.e., to impose the condition $2k(x)\mathscr{k}(x)=\omega$. This condition agrees with the results of Sec. III, studying autoresonant standing waves. Indeed, for the standing waves, $v\equiv 0$ and therefore purely spatial modulations ($\omega\equiv 0$) of the external forcing used in Sec. III satisfy the above-mentioned condition. Finally, note that in the case of standing waves, $\mathscr{k}(x)$ must be sufficiently slow, but otherwise arbitrary (in contrast to the traveling waves).

The assumed presence of the two different space-time scales allows one to develop the theory of autoresonance in our driven system by using Whitham's averaged variational principle [17]. To this end, we observe that Eqs. (26) can be obtained from the variational principle $\delta\int\int L(a,a_x;\Phi,\Phi_x,\Phi_1)dx\,dt=0$, where the two-field Lagrangian is of the form $L=L_0+\varepsilon(x,t)a\cos\Phi$ and the unperturbed part is

$$L_0(a,a_x;\Phi_x,\Phi_t)\equiv\tfrac{1}{2}[a_x^2+a^2(\Phi_x^2+\Phi_t+v\Phi_x)]+\tfrac{1}{2}\Lambda a^2-\tfrac{1}{4}a^4. \tag{27}$$

Note that Eq. (27) does not depend on the field Φ, but only on its space-time derivatives and therefore, in the unperturbed problem, Φ is a *potential* (the analog of the cyclic variable in the dynamical problem). The averaged variational principle for studying autoresonance in general driven two-field problems with L_0 of this type, but for a different form of the perturbation, was constructed recently [2]. Although we can use many ingredients of that theory in our application, we shall repeat the main steps of the construction, for completeness.

As a starting point, we introduce the following two-scale *representation* of the solutions [compare to Eqs. (15) and (16) in the dynamical problem in Sec. III]

$$a\equiv U(\Theta,x,t),\quad \Phi\equiv\xi(x,t)+V(\Theta,x,t), \tag{28}$$

where the explicitly shown x,t dependence is slow, while $\Theta(x,t)$ is a fast angle variable and both U and V are assumed to be 2π periodic in Θ. Also, one assumes that the frequency $\Omega(x,t)\equiv-\Theta_t$ and wave vector $K(x,t)\equiv\Theta_x$, as well as the derivatives $\gamma(x,t)\equiv-\xi_t$ and $\beta(x,t)\equiv\xi_x$ are slow functions of space-time. The secular component ξ in Φ (the term of this type is absent in a) is necessary for properly representing the potential field and the quantities γ and β are the slow Θ averages of the derivatives of Φ. We have already seen these ingredients in the theory when studying standing autoresonant waves in Sec. III.

At this stage, we introduce the averaged variational principle [17], i.e., instead of the original variational principle, use $\delta\int\int\mathcal{L}\,dx\,dt=0$, where $\mathcal{L}$ is the averaged Lagrangian

$$\mathcal{L}\equiv(2\pi)^{-1}\int_0^{2\pi}L\,d\Theta=\mathcal{L}_0+\mathcal{L}_1 \tag{29}$$

split into the unperturbed part and $O(\varepsilon)$ perturbation. The averaging in Eq. (29) is defined by holding the slow variables in the integrand *fixed* at a given position and time. The calculation of $\mathcal{L}_{0,1}$ is the next necessary step in the theory. As mentioned earlier, we neglect the slow variations of all the dependent variables and parameters in this calculation. We use $\Phi_x=\beta+KV_\Theta$, to express $V_\Theta=K^{-1}(\Phi_x-\beta)$, and $\Phi_t=-\gamma-\Omega V_\Theta=\delta-(\Omega/K)\Phi_x$, where $\delta\equiv-\gamma+(\Omega/K)\beta$. Then the unperturbed Lagrangian (27) can be written as

$$L_0=\tfrac{1}{2}(a_x^2+a^2\Phi_x^2)+\tfrac{1}{2}ha^2\Phi_x+\tfrac{1}{2}(\Lambda+\delta)a^2-\tfrac{1}{4}a^4, \tag{30}$$

where $h\equiv v-\Omega/K$ is the *velocity mismatch* and Ω, K, and δ are evaluated at their local (still unknown) values. Now we observe that, similarly to the case $v=\Omega=\delta=0$ studied in

Sec. III, the unperturbed Lagrangian (30) describes a two degrees of freedom dynamical problem, where x plays the role of time. Therefore, again, in solving this *dynamical* problem [the solution is necessary for performing the averaging in Eq. (29)], we use the Hamiltonian formulation. We define the usual canonical momenta

$$p^a \equiv \partial L_0/\partial a_x = a_x, \quad p^\Phi \equiv \partial L_0/\partial \Phi_x = a^2(\Phi_x + \tfrac{1}{2}h) \tag{31}$$

and observe that, since Eq. (30) is independent of Φ, $p^\phi \equiv B(x,t)$ is a slow variable. Next we write the unperturbed Hamiltonian $H_0 = a_x p^a + \Phi_x B - L_0$ or, after some algebra,

$$H_0 = \tfrac{1}{2}[a_x^2 + (B/a)^2] - \tfrac{1}{2}hB + V_{\text{eff}}(a), \tag{32}$$

where

$$V_{\text{eff}}(a) = -\tfrac{1}{2}Ra^2 + \tfrac{1}{4}a^4 \tag{33}$$

and $R \equiv \Lambda - \gamma + (v-h)\beta - \frac{1}{4}h^2$. Since only the slow space-time dependence enters the Hamiltonian explicitly, the energy $H_0 \equiv A(x,t)$ is the second slow dependent variable in the problem, in addition to $B(x,t)$. Now, according to Eq. (32), the fast spatial variation of a in the unperturbed problem can be found by solving

$$\tfrac{1}{2}[a_x^2 + (B/a)^2] + V_{\text{eff}}(a) = A', \quad A' \equiv A + \tfrac{1}{2}hB. \tag{34}$$

Equations (34) show that a can be interpreted as the radial component of the planar motion of a quasiparticle of unit mass, energy A', and angular momentum B under the action of the *central* force characterized by the potential V_{eff}. Note that if B, h, β, and γ are all small, this planar motion is nearly the same as that studied for spatially autoresonant oscillations in Sec. III [see the Hamiltonian (11)].

At this point, we identify the fast variable Θ in Eq. (28) with the canonical angle variable of the radial oscillations in the unperturbed dynamical problem described above (the motion in the quasipotential V_{eff}) and average the expression $H_0 \equiv a_x p^a + \Phi_x B - L_0 = A' - \frac{1}{2}hB$ with respect to Θ between 0 and 2π, yielding

$$\mathcal{L}_0 = KJ(A',B,R) - A' + B(\beta + \tfrac{1}{2}h), \tag{35}$$

where [compare to Eq. (14)]

$$J(A',B,R) \equiv (2\pi)^{-1} \oint \{2[A' - \tfrac{1}{2}(B/a)^2 - V_{\text{eff}}(R,a)]\}^{1/2} da \tag{36}$$

is the canonical action variable associated with the radial oscillations. Later we shall use the following partial derivatives of the action:

$$J_{A'} = (2\pi)^{-1}\lambda(A',B,R), \quad J_B = -BJ_{A'}\langle a^{-2}\rangle_{\text{av}}, \quad I_R = 0.5 I_{A'}\langle a^2\rangle_{\text{av}}, \tag{37}$$

where

$$\lambda(A',B,R) \equiv \oint \{2[A' - \tfrac{1}{2}(B/a)^2 - V_{\text{eff}}(R,a)]\}^{-1/2} da \tag{38}$$

is the wavelength associated with the radial oscillations and $\langle \cdots \rangle_{\text{av}} \equiv (2\pi)^{-1}\int_0^{2\pi}(\cdots)d\Theta$.

Next we calculate the perturbed part

$$\mathcal{L}_1 = \langle \{\varepsilon_0 + \varepsilon_1 \cos[\Psi(x) - \omega t)]\} U \cos(\xi + V) \rangle_{\text{av}},$$

of the averaged Lagrangian (29). By making the same double resonance approximation as in deriving the perturbed part in Eq. (18), i.e., by assuming that V and ξ are continuously small and that the phase mismatch $\mu \equiv \Theta - (\Psi - \omega t) + \pi$ is slow (this is our phase locking assumption), we find

$$\mathcal{L}_1 \approx (\varepsilon_0\alpha_0 - \varepsilon_1\alpha_1 \cos\mu)\cos\xi, \tag{39}$$

where the coefficients $\alpha_{0.1}$ are again associated with the zero and first harmonics in the Fourier expansion $U(\Theta,J,B)\cos V(\Theta,J,B) = \Sigma_n \alpha_n(J,B)\exp(in\Theta)$. Thus the averaged Lagrangian in the driven NLS problem of interest becomes

$$\mathcal{L} = KJ(A',B,R) - A' + B(\beta + \tfrac{1}{2}h) + (\varepsilon_0\alpha_0 - \varepsilon_1\alpha_1\cos\mu)\cos\xi. \tag{40}$$

At this stage, we can write the variational evolution equations in our problem. We observe that the averaged Lagrangian depends on four field variables, i.e., $\mathcal{L} = \mathcal{L}(A'; B; \Theta, \Theta_x = K, \Theta_t = -\Omega; \xi, \xi_x = \beta, \xi_t = -\gamma)$. By taking the variations in $\delta\int\int \mathcal{L}\, dx\, dt = 0$ with respect to these dependent fields, we obtain four evolution equations. For example, the variation with respect to A' yields

$$KJ_{A'} - 1 + (\varepsilon_0\alpha_{0A'} - \varepsilon_1\alpha_{1A'}\cos\mu)\cos\xi = 0 \tag{41}$$

or, since $\mu_x = K - \mathscr{k}(x)$ and $J_{A'} = (2\pi)^{-1}\lambda(A',B,R)$ [see the first equation of Eqs. (37)],

$$\mu_x = 2\pi\lambda^{-1} - \mathscr{k}(x) - 2\pi\lambda^{-1}(\varepsilon_0\alpha_{0A'} - \varepsilon_1\alpha_{1A'}\cos\mu)\cos\xi. \tag{42}$$

The variation with respect to B gives $KJ_B + \beta + \frac{1}{2}h + (\varepsilon_0\alpha_{0B} - \varepsilon_1\alpha_{1B}\cos\mu)\cos\xi = 0$ or, by using the second equation of Eqs. (37),

$$\xi_x = BKJ_{A'}\langle a^{-2}\rangle_{\text{av}} - \tfrac{1}{2}h - (\varepsilon_0\alpha_{0B} - \varepsilon_1\alpha_{1B}\cos\mu)\cos\xi. \tag{43}$$

Finally, the variations with respect to Θ and ξ yield

$$(J + KJ_R R_K + \tfrac{1}{2}B\Omega/K^2)_x - (KJ_R R_\Omega - \tfrac{1}{2}B/K)_t = \varepsilon_1\alpha_1\cos\xi\sin\mu \tag{44}$$

and

$$(\Omega J_R + B)_x + (KJ_R)_t = -(\varepsilon_0\alpha_0 - \varepsilon_1\alpha_1\cos\mu)\sin\xi, \tag{45}$$

where J_R is given by the third equation of Eqs. (37). Equations (42)–(45) are the desired evolution equations for studying the autoresonant cnoidal wave solutions of the driven NLS equation.

V. APPLICATIONS OF THE VARIATIONAL APPROACH

A. Transition to the purely *x*-dependent case

As a first application, we check the transition to the purely *x*-dependent case studied in Sec. III. This transition is obtained by setting Ω, γ, ω, v, and all the time derivatives in Eqs. (44) and (45) to zero. The resulting system is

$$\begin{aligned} \mu_x &= 2\pi\lambda^{-1} - k(x) \\ &\quad - 2\pi\lambda^{-1}(\varepsilon_0\alpha_{0A'} - \varepsilon_1\alpha_{1A'}\cos\mu)\cos\xi, \\ \xi_x &= -KJ_B - (\varepsilon_0\alpha_{0B} - \varepsilon_1\alpha_{1b}\cos\mu)\cos\xi, \\ J_x &= \varepsilon_1\alpha_1\cos\xi\sin\mu, \\ B_x &= -(\varepsilon_0\alpha_0 - \varepsilon_1\alpha_1\cos\mu)\sin\xi. \end{aligned} \tag{46}$$

Here, on the RHS in the first equation of Eqs. (46), one can replace $(\cdots)_{A'} = J_{A'}(\cdots)_J = \lambda(2\pi)^{-1}(\cdots)_J$. Also, in the second equation, $KJ_B = KJ_{A'}(J_B/J_{A'}) = -KJ_{A'}\partial A'(J,B)/\partial B = -KJ_{A'}\beta(J,B)$ and since, from Eq. (41), $KJ_{A'} = 1 + O(\varepsilon)$, we have $KJ_B \approx -\beta(A',B,R)$. With these substitutions, Eqs. (46) coincide with Eqs. (19) of Sec. III.

B. Stability of autoresonant standing waves

Our second application is devoted to studying the stability of the autoresonant standing wave solutions of Sec. II with respect to small periodic *temporal* modulations. The latter can be introduced in the problem via a periodic time dependence in the boundary conditions. For the standing waves, we set $\omega = v = 0$, but now keep the time derivatives in Eqs. (45) and (46) and include nonvanishing Ω and γ. Also, for simplicity, we shall focus on the autoresonant evolution stage, where, to lowest order in ε (see below), one can neglect the interaction terms in Eqs. (45) and (46). Furthermore, we shall neglect the space variation of the wave vector k in our analysis.

We proceed by making the following ordering assumptions, subject to *a posteriori* verification. By definition, $K = k + \mu_x$ and $\Omega = -\mu_t$. We shall view both μ_x and μ_t in these expressions as small and of $O(\varepsilon^{1/2})$. This is our phase locking assumption between the radial oscillations and the modulation of the driving wave. In addition, we assume that the slow variable B (as in the case of the standing waves in Sec. III), $\beta = \xi_x$, and $\gamma = -\xi_t$ (and therefore δ) are also small and of $O(\varepsilon^{1/2})$. Finally, we view both the radial phase mismatch μ and the azimuthal shift ξ as being *sufficiently* small for approximating $\cos(\mu,\xi)\approx 1$ and $\sin(\mu,\xi)\approx(\mu,\xi)$ in our evolution equations. With these assumptions, to lowest significant order in ε, Eqs. (42)–(45) become

$$\begin{aligned} \mu_x &\approx K^0 - k + K_R^0\xi_t, \\ \xi_x &\approx Bp^0 - (2k)^{-1}\mu_t, \\ J_{A'}^0 A_x' &+ (2k)^{-1}(B - J_R^0\mu_t)_t \approx \varepsilon_1\alpha_1^0\mu, \\ B_x &+ k(J_{RA'}^0 A_t' + J_{RR}^0\xi_{tt}) \approx -(\varepsilon_0\alpha_0^0 - \varepsilon_1\alpha_1^0)\xi. \end{aligned} \tag{47}$$

Here we have used the expansion $2\pi\lambda^{-1} \equiv K(A',B,R) \approx K^0 - \gamma K_R^0$, while $p^0 \equiv \langle a^{-2}\rangle_{\rm av}$ and all other objects with the zero subscript are evaluated at $B=0$ and $R=\Lambda$.

Now we observe that Eqs. (47) have a trivial solution $\bar{\mu} = \bar{\xi} = \bar{B} = 0$ and $A' = \bar{A}$ given by $K^0(\bar{A}) - k = 0$. This is the space-time averaged autoresonant solution described in Sec. III. In studying the stability of this solution, we add a small perturbation $A' = \bar{A} + \delta A$ and linearize Eqs. (47) with respect to δA. Then the first and third equations in this system become

$$\mu_x \approx \bar{K}_A\delta A + \bar{K}_R\xi_t, \tag{48}$$

$$\bar{J}_A\delta A_x + (2k)^{-1}(B - \bar{J}_R\mu_t)_t \approx \varepsilon_1\bar{\alpha}_1\mu, \tag{49}$$

where $(\bar{\ })$ means evaluations at $A' = \bar{A}$, $B=0$, and $R=\Lambda$. The remaining two equations of Eqs. (47), to lowest order, remain the same, but with the coefficients evaluated at $\bar{A}$, i.e.,

$$\xi_x \approx B\bar{p} - (2k)^{-1}\mu_t, \tag{50}$$

$$B_x + k(\bar{J}_{RA}\delta A_t + \bar{J}_{RR}\xi_{tt}) \approx -(\varepsilon_0\bar{\alpha}_0 - \varepsilon_1\bar{\alpha}_1)\xi. \tag{51}$$

The solution of the linear homogeneous system (48)–(51) has the form $(\delta A,B,\mu,\xi) \sim \exp[i(\kappa x - \nu t)]$, with κ and ν satisfying the dispersion relation

$$[\kappa^2 + d_1\nu^2 + (\bar{K}_A/\bar{J}_A)\varepsilon_1\bar{\alpha}_1][\kappa^2 + d_2\nu^2 - \bar{p}(\varepsilon_0\bar{\alpha}_0 - \varepsilon_1\bar{\alpha}_1)] + d_3 d_4(\nu\kappa)^2 = 0, \tag{52}$$

where

$$\begin{aligned} d_1 &= (0.5\bar{K}_A/k\bar{J}_A)[(2k\bar{p})^{-1} - \bar{J}_R], \\ d_2 &= k\bar{p}(\bar{J}_{RR} - \bar{K}_R\bar{J}_{RA}/\bar{K}_A), \\ d_3 &= \bar{K}_R - \bar{K}_A(2k\bar{J}_A\bar{p})^{-1}, \\ d_4 &= (2k)^{-1} + k\bar{J}_{RA}\bar{p}/\bar{K}_A. \end{aligned} \tag{53}$$

Note that Eq. (52) justifies the assumed scalings $\kappa,\nu \sim O(\varepsilon_0^{1/2})$. Also, from Eqs. (48) and (50) we obtain $\delta A, B \sim O(\varepsilon_0^{1/2})$, justifying another assumption in the theory. Now we proceed to the stability problem. In the case $\nu = 0$, Eq. (52) yields two solutions

$$\begin{aligned} \kappa_1^2 &= -(\bar{K}_A/\bar{J}_A)\varepsilon_1\bar{\alpha}_1, \\ \kappa_2^2 &= \bar{p}(\varepsilon_0\bar{\alpha}_0 - \varepsilon_1\bar{\alpha}_1), \end{aligned} \tag{54}$$

in agreement with the predictions of the theory in Sec. III [see Eqs. (23) and (24)]. Since the right-hand sides of Eqs. (54) are positive, *purely* spatial modulations are stable. Then, by continuity, sufficiently slow (small ν) temporal modulations of autoresonant standing waves are also stable until, as ν increases, one reverses the signs in the solutions of Eq. (52) for κ^2.

C. Autoresonant cnoidal waves

In this application, we study purely x-dependent solutions of the slow variational evolution equations (42)–(45) for the case when the driving term in Eq. (25) has the form $\{\varepsilon_0 + \varepsilon_1\cos[\Psi(x)-\omega t]\}\exp\{i[\phi(x)-\omega_0 t)]\}$ with constant ω and ω_0, but slowly varying wave vectors $\Psi_x=\mathscr{k}(x)$ and $\phi_x = k(x)$. Also, we shall assume that the phase velocity $\omega/\mathscr{k}(x)$ of the amplitude modulation satisfies the relation $\omega/\mathscr{k}(x)=2k(x)$. We shall see that this matching allows one to excite a stable, adiabatically varying cnoidal solution with a having the form of a quasiperiodic wave moving with the velocity $v(x)=2k(x)$. In this purely x-dependent limit we set $\Omega=\omega=\text{const}$ and $\gamma=0$ in Eqs. (42)–(45), yielding the slow evolution system

$$\begin{aligned}
\mu_x &= 2\pi\lambda^{-1}-\mathscr{k}(x)-2\pi\lambda^{-1}(\varepsilon_0\alpha_{0A'}-\varepsilon_1\alpha_{1A'})\cos\xi,\\
\xi_x &= BKJ_{A'}p-\tfrac{1}{2}h-(\varepsilon_0\alpha_{0B}-\varepsilon_1\alpha_{1B})\cos\xi,\\
(J+KJ_RR_K+\tfrac{1}{2}B\omega/K^2)_x &= \varepsilon_1\alpha_1\sin\mu,\\
(\omega J_R+B)_x &= -(\varepsilon_0\alpha_0-\varepsilon_1\alpha_1)\sin\xi,
\end{aligned} \tag{55}$$

where, again, assuming the strong *double* phase locking, we will replace $\cos(\mu,\xi)\to 1$ and $\sin(\mu,\xi)\to(\mu,\xi)$. In addition, as for the standing waves, we shall view μ_x, ξ_x and B in Eqs. (55) as small and $O(\varepsilon^{1/2})$. Next we recall that $K\equiv\mathscr{k}+\mu_x$ and $h\equiv v-\omega/K\approx\omega\mu_x/\mathscr{k}^2\sim O(\varepsilon^{1/2})$, so $R=\Lambda(x)+\omega\beta/\mathscr{k}+O(\varepsilon)$. Then, to lowest order in ε, Eqs. (55) become

$$\begin{aligned}
\mu_x &= K^0-\mathscr{k}+K_R^0\omega\xi_x/\mathscr{k},\\
\xi_x &= Bp-\tfrac{1}{2}\omega\mu_x/\mathscr{k}^2,\\
(J^0+\tfrac{1}{2}B\omega/\mathscr{k}^2)_x &= \varepsilon_1\alpha_1^0\mu,\\
(\omega J_R^0+\omega^2J_{RR}^0\xi_x/\mathscr{k}+B)_x &= -(\varepsilon_0\alpha_0^0-\varepsilon_1\alpha_1^0)\xi,
\end{aligned} \tag{56}$$

where the objects with the zero subscript are evaluated at $B=0$ and $R=\Lambda(x)=k^2(x)-\omega_0$. Finally, we write $A'=\bar{A}(x)+\delta A$, where $\bar{A}$ is defined via $K^0[\bar{A}(x),R(x)]=\mathscr{k}(x)$, view δA as being $O(\varepsilon^{1/2})$, and linearize Eqs. (56), yielding

$$\begin{aligned}
\mu_x &= \bar{K}_A\delta A+K_R^0\omega\xi_x/\mathscr{k},\\
\xi_x &= B\bar{p}-\tfrac{1}{2}\omega\mu_x/\mathscr{k}^2,\\
(\bar{J}_A\delta A+\tfrac{1}{2}B\omega/\mathscr{k}^2)_x &= -\bar{J}_x+\varepsilon_1\bar{\alpha}_1\mu,\\
(\omega\bar{J}_{RA}\delta A+\omega^2\bar{J}_{RR}^0\xi_x/\mathscr{k}+B)_x &= -\omega\bar{J}_{Rx}-(\varepsilon_0\bar{\alpha}_0-\varepsilon_1\bar{\alpha}_1)\xi,
\end{aligned} \tag{57}$$

where $(\bar{\ })$ denotes evaluations at $A'=\bar{A}(x)$, $B=0$, and $R=\Lambda(x)$.

The autoresonant solution of Eqs. (57) corresponds to small oscillations of μ and ξ around the slow averages $\bar{\mu}$ and $\bar{\xi}$ given by [see the RHS of the last two equations of Eqs. (57)]

$$-\bar{J}_x+\varepsilon_1\bar{\alpha}_1\bar{\mu}\equiv 0,\quad -\omega\bar{J}_{Rx}-(\varepsilon_0\bar{\alpha}_0-\varepsilon_1\bar{\alpha}_1)\bar{\xi}\equiv 0. \tag{58}$$

Note that the smallness of $\bar{\mu}$ and $\bar{\xi}$ requires the smallness of $\chi\equiv\sigma/\varepsilon$, where σ is the dimensionless parameter characterizing the space variation of the wave vectors k and $\mathscr{k}$ in the driver. We shall assume that $\chi\ll 1$ in the following (this is our adiabaticity condition), so $\bar{\mu},\bar{\xi}\sim O(\chi)$. Similarly to μ and ξ, the variables δA and B are also viewed as oscillating, but having negligible averages since, as follows from the first two equations of Eqs. (57), these averages are $O(\sigma\chi)$. Then, for the oscillating components, Eqs. (57) yield

$$\begin{aligned}
\delta\mu_x &= \bar{K}_A\delta A+K_R^0\omega\,\delta\xi_x/\mathscr{k},\\
\delta\xi_x &= B\bar{p}-\tfrac{1}{2}\omega\,\delta\mu_x/\mathscr{k}^2,\\
\bar{J}_A\delta A_x+\tfrac{1}{2}B_x\omega/\mathscr{k}^2 &= \varepsilon_1\bar{\alpha}_1\delta\mu,\\
\omega\bar{J}_{RA}\delta A_x+\omega^2\bar{J}_{RR}^0\delta\xi_{xx}/\mathscr{k}+B_x &= -(\varepsilon_0\bar{\alpha}_0-\varepsilon_1\bar{\alpha}_1)\delta\xi,
\end{aligned} \tag{59}$$

where $\delta\mu=\mu-\bar{\mu}$ and $\delta\xi=\xi-\bar{\xi}$. This linear system gives the following *local* characteristic equation for the wave number κ of the autoresonant oscillations:

$$(c_1\kappa^2+e_1)(c_2\kappa^2+e_2)+c_3\kappa^4=0, \tag{60}$$

where $e_1=\varepsilon_1\bar{\alpha}_1$, $e_2=\varepsilon_0\bar{\alpha}_0-\varepsilon_1\bar{\alpha}_1$, and

$$\begin{aligned}
c_1 &= (\bar{J}_A/\bar{K}_A)+\tfrac{1}{2}\omega/(\bar{p}\mathscr{k}^2),\\
c_2 &= -\bar{p}^{-1}-(\omega^2/\mathscr{k})(\bar{J}_{RR}-\bar{J}_{RA}\bar{K}_R/\bar{K}_A),\\
c_3 &= \tfrac{1}{2}\omega^2(\bar{p}\mathscr{k}^2)^{-2}(1+2\bar{p}\mathscr{k}^2\bar{J}_{RA}/\bar{K}_A)(1-\mathscr{k}\bar{J}_A\bar{K}_R/\bar{K}_A).
\end{aligned} \tag{61}$$

We observe that if $\omega=0$ (this is the case of standing autoresonant waves), $c_3=0$ and the solutions of Eq. (60) are $\kappa_1^2=-(\bar{K}_A/\bar{J}_A)\varepsilon_1\bar{\alpha}_1$ and $\kappa_2^2=\bar{p}(\varepsilon_0\bar{\alpha}_0-\varepsilon_1\bar{\alpha}_1)$, which coincides with our previous results [see Eqs. (54)] for this case. Since both $\kappa_{1,2}^2$ are positive, the driven standing waves of the NLS equation are stable. However, the positiveness of $\kappa_{1,2}^2$ for the $\omega=0$ case guarantees the stability of the autoresonant cnoidal waves for sufficiently small values of $v=\omega/\mathscr{k}$.

VI. CONCLUSIONS

We have studied the problem of excitation and control of cnoidal solutions of the NLS equation by driving the system by oscillations/waves with adiabatically varying parameters. We have used the autoresonance effect for the excitation purpose, i.e., the state in which the excited wave self-adjusts its parameters to remain in an approximate resonance with the driver despite the variation of the driving frequency and/or wave vector. Different scenarios of entering and sustaining the autoresonance in the NLS system were considered.

The simplest autoresonant solution of the NLS equation is obtained if one starts from a small uniform initial condition, drives the system by a wave of the form $\varepsilon\exp\{i[k_0x-k_0^2t+\phi(t)]\}$, where the frequency $k_0^2-\Lambda(t)$ is slowly increasing in time ($\Lambda\equiv d\phi/dt$, $d\Lambda/dt<0$), and passes the linear resonance point $\Lambda=0$. Beyond this point, one excites a quasiperiodic solution of form (see Sec. II) $\psi=a_0(t)\exp\{i[k_0x-k_0^2t+\theta(t)]\}$, where $a_0\approx\Lambda^{1/2}$ and $\theta(t)\approx\phi(t)$, i.e., ψ is phase locked with the driving wave. We have studied the stability of this solution with respect to spatially periodic perturbations and, in contrast to the plane wave solutions of the free ($\varepsilon=0$) NLS equation, found a long-wavelength stability window for the autoresonant plane waves.

We have shown (see Sec. III) that the aforementioned autoresonant plane waves can serve as boundary conditions for exciting autoresonant *standing* waves of the form $\psi=a(x)\exp(i\Lambda_0 t)$ in the NLS system. This goal is achieved by using the driver of the form $\varepsilon(x)\exp(i\Lambda_0 t)$, where $\varepsilon(x)\equiv\varepsilon_0+\varepsilon_1\cos[\Psi(x)]$ and $\mathscr{k}(x)\equiv\Psi_x$ is a slowly varying function of position. Therefore, the excitation of autoresonant standing waves requires forcing by a superposition of an oscillation and an adiabatically varying standing wave. We have studied the autoresonance in this driven system and in the associated dynamical problem. The *double* frequency dynamic autoresonance in this problem and its relation to the driven NLS equation were investigated in detail.

We have also constructed Whitham's averaged variational principle for the resonantly driven NLS equation (see Sec. IV) and used it for testing the temporal stability of the spatially autoresonant standing waves as well as for studying autoresonant excitation and control of more general, cnoidal waves. The averaged variational principle yields a system of slow evolution equations (42)–(45) describing the *space-time* evolution of two pairs of canonical actions and angles in the associated dynamical problem.

We have studied several applications of the averaged variational theory in Sec. V. For instance, we have shown that the autoresonant standing waves are stable with respect to sufficiently slow temporal modulations. We have shown that more general autoresonant cnoidal solutions in the driven NLS system can also be excited by using plane autoresonant solutions as a boundary condition. However, the driver, in this case, must have the form $\{\varepsilon_0+\varepsilon_1\cos[\Psi(x)-\omega t]\}\exp\{i[\phi(x)-\omega_0 t)]\}$, where the wave vectors $\Psi_x=\mathscr{k}(x)$ and $\phi_x=k(x)$ are both slowly varying functions of position. Thus the driver comprises a *superposition* of waves with adiabatically varying parameters. Furthermore, we have seen that, while the frequencies ω and ω_0 in the driver can be arbitrary, the stability of the autoresonant cnoidal waves requires an additional phase velocity matching, i.e., $\omega/\mathscr{k}(x)=2k(x)$, in the system.

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