Resonant excitation and control of high order dispersive nonlinear waves

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Autoresonant excitation of high order nonlinear waves with space–time varying parameters is investigated. A class of driven, two-component nonlinear waves described by the variational principle is studied in detail. The autoresonance in the system proceeds as an external eikonal pump wave excites a nonlinear daughter wave after crossing the linear resonance surface. Beyond the linear resonance, the pump and the daughter waves stay phase locked in an extended region of space–time despite the variation of the system's parameters. The theory of the autoresonance is developed on the bases of the averaged variational principle and comprises a generalization of the formalism for scalar fields. The relation of the wave autoresonance problem to an associated two degrees of freedom problem in nonlinear dynamics is discussed. The conditions for the stable autoresonant solutions are (a) the adiabaticity of the driven system and (b) a sufficient nonlinearity. The theory is applied to the problem of resonant excitation and control of a Korteweg–de Vries (KdV) wave by means of launching an external pump wave with space–time varying frequency and wave vector. Numerical examples for temporal and spatial autoresonance in this system are presented. © *1998 American Institute of Physics.* [S1070-664X(98)01103-3]

I. INTRODUCTION

The problem of excitation and control of nonlinear waves having well defined properties in systems with space–time varying parameters is of interest to many branches of physics. Recently (see Ref. 1 and references therein), a solution to this problem was suggested via the autoresonant nonlinear mode conversion, i.e., by driving the adiabatically varying system by an external eikonal (pump) wave and generating a nonlinear daughter wave as the two waves resonate in a region of space–time. This excitation process does not require seeding the daughter wave outside the resonant region and evolves similarly to the linear mode conversion,[2] as long as the amplitude of the daughter wave is small enough. Nevertheless, a new phenomenon may take place when the amplitude grows and the nonlinearity becomes important, i.e., the self-adjustment of the daughter wave parameters leading to the preservation of the daughter-pump wave resonance in an extended region of space–time despite the variation of system's parameters. This phenomenon is the main signature of the wave autoresonance in the system. In the autoresonance the daughter and the pump waves are phase locked and, therefore, by shaping the space–time variation of the frequency and/or the wave vector of the pump, one can control the characteristic parameters of the nonlinear wave. The conditions for entering and sustaining the autoresonance were studied in detail in Ref. 1.

The general formalism[1] of the wave autoresonance was developed for a class of driven nonlinear waves described by the variational principle,

$$\delta\left(\int \int L dx dt \right) = 0, \tag{1}$$

with the Lagrangian

$$\mathbf{L} = L(u_t, u_x, u, q) + \varepsilon u b \cos \psi(x,t). \tag{2}$$

Here $u=u(x,t)$ is a scalar wave field, $q=q(x,t)$ represents a set of adiabatically varying parameters in the nonlinear system, $b \cos \psi$ describes the eikonal pump wave with adiabatically varying amplitude $b(x,t)$, wave vector $k_0(x,t) \equiv \psi_x$ and frequency $\omega_0(x,t) \equiv -\psi_t$, and ε is a small coupling parameter. Lagrangian (2) yields the following variational evolution equation for the field variable:

$$\partial_t(L_{u_t}) + \partial_x(L_{u_x}) - L_u = \varepsilon b \cos \psi, \tag{3}$$

i.e., a second order partial differential equation (PDE). Therefore, the theory of Ref. 1 cannot be used in studying driven wave problems involving higher order space–time derivatives of the wave field, or when one deals with driven systems of equations involving several fields. These higher order problems, in many cases, can still be formulated in terms of the variational principle with the unperturbed Lagrangian depending on $N(N>1)$ field variables and having the form

$$\mathbf{L} = L(u_t^1, u_x^1, u^1; u_t^2, u_x^2, u^2; \ldots; u_t^N, u_x^N, u^N; q). \tag{4}$$

The corresponding variational evolution system is

$$\partial_t(L_{u_t^i}) + \partial_x(L_{u_x^i}) - L_{u^i} = \varepsilon \delta_{mi} b \cos \psi; \quad i=1,\ldots,N, \tag{5}$$

where, for simplicity, only the mth equation is driven externally. The driven ion acoustic wave problem in a plasma is a well known example allowing the variational formulation (4). Indeed, in terms of the one-dimensional, time dependent cold fluid model for plasma ions and Maxwellian electrons, this problem is described by the following normalized momentum, continuity and Poisson equations:[3]

$$\begin{aligned} &u_t + u u_x = -\phi_x + \varepsilon b \cos \psi, \\ &n_t + (nu)_x = 0, \\ &\phi_{xx} = e^{\phi} - n. \end{aligned} \tag{6}$$

The driving term in the first equation in (6) may represent, for instance, the ponderomotive effect of external high frequency electromagnetic pump waves. By introducing the potential representation $u \equiv \Psi_x$, one finds that system (6) follows from the variational principle (1) for the three field variables n, Ψ, and ϕ with the Lagrangian $\mathbf{L}=L+\varepsilon ub\cos\psi$, where

$$L(n;\Psi_t,\Psi_x;\phi_t,\phi_x,\phi) = n[\phi+\Psi_t+\tfrac{1}{2}(\Psi_x)^2+C]-\tfrac{1}{2}(\phi_x)^2-e^{\phi}, \tag{7}$$

and C is an arbitrary constant. Another famous example is the third order driven Korteweg–de Vries (KdV) equation which, at certain conditions,[3] can be reduced from (6),

$$\eta_t+\eta_x+6\eta\eta_x+\eta_{xxx}=\varepsilon b\cos\psi. \tag{8}$$

If, again, one introduces the potential representation $\eta \equiv v_x$ in Eq. (8) and defines the second field variable $u \equiv v_{xx}$, one arrives at the variational problem (1) for the fields u and v with $\mathbf{L}=L+\varepsilon vb\cos\psi$, where the unperturbed Lagrangian is

$$L(u,u_x;v_t,v_x)=\tfrac{1}{2}v_tv_x+\tfrac{1}{2}v_x^2+v_x^3+v_xu_x+\tfrac{1}{2}u^2. \tag{9}$$

Autoresonant excitation of nonlinear waves in the driven KdV system (8) was first studied by Aranson, Meerson, and Tajima[4] for the case of a slow temporal variation of the driving frequency and a constant wave vector. The conventional perturbation expansion approach based on the adiabaticity of the frequency and the smallness of the coupling parameter ε was used in solving the problem. Autoresonant transition to solitary wave solutions was predicted by the theory and observed numerically. In the present work, we shall study the autoresonance in more general driven two-field systems described by Lagrangians of the form

$$\mathbf{L}=L(u_t,u_x,u;v_t,v_x;q)+\varepsilon bv\cos\psi(x,t) \tag{10}$$

and, similarly to the approach of Ref. 1, apply the Whitham's averaged variational principle[5] in the problem. The choice of one of the field variables to enter the unperturbed Lagrangian in (10) via its derivatives only (the potential field) is important, since fields of this type lead to nontrivial consequences in the theory (see below). Other driven wave problems involving more field variables of both types [e.g., system (6)], as well as, higher dimensional cases and different types of couplings to the pump wave can be treated similarly. Our goal is to find conditions for generating nonlinear waves comprising quasiuniform wavetrain solutions of the unperturbed ($\varepsilon=0$) problem (5), which are phase locked with the pump wave, i.e., have almost the same frequencies and wave vectors as the pump despite the space–time variation of the system's parameters. We shall see that the method based on the averaged variational principle[5] is a natural tool in studying the multidimensional autoresonance since it reveals the underlying Hamiltonian structure of the driven wave problem by relating it to that in the Hamiltonian dynamics.

The scope of the theory will be as follows. In Sec. II we shall construct the averaged variational principle for studying the autoresonance in systems described by Lagrangians of the form (10). We shall apply this principle in Sec. III and make the above-mentioned dynamical connection in our problem. The slow evolution equations for the driven two-field system will be also derived in Sec. III. Section IV will be devoted to finding autoresonant solutions of the slow equations and studying the stability of these solutions. We shall illustrate our variational theory by applying it to the driven KdV system in Sec. V. In the same section we shall present numerical examples and comparisons with the predictions of the weakly nonlinear theory. Finally, Sec. VI gives our conclusions.

II. TWO-SCALE REPRESENTATION OF RESONANTLY DRIVEN WAVES

The Lagrangian (10) yields the following variational system of equations for the field variables u and v:

$$\partial_t(L_{u_t})+\partial_x(L_{u_x})-L_u=0, \tag{11}$$

$$\partial_t(L_{v_t})+\partial_x(L_{v_x})=\varepsilon b\cos\psi. \tag{12}$$

We assume the existence of two small parameters in this system, i.e., the coupling parameter ε and the dimensionless parameter σ characterizing the adiabaticity of q and of the frequency and wave vector ω_0,k_0 of the pump wave. It will be shown later that, in the autoresonance, $\sigma \ll \varepsilon$ and, therefore, our theory is, basically, the perturbation analysis in terms of the largest small parameter ε. Thus, following Whitham,[5] we introduce the two-scale representation of the field u,

$$u(x,t)=U(\theta,X,T). \tag{13}$$

Here $X=\varepsilon x$, $T=\varepsilon t$ are the slow variables and θ is the fast phase variable, which will be identified later with the canonical angle variable of a certain dynamical system. We shall also assume that the daughter and the pump waves are phase locked, i.e., $\theta=\psi(t,x)+\Phi$, where $|\Phi|<2\pi$ and scales with x,t as $\Phi=\Phi(\varepsilon^{1/2}t,\varepsilon^{1/2}x)$. This scaling is one of the characteristic features of the autoresonance (see Sec. IV) and guarantees that the frequency $\omega \equiv -\theta_t=\omega_0(T,X)+\delta\omega$ and the wave vector $k \equiv \theta_x = k_0(X,T)+\delta k$ of the nonlinear wave, to leading order, are as those of the pump and that the frequency and the wave vector shifts $\delta\omega \equiv -\Phi_t$ and $\delta k \equiv \Phi_x$ are of $O(\varepsilon^{1/2})$, but, at the same time, their space–time derivatives are of $O(\varepsilon)$.

In contrast to u, the potential field v is chosen to have the form

$$v(x,t)=\xi(x,t)+V(\theta,X,T), \tag{14}$$

where only the wave-like part $V(\theta,X,T)$ depending on θ is phase locked with the pump wave. At the same time, $\xi(x,t)$ is a fast secular component, with $\beta \equiv \xi_x$ and $\gamma \equiv -\xi_t$ viewed as functions of the slow variables only, i.e., $\beta=\beta(X,T)$ and $\gamma=\gamma(X,T)$. We shall see, after making the dynamical connection in our problem, that this difference between u and v is a natural result of a certain canonical transformation. The presence of the secular part $\xi(x,t)$ in v, i.e., of another dependent variable in the problem, is the main addition in the theory characteristic of potential fields only.

At this stage, by differentiation, we have

$$
\begin{aligned}
u_t &= -\omega U_\theta + \varepsilon U_T, \\
u_x &= k U_\theta + \varepsilon U_X, \\
v_t &= -\gamma - \omega V_\theta + \varepsilon V_T, \\
v_x &= \beta + k V_\theta + \varepsilon V_X.
\end{aligned} \tag{15}
$$

This result can be used to rewrite the evolution Eqs. (11), (12) in the form

$$-\omega L_{1\theta} + k L_{2\theta} - L_0 + \varepsilon (L_{1T} + L_{2X}) = 0, \tag{16}$$

$$-\omega L_{3\theta} + k L_{4\theta} + \varepsilon (L_{3T} + L_{4X}) = \varepsilon b \cos \psi, \tag{17}$$

where $L_{0,1,2} \equiv L_u, L_{u_t}, L_{u_x}$ and $L_{3,4} \equiv L_{v_t}, L_{v_x}$. Note that the arguments of L_i are as in

$$
\begin{aligned}
&L_i(-\omega U_\theta + \varepsilon U_T, k U_\theta + \varepsilon U_X, U; \\
&\quad -\gamma - \omega V_\theta + \varepsilon V_T, \beta + k V_\theta + \varepsilon V_X; q),
\end{aligned} \tag{18}
$$

so, formally, $L_i = L_i(\theta, X, T)$ (recall that $\omega \equiv -\theta_t$ and $k \equiv \theta_x$).

The next two important steps of the Whitham's method are (a) viewing θ, X, and T as independent variables and (b) assuming 2π-periodicity of U and V with respect to θ. The reasoning behind the periodicity assumption will be given below, when we shall identify θ with the canonical angle variable of a certain dynamical problem. Now, we multiply Eq. (16) by U_θ and Eq. (17) by V_θ and add the results, yielding

$$
\begin{aligned}
&[U_\theta(-\omega L_1 + k L_2) + V_\theta(-\omega L_3 + k L_4) - L]_\theta \\
&\quad + \varepsilon[(U_\theta L_1 + V_\theta L_3)_T + (U_\theta L_2 + V_\theta L_4)_X] = \varepsilon V_\theta b \cos \psi.
\end{aligned} \tag{19}
$$

Similarly, Eq. (17) can be rewritten as

$$(-\omega L_3 + k L_4)_\theta + \varepsilon (L_{3T} + L_{4X}) = \varepsilon b \cos \psi. \tag{20}$$

At this stage, we replace ψ by $\theta - \Phi$ in the right-hand side (RHS) in Eqs. (19) and (20), average with respect to θ and use the periodicity assumption, obtaining

$$(I_1 + I_3)_T + (I_2 + I_4)_X = b \langle V_\theta \cos(\theta - \Phi) \rangle \tag{21}$$

and

$$J_{3T} + J_{4X} = 0, \tag{22}$$

where

$$
\begin{aligned}
I_{1,2} &\equiv \langle U_\theta L_{1,2} \rangle, \\
I_{3,4} &\equiv \langle V_\theta L_{3,4} \rangle, \\
J_{3,4} &\equiv \langle L_{3,4} \rangle,
\end{aligned} \tag{23}
$$

and $\langle \cdots \rangle \equiv (2\pi)^{-1} \int_0^{2\pi} (\cdots) d\theta$. Finally, we expand $V(\theta, X, T) = \Sigma a_n(X, T) \exp(in\theta)$ and evaluate the average in the RHS in Eq. (21),

$$(I_1 + I_3)_T + (I_2 + I_4)_X = -\alpha \sin(\Phi + \psi_0), \tag{24}$$

where $\alpha \equiv b |a_1|$ and $\psi_0 \equiv \mathrm{Arg}(a_1)$. At this stage, formally, one can include the slow angle ψ_0 in (24) in the phase mismatch Φ and, therefore, without the loss of generality, we set $\psi_0 \equiv 0$ in the following.

Equations (22) and (24) describe the wave evolution on the slow variation scale and comprise an important pair of evolution equations in the theory of autoresonance. Additional slow equations can be obtained by constructing the averaged variational principle. To this end, we observe that an alternative formulation of the evolution Eqs. (16) and (17) is via the variational principle

$$\delta \int \int \langle \mathbf{L} \rangle dX dT = 0, \tag{25}$$

where

$$
\begin{aligned}
\langle \mathbf{L} \rangle = (2\pi)^{-1} \int_0^{2\pi} &[L(-\omega U_\theta + \varepsilon U_T, k U_\theta \\
&+ \varepsilon U_X, U; -\gamma - \omega V_\theta + \varepsilon V_T, \beta + k V_\theta + \varepsilon V_X; q) \\
&- \varepsilon b V_\theta \sin(\theta - \Phi)] d\theta.
\end{aligned} \tag{26}
$$

Equation (25) comprises the exact form of the Whitham's averaged variational principle[5] for our problem. Finding the remaining slow evolution equations for studying the autoresonance in the system involves using (25) with the averaged Lagrangian (26) evaluated to first order in the small parameter ε. We shall present this perturbation analysis in the following section.

III. SLOW EVOLUTION EQUATIONS

We proceed our analysis from studying the unperturbed, constant parameters problem described by the same Lagrangian as in the integrand in (26), but with $\varepsilon = 0$, i.e.,

$$L^0 = L(-\omega U^0_\theta, k U^0_\theta, U; -\gamma - \omega V^0_\theta, \beta + k V^0_\theta; q), \tag{27}$$

where $U^0(\theta, X, T)$ and $V^0(\theta, X, T)$ denote the fields of this unperturbed problem. The Lagrangian L^0 yields the following evolution equations:

$$-\omega L^0_{1\theta} + k L^0_{2\theta} - L^0_0 = 0, \tag{28}$$

$$-\omega L^0_{3\theta} + k L^0_{4\theta} = 0, \tag{29}$$

where $L^0_{i\theta}$ $(i = 1-4)$ represent evaluations of L_i at $u_t = -\omega U^0_\theta$, $u_x = k U^0_\theta$, $v_t = -\gamma - \omega V^0_\theta$, and $v_x = \beta + k V^0_\theta$.

Note that Eqs. (22) and (23) comprise a set of ordinary differential equations in θ, while T and X are held fixed in these equations. It is convenient to treat this one-dimensional (in θ) two degrees of freedom problem (u and v) by the usual Hamiltonian formalism. To this end, we introduce an auxiliary time variable τ of the unperturbed problem via $\theta_\tau = -\omega$ (then, $U^0_\tau \equiv -\omega U^0_\theta$ and $V^0_\tau \equiv -\omega V^0_\theta$), view L^0 as a function of U, U^0_τ, and V^0_τ only, and define the canonical momenta

$$p^u \equiv \frac{\partial L^0}{\partial U^0_\tau} = L^0_1 - (k/\omega) L^0_2, \quad p^v \equiv \frac{\partial L^0}{\partial V^0_\tau} = L^0_3 - (k/\omega) L_4. \tag{30}$$

Then, we use (30) for expressing U^0_τ and V^0_τ via p^u and p^v and construct the Hamiltonian

$$H^0(p^u, p^v, U^0) \equiv U^0_\tau p^u + V^0_\tau p^v - L^0. \tag{31}$$

Now, since H^0 is independent of V^0 (V^0 is a cyclic variable), p^v is conserved, i.e.,

$$p^v = L^0_3 - (k/\omega) L_4 = B(X, T). \tag{32}$$

Furthermore, since H^0 is independent of τ (autonomous problem), it is also conserved,

$$U^0_\tau p^u + V^0_\tau p^v - L^0 = A(X,T). \tag{33}$$

Consequently, the unperturbed Hamiltonian problem is integrable.

The most convenient representation of the solutions of our unperturbed problem is obtained by using the constancy of p^v and transforming to the canonical action-angle variables associated with the reduced problem for U^0. The desired canonical transformation is achieved by using the following mixed variables generating function $F(U^0,I;V^0,B) = S(U^0,I,B) + V^0 B$ with S defined as

$$S(U^0,I,B) \equiv \int^{U^0} p^{u*}[A(I,B),B,U]\,dU. \tag{34}$$

Here $p^{u*}(A,B,U)$ is the solution of (33) (where one substitutes $p^v = B$) for p^u, the action $I = I(A,B)$ is defined by integrating over one oscillation of the unperturbed problem

$$I \equiv (2\pi)^{-1} \oint p^{u*}(A,B,U)\,dU = I(A,B), \tag{35}$$

and we invert (35) by writing $A = A(I,B)$ in the integrand in Eq. (34). The generating function F defines the canonical angle variable of the transformed problem, i.e., $\theta_0 \equiv F_I = S_I$, while V^0 is transformed into

$$\tilde{V}^0 \equiv F_B = V^0 - f(I,B,U^0), \tag{36}$$

where $f \equiv -S_B$. Since F does not depend on τ explicitly, the dependent variables of the transformed problem, i.e., I, B, θ^0 and $\tilde{V}^0$ are governed by the same Hamiltonian (31), where one simply expresses the old via the new variables, i.e., $H^0 \to \tilde{H}^0 = A(I,B)$. This Hamiltonian is independent of θ^0 and $\tilde{V}^0$ and, therefore, I and B, as well as, the frequency $\theta^0_\tau = A_I$ and $\tilde{V}^0_\tau = A_B$ in the unperturbed problem are constant of τ (but parametrically depend on X,T). Finally, we go back to the original field variables of the unperturbed problem and write them in the form [see Eq. (36)],

$$U^0 = U^0(\theta_0,I,B,Q), \tag{37}$$

$$V^0 = \tilde{V}^0 + f(\theta_0,I,B,Q), \tag{38}$$

where Q represents the set $[q,\gamma,\beta,(k/\omega)]$.

Note that, by definition, the generating function $S(U^0,I,B)$ changes by $\Delta S = 2\pi I$ during one oscillation of U^0, and, therefore, θ^0 changes by $\Delta\theta_0 = (\Delta S)_I = 2\pi$, while f in (38) does not change, since $\Delta f = (\Delta S)_B = 0$. Thus, both U^0 and f in Eqs. (37) and (38) are 2π-periodic functions of θ^0. Recall that, originally, we have assumed that U^0 is 2π-periodic function of θ and, therefore, naturally, we identify $\theta = \theta^0$. Then,

$$\theta^0_\tau = A_I = -\omega. \tag{39}$$

Furthermore, V^0 was also assumed to be a 2π-periodic function, but in (38) it has an aperiodic (secular) component $\tilde{V}^0$. Nevertheless, we have already introduced the secular term ξ in (14) for field v, so, now, we set $\tilde{V}^0 = 0$ and, therefore,

$$\tilde{V}^0_\tau = A_B = 0. \tag{40}$$

At this stage, one can justify the presence of a secular term in the solution (14) for the potential field v in our original problem. Simply, in the unperturbed case, the potential field corresponds to a cyclic variable in the associated dynamical problem, which, as shown above, has a monotonic secular part in addition to the periodic component.

At this stage, we return to the perturbed problem. Here, in the variational principle (25), we use a class of functions $U(\theta,X,T)$ and $V(\theta,X,T)$, which are 2π-periodic in θ and slow in x,t ($X = \epsilon x$, $T = \epsilon t$). But the above-mentioned solutions $U^0(\theta_0,I,B,Q)$ and $V^0(\theta_0,I,B,Q)$ of the unperturbed problem have exactly these properties, if one again identifies θ with θ^0. Furthermore, U^0 and V^0 solve the $O(\epsilon^0)$ problem and, therefore, to $O(\epsilon)$, the solution of the perturbed problem must lie in the ϵ-vicinity of U^0 and V^0. Thus, to $O(\epsilon)$, one can use the following trial functions in evaluating (26),

$$\begin{aligned} U(\theta,X,T) &= U^0(\theta,I,B,Q) + \epsilon U^1(\theta,X,T), \\ V(\theta,X,T) &= V^0(\theta,I,B,Q) + \epsilon V^1(\theta,X,T), \end{aligned} \tag{41}$$

where U^1 and V^1 are arbitrary, 2π-periodic in θ functions of X,T.

Next, we use (41) in evaluating the averaged Lagrangian (26) to $O(\epsilon)$. Note that, since U^0 and V^0 minimize the action functional in (25) in the solutions function space, the $O(\epsilon)$ additions in (41) will lead to $O(\epsilon^2)$ corrections in the averaged Lagrangian (26). Thus, to $O(\epsilon)$, the averaging in (26) can be accomplished by using the unperturbed solutions U^0 and V^0 only. This is an important advantage of the variational method. By expanding to $O(\epsilon)$ in (26) and neglecting the contributions of U^1 and V^1, as explained above, we obtain

$$\langle \mathbf{L} \rangle = \mathscr{L} + \epsilon\alpha \cos\Phi + R, \tag{42}$$

where $\mathscr{L} \equiv \langle L^0 \rangle$ and

$$R \equiv \epsilon[\langle U^0_T L^0_1\rangle + \langle U^0_X L^0_2\rangle + \langle V^0_T L^0_3\rangle + \langle V^0_X L^0_4\rangle]. \tag{43}$$

The averaged Lagrangian $\mathscr{L}$ can be evaluated by averaging in (33), where one substitutes $U^0_\tau \equiv -\omega U^0_\theta$, $V^0_\tau \equiv -\omega V^0_\theta$, $p^u = p^{u*}(A,B,Q,U^0)$, and $p^v = B$,

$$\mathscr{L} = -\omega I(A,B,Q) - A. \tag{44}$$

Alternatively, by using the definitions (23) and (30), one can write

$$\mathscr{L} = -\omega Y + kZ - A, \tag{45}$$

where

$$Y \equiv I^0_1 + I^0_3 = -\mathscr{L}_\omega, \quad Z \equiv I^0_2 + I^0_4 = \mathscr{L}_k. \tag{46}$$

Note that $-\omega I = -\omega I^0_1 + k I^0_2$, while $-\omega I^0_3 + k I^0_4 = 0$. Furthermore, formally, the zero order averaged Lagrangian $\mathscr{L}$ in (42) is the same as that used in the theory of free (undriven) adiabatic modulations of the field equations.[5] Therefore, in studying the autoresonance, one can use the expression for $\mathscr{L}$ (if known) from that theory.

We observe that $\langle \mathbf{L} \rangle$ [see (42)] is an object depending on A,B,ξ,Φ and their derivatives (recall that $\beta \equiv \xi_x$, $\gamma \equiv -\xi_t$, $\omega \equiv \omega_0 - \Phi_t$, and $k \equiv k_0 + \Phi_x$). Variation of $\langle \mathbf{L} \rangle$ with respect to these dependent variables yields the desired system of slow evolution equations in our problem. For instance, variations with respect to A and B provide a pair of equations,

$$\mathscr{L}_A-\epsilon\alpha_A\cos\Phi-\Gamma^A=0, \tag{47}$$
$$\mathscr{L}_B-\epsilon\alpha_B\cos\Phi-\Gamma^B=0,$$

where the $O(\epsilon)$ terms Γ^r $(r=A,B)$ are

$$\Gamma^r\equiv R_r-(R_{r_t})_t-(R_{r_x})_x. \tag{48}$$

We shall see in the next section that Γ^r can be neglected in Eqs. (47) when studying the autoresonance. In contrast, the terms $\epsilon\alpha_{A,B}\cos\Phi$ play an important role in the initial autoresonant wave excitation stage despite the smallness of ϵ, since $\alpha_{A,B}$ may be large in this stage. Indeed, let a be the characteristic amplitude of weakly nonlinear oscillations of the fields u and v. Then, to lowest order in the amplitude, A and B are quadratic in a [see the definitions in (32) and (33)]. At the same time, α scales as a. Therefore, $\epsilon\alpha_{A,B}\sim O(\epsilon/a)$ can be large in the initial excitation stage, when a is still small. Note that, without Γ^r, Eqs. (47) are algebraic in terms of A, B, β, γ, ω, k, and Φ. Nevertheless, these equations involve partial derivatives of Φ implicitly via $\omega\equiv\omega_0-\Phi_t$ and $k=k_0+\Phi_x$. This dependence can be made explicit by using the assumed smallness of $\Phi_{x,t}$ [$\sim O(\epsilon^{1/2})$, see below] and expanding in (47) around ω_0,k_0,

$$\begin{aligned}Y_A\Phi_t+Z_A\Phi_x&=\omega_0Y_A-k_0Z_A+1+\epsilon\alpha_A\cos\Phi+\Gamma^A,\\ Y_B\Phi_t+Z_B\Phi_x&=\omega_0Y_B-k_0Z_B+\epsilon\alpha_B\cos\Phi+\Gamma^B,\end{aligned} \tag{49}$$

where, to lowest order, all the objects are evaluated at ω_0,k_0.

An additional pair of slow equations in the problem is obtained by taking the variations with respect to Φ and ξ yielding, to $O(\epsilon)$,

$$Y_t+Z_x=-\epsilon\alpha\sin\Phi, \tag{50}$$

$$-\mathscr{L}_{\gamma t}+\mathscr{L}_{\beta x}=0. \tag{51}$$

Remarkably, these equations are not new and correspond to $O(\epsilon)$ approximations of (21) and (22), since, to lowest order, $\mathscr{L}_\gamma=-J_3$, $\mathscr{L}_\beta=J_4$. Note that instead of one of the slow PDE's, say Eq. (51), one can use the algebraic equation,

$$\omega(B+\mathscr{L}_\gamma)+k\mathscr{L}_\beta=0, \tag{52}$$

which is obtained by averaging in (32). Finally, it may be convenient to view β and γ as two dependent variables instead of a single variable ξ. Then, we must complete the system by the consistency condition

$$\gamma_x+\beta_t=0 \tag{53}$$

and Eqs. (47) [or, alternatively, (49)], (50), (52) and (53) comprise a complete set for A, B, γ, β, and Φ.

IV. AUTORESONANT SOLUTIONS AND STABILITY

At this stage we proceed to studying the solutions of the slow system (49), (50), (52), and (53). We postpone the discussion of the initial driven wave excitation stage and view the terms $\epsilon\alpha_{A,B}\cos\Phi$ in Eqs. (49) as being $O(\epsilon)$. Then, the necessary condition for the assumed slowness of variation of the solutions A, B, β, γ, and Φ in our system is the near cancellation of the large algebraic terms in $\omega_0Y_A-k_0Z_A+1$ and $\omega_0Y_B-k_0Z_B$ in the RHS of Eqs. (49), since other terms in these equations are small. Consequently, in discussing the existence of the slowly varying solutions, we proceed from the question of existence of auxiliary functions, $\bar{A},\bar{B},\bar{\beta},\bar{\gamma}$ satisfying

$$\omega_0\bar{Y}_A-k_0\bar{Z}_A+1=0, \tag{54}$$

$$\omega_0\bar{Y}_B-k_0\bar{Z}_B=0, \tag{55}$$

$$\omega_0(\bar{B}+\bar{\mathscr{L}}_\gamma)+k_0\bar{\mathscr{L}}_\beta=0, \tag{56}$$

$$\bar{\gamma}_x+\bar{\beta}_t=0, \tag{57}$$

where $\bar{Y}_A,\bar{Z}_A,\bar{Y}_B,\bar{Z}_B,\bar{\mathscr{L}}_\gamma,\bar{\mathscr{L}}_\beta$ denote the evaluations at $\bar{A}$, $\bar{B}$, $\bar{\beta}$, and $\bar{\gamma}$ and by using the frequency and the wave vector ω_0, k_0 of the pump wave. Let us show that, Eqs. (54)–(57) yield slowly varying solutions in an extended region of space–time, provided these equations are satisfied on a line in the x,t-plane (the boundary line in the following). Indeed, one can eliminate $\bar{A}$ and $\bar{B}$ from the system by using Eqs. (54) and (55) and, then, use (56) for expressing $\bar{\gamma}=G(\bar{\beta};x,t)$, where the x,t dependence enters via the slowly varying parameters ω_0,k_0,q in the problem. On substituting G into Eq. (57) we obtain a PDE for $\bar{\beta}$,

$$G_{\bar{\beta}}\bar{\beta}_x+\bar{\beta}_t=-G_x, \tag{58}$$

which can be solved along the characteristics originating on the boundary line defined above. The smallness of the RHS in Eq. (58) guarantees the slowness of variation of $\bar{\beta}$ along the characteristics, i.e., in the entire x,t-region accessible by the characteristics.

The solutions $\bar{A}$, $\bar{B}$, $\bar{\beta}$, and $\bar{\gamma}$ defined above have an important physical meaning. They represent the characteristic parameters of a uniform wavetrain solution in the unperturbed wave problem U^0,V^0 [see the definition in (41)], should this wavetrain, at any point x,t, have the frequency ω and wave vector k equal to those of our pump wave evaluated at this point, i.e., $\omega=\omega_0(x,t)$ and $k=k_0(x,t)$. In other words, we have defined an auxiliary globally resonant quasiuniform wavetrain, with the characteristic parameters $\bar{A}$, $\bar{B}$, $\bar{\beta}$, and $\bar{\gamma}$ adjusted in space–time continuously to preserve the exact resonance with the pump wave.

Now, we return to our original system (49), (50), (52), and (53) for A, B, γ, β, and Φ. We shall see that, at certain conditions, this system yields solutions in the vicinity of $\bar{A}$, $\bar{B}$, $\bar{\gamma}$, $\bar{\beta}$, and 0 (mod π), respectively. We shall use the term autoresonant solution to represent the driven nonlinear wave in this case. It follows from the discussion above that the proof of the existence of the autoresonant solutions must include two steps. The first is showing the existence of the boundary line, where our solution is close to the globally resonant wavetrain solution defined above, i.e., where all the differences $\Delta A\equiv A-\bar{A}$, $\Delta B\equiv B-\bar{B}$, $\Delta\gamma\equiv\gamma-\bar{\gamma}$, $\Delta\beta\equiv\beta-\bar{\beta}$, as well as Φ (mod π), are small. The second step is demonstrating that ΔA, ΔB, $\Delta\gamma$, $\Delta\beta$ and the change in Φ remain small in the region of existence of the globally resonant wavetrain solution (the autoresonant region in the following). In other words, one must prove the stability of the autoresonant solution.

Let us address the problem of existence of the resonant boundary line, first. We are interested in the internal excitation of the daughter wave, i.e., when the pump wave is launched from an external boundary, where it is out of resonance with the daughter wave (the latter is assumed to be negligible on this boundary). Then, the initial excitation process is that of generating a small amplitude daughter wave as the pump propagates from the boundary. Such initial excitation can be treated within a weakly nonlinear theory. In studying this limit, we can use the set of variational equations derived above. Nevertheless, the conventional harmonic decomposition approach (see the example in the Appendix) is more convenient. The main formal difference between the weakly nonlinear regime and the fully nonlinear autoresonant evolution (see the discussion at the end of Sec. III) is in the role of the interaction terms $\epsilon\alpha_{A,B}\cos\Phi$ in Eqs. (49). In the nearly linear regime, these terms scale as ϵ/a, where a is the amplitude of the oscillations. We identify these singular interaction terms in Eqs. (49), as the main reason for the phase locking [i.e., $\Phi\rightarrow 0 (\mathrm{mod}\ \pi)$] between the daughter and the pump waves which takes place as the pump approaches the region where it resonants with a linear daughter wave (a line in the x,t dependent case studied here). This phase trapping effect was discussed in the single-field theory[1] and we expect the same phenomenon take place in the two-field case. In this work, we shall illustrate this phenomenon in application to the driven KdV equation (see the Appendix), but a similar treatment can be performed in the general case. We shall see that, as the pump wave crosses the linear resonance line, in addition to the phase trapping, the objects $\mathscr{L}_A$ and $\mathscr{L}_B$ evaluated at $\omega=\omega_0$, $k=k_0$ become small, while the initial inequality $\epsilon\alpha_{A,B}\gg 1$ may be reversed. In this case, the linear resonance line serves as the resonant boundary line discussed above in the context of autoresonance and we can proceed to the question of stability of autoresonant solutions beyond the linear resonance, where the interaction terms in Eqs. (49) can be neglected.

In studying the stability, we assume that the set $\bar{A}$, $\bar{B}$, $\bar{\beta}$, and $\bar{\gamma}$ is already found and write the solutions of our evolution equations as $A=\tilde{A}+\delta A$, $B=\tilde{B}+\delta B$, $\beta=\tilde{\beta}+\delta\beta$, $\gamma=\tilde{\gamma}+\delta\gamma$, and $\Phi=\tilde{\Phi}+\delta\Phi$, where $\tilde{A}$, $\tilde{B}$, $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{\Phi}$ are slow and smooth (nonoscillating), while $\delta(\cdots)\sim\exp[i\phi(x,t)]$ represent small oscillating eikonal type components. In addition, by defining the local wave vector and frequency of these oscillations via $\nu\equiv-\phi_t$ and $\kappa\equiv\phi_x$, we shall assume that the smooth objects vary on the space–time scales much larger than those given by κ^{-1} and ν^{-1} (this is the usual WKB approximation). Then, one can separate the smooth (averaged) and oscillating components in our system of Eqs. (49), (50), (52), and (53). We use the definitions $\omega=\omega_0-\tilde{\Phi}_t+i\nu\delta\Phi\approx\omega_0+i\nu\delta\Phi$ and $k=k_0+\tilde{\Phi}_x+i\kappa\delta\Phi\approx k_0+i\kappa\delta\Phi$ and write the lowest order averaged equations

$$\omega_0\tilde{Y}_A-k_0\tilde{Z}_A+1\approx 0, \tag{59}$$

$$\omega_0\tilde{Y}_B-k_0\tilde{Z}_B\approx 0, \tag{60}$$

$$\tilde{Y}_t+\tilde{Z}_x=-\epsilon\alpha\sin\tilde{\Phi}, \tag{61}$$

$$\omega_0(\tilde{B}+\tilde{\mathscr{L}}_\gamma)+k_0\tilde{\mathscr{L}}_\beta=0, \tag{62}$$

$$\tilde{\gamma}_x+\tilde{\beta}_t=0, \tag{63}$$

where the objects with tildes are evaluated at $\omega_0,k_0,\tilde{A},\tilde{B},\tilde{\beta},\tilde{\gamma}$. Similarly, the lowest order linearized equations for the oscillating components are

$$-i(\nu\tilde{Y}_A-\kappa\tilde{Z}_A)\delta\Phi=\sum_p d_{Ap}\delta p+\delta\Gamma^A, \tag{64}$$

$$-i(\nu\tilde{Y}_B-\kappa\tilde{Z}_B)\delta\Phi=\sum_p d_{Bp}\delta p+\delta\Gamma^B, \tag{65}$$

$$-i\sum_p(\nu\tilde{Y}_p-\kappa\tilde{Z}_p)\delta p+[\nu^2\tilde{Y}_\omega-\kappa^2\tilde{Z}_k+\nu\kappa(\tilde{Y}_k-\tilde{Z}_\omega)]\delta\Phi$$
$$=-\epsilon\alpha\delta\Phi, \tag{66}$$

$$\omega_0\delta B-i[\nu(\tilde{B}+\tilde{\mathscr{L}}_\gamma)-\kappa\tilde{\mathscr{L}}_\beta]\delta\Phi+\sum_p(\omega_0\tilde{\mathscr{L}}_{\gamma p}+k_0\tilde{\mathscr{L}}_{\beta p})\delta p$$
$$=0, \tag{67}$$

$$\nu\delta\gamma-\kappa\delta\beta=0, \tag{68}$$

where we have assumed $\tilde{\Phi}\ll 1$, and used the notations $p\equiv\{A,B,\beta,\gamma\}$, $\delta p\equiv\{\delta A,\delta B,\delta\beta,\delta\gamma\}$, $d\equiv\omega_0\tilde{Y}-k_0\tilde{Z}$, and $(\cdots)_p\equiv\partial(\ldots)/\partial p$. Note that the lowest order oscillating components $\delta\Gamma^{A,B}$ in Eqs. (64), (65) comprise linear combinations of x,t-derivatives of δp, $\delta\omega=-i\nu\delta\Phi$, and $-\delta k=i\kappa\delta\Phi$ and, thus, have the structure [see (48)]

$$\delta\Gamma^{A,B}=i\sum_p C_p^{A,B}(\nu,k)\delta p+D^{A,B}(\nu,k)\delta\Phi, \tag{69}$$

where $C^{A,B}(\nu,k)$ and $D^{A,B}(\nu,k)$ are homogeneous first and second order polynomials of ν and κ, respectively, with coefficients being slow functions of $\{\tilde{A},\tilde{B},\tilde{\beta},\tilde{\gamma}\}$.

We discuss the system (59)–(63) for the averaged components first. The comparison of Eqs. (59), (60), (62), and (63) in this system with Eqs. (54)–(57) shows that $\tilde{A},\tilde{B},\tilde{\beta},\tilde{\gamma}$ only slightly differ from the autoresonant solutions $\bar{A},\bar{B},\bar{\beta},\bar{\gamma}$ defined above. Then, to lowest order, Eq. (61) yields

$$\tilde{\Phi}\approx-(\epsilon\alpha)^{-1}(\bar{Y}_t+\bar{Z}_x), \tag{70}$$

i.e., $\tilde{\Phi}\sim O(\sigma/\epsilon)$, where, as defined earlier, $\sigma\ll 1$ is a dimensionless parameter characterizing the adiabaticity of the system's parameters [i.e., $(\bar{Y}_t+\bar{Z}_x)\sim\sigma$]. Therefore, our assumption $\tilde{\Phi}\ll 1$ yields the condition

$$\sigma/\epsilon\ll 1, \tag{71}$$

which will be assumed to be satisfied in the following.

Next, we proceed to Eqs. (64)–(68) for the oscillating components. We observe that ϵ in the RHS in Eq. (66) is the only explicit small parameter in this system and rescale the problem by introducing $\delta\hat{p}\equiv\delta p/\epsilon^{1/2}$, $\hat{\nu}\equiv\nu/\epsilon^{1/2}$, and $\hat{\kappa}\equiv\kappa/\epsilon^{1/2}$. Then the oscillating equations become

$$-i(\hat{\nu}\widetilde{Y}_A-\hat{\kappa}\widetilde{Z}_A)\delta\Phi=\sum_p d_{Ap}\delta\hat{p}+\epsilon^{1/2}\delta\hat{\Gamma}^A, \tag{72}$$

$$-i(\hat{\nu}\widetilde{Y}_B-\hat{\kappa}\widetilde{Z}_B)\delta\Phi=\sum_p d_{Bp}\delta\hat{p}+\epsilon^{1/2}\delta\hat{\Gamma}^B, \tag{73}$$

$$-i\sum_p (\hat{\nu}\widetilde{Y}_p-\hat{\kappa}\widetilde{Z}_p)\delta\hat{p}-[\hat{\nu}^2\widetilde{Y}_\omega+\hat{\kappa}^2\widetilde{Z}_k-\hat{\nu}\hat{\kappa}(\widetilde{Y}_k+\widetilde{Z}_\omega)]\delta\Phi =-\alpha\delta\Phi, \tag{74}$$

$$\omega_0\delta\hat{B}-i[\hat{\nu}(\widetilde{B}+\widetilde{\mathscr{L}}_\gamma)-\hat{\kappa}\widetilde{\mathscr{L}}_\beta]\delta\Phi+\sum_p (\omega_0\widetilde{\mathscr{L}}_{\gamma p}+k_0\widetilde{\mathscr{L}}_{\beta p})\delta\hat{p} =0, \tag{75}$$

$$\hat{\nu}\delta\hat{\gamma}-\hat{\kappa}\delta\hat{\beta}=0, \tag{76}$$

where, see (69), $\delta\hat{\Gamma}^{A,B}=i\Sigma_p C_p^{A,B}(\hat{\nu},\hat{k})\delta\hat{p}+D^{A,B}(\hat{\nu},\hat{k})\delta\Phi$. Next, we assume that

$$\nu,\kappa\sim O(\epsilon^{1/2}), \tag{77}$$

i.e., $\hat{\nu}$ and $\hat{\kappa}$ are of $O(1)$. Then, Eq. (74) yields the scaling

$$\delta\Phi\sim O(\delta\hat{p}), \tag{78}$$

so $\delta\Gamma^{A,B}\sim O(\delta\hat{p})$ and, therefore, one can neglect $O(\epsilon^{1/2})$ terms $\epsilon^{1/2}\delta\hat{\Gamma}^{A,B}$ in Eqs. (72) and (73). The resulting system yields real local dispersion relation for ν and κ, which has the form $D(\hat{\nu},\hat{\kappa};x,t)=D(\nu/\epsilon^{1/2},\kappa/\epsilon^{1/2};x,t)=0$ and, therefore, justifies the assumed scaling (77). Furthermore, this dispersion relation is a first order PDE for the eikonal function ϕ (recall that $\nu\equiv-\phi_t$ and $\kappa\equiv\phi_x$). It can be solved along the characteristics (the rays of the WKB approximation) starting at the linear resonance line. Since $D(\hat{\nu},\hat{\kappa};x,t)$ is real, the ray equations are also real, yielding real solutions for ν and κ. This proves the stability of the autoresonant solutions in the region accessible by the characteristics beyond the linear resonance line.

We conclude this section by making the following two remarks regarding the above-mentioned scalings of various objects in the autoresonant region beyond the liner resonance line. Firstly, on using (78), one finds that even for not very small values of $\delta\Phi$, say $|\delta\Phi|\leqslant\pi/6\approx0.5$ [which still allow the use of our approximation $\sin(\delta\Phi)\approx\delta\Phi$ to the accuracy of more than 1%] the amplitudes of the oscillations δp are smaller than $\delta\Phi$ by factor $\epsilon^{1/2}$. This guarantees the validity of our expansions in powers of δp, i.e., the validity of the assumptions $\delta p/p\ll1$, for $p\sim O(1)$. Secondly, in discussing the scalings of ν and κ, we have assumed that all other coefficients in Eqs. (72)–(76) are of $O(1)$. This is the strong nonlinearity assumption. Now, we can relax it to include weakly nonlinear situations. Indeed, the objects Y_p and Z_p in Eqs. (72) and (73) involve first derivatives of the averaged Lagrangian with respect to the dependent variables p. These objects are of $O(1)$ even in linear problems. On the other hand, the coefficients d_{Ap}, d_{Bp} in these equations involve second derivatives of the averaged Lagrangian with respect to p and, thus, vanish in the linear limit. Therefore, Eqs. (72) and (73) yield the scaling

$$\nu,\kappa\sim O(\epsilon\alpha\mathscr{d}/\mathscr{Y}_p), \tag{79}$$

where $\mathscr{d}$ and $\mathscr{Y}_p$ are typical values of d_{Ap},d_{Bp} and Y_p,Z_p, respectively. On the other hand, Eq. (66) gives

$$\delta p/p\sim(\epsilon\alpha\delta\Phi)/(\nu\mathscr{Y}_p p), \tag{80}$$

which, on using (79), becomes

$$\delta p/p\sim p^{-1}(\epsilon\alpha/d\mathscr{Y}_p)^{1/2}\delta\Phi. \tag{81}$$

Therefore, since $\delta\Phi$ may be of $O(1)$ (see above), the condition $\delta p/p\ll1$ requires

$$p^{-1}(\epsilon\alpha/d\mathscr{Y}_p)^{1/2}\ll1. \tag{82}$$

We shall refer to the last inequality as the moderate nonlinearity condition. For instance, in a weakly nonlinear case, when $p\sim a^2$ (a being the characteristic amplitude of the oscillating fields), $\alpha\sim a$ and $Y_m\sim1$, condition (82) becomes

$$(\epsilon/da^3)^{1/2}\ll1 \tag{83}$$

and, obviously requires a sufficiently large amplitude.

Finally, we recall that, in addition to the sufficient nonlinearity requirement, we based our stability analysis on the adiabaticity assumption for the autoresonant oscillations δp. In other words, we have assumed a weak variation of the coefficients in Eqs. (72)–(75) on the space–time scales associated with ν and κ. Then, (77) yields the condition $\sigma\epsilon^{-1/2}\ll1$, which is satisfied because a stronger inequality (71) is already imposed. This completes our analysis of the autoresonance in a 2-field system given by Lagrangian of form (10) and we proceed to an application of the theory.

V. RESONANTLY DRIVEN KdV EQUATION

This section presents the application of the averaged variational principle approach developed in Sec. II–IV to studying autoresonant solutions of the driven KdV equation (8). This is a two-field problem with the Lagrangian [see Eqs. (9), (10)],

$$\mathbf{L}(u,u_x;v_t,v_x;x,t)=L+\epsilon v\cos\psi(xt), \tag{84}$$

where $L=\frac{1}{2}v_tv_x+\frac{1}{2}v_x^2+v_x^3+v_xu_x+\frac{1}{2}u^2$, the fields v and u are related to the original field variable η via $\eta=v_x$ and $u=v_{xx}=\eta_x$, and we have set $b=1$ for simplicity. The averaged Lagrangian for studying the autoresonance in this system can be written in the form [see Eq. (42)],

$$\langle\mathbf{L}\rangle=\mathscr{L}+\epsilon\alpha\cos\Phi, \tag{85}$$

where we have neglected R, since (see Sec. IV) this term leads to only a slight shift of the smooth autoresonant solutions. As mentioned earlier, formally, the averaged Lagrangian $\mathscr{L}=\langle L^0\rangle$ of the unperturbed problem in (85) is the same as that characterizing free modulations of the KdV equation[5] and we can use the expression for $\mathscr{L}$ from that theory. Nevertheless, in order to use notations consistent with the developments in Sec. III, as well as, for the completeness of presentation, we give a brief derivation of $\mathscr{L}$.

On using the definition [see Eq. (15)],

$$v_x=\beta+kV_\theta^0 \tag{86}$$

to express

$$v_t=-\gamma-\omega V_\theta^0=\delta-(\omega/k)v_x, \tag{87}$$

where $\delta\equiv-\gamma+(\omega/k)\beta$, and defining $h\equiv(\omega/k)-1$, we rewrite L^0 as a function of u, u_x and v_x only,

$$L^0=\frac{\delta}{2}v_x-\frac{h}{2}v_x^2+v_x^3+v_xu_x+\frac{1}{2}u^2=L^0(u,u_x,v_x). \tag{88}$$

Note a slight difference from the general theory of Sec. III, where we have viewed L^0 as a function of U^0, U^0_τ and V^0_τ. Next, similarly to (30), we define the canonical momenta

$$p^u\equiv L^0_{u_x}=v_x \tag{89}$$

and

$$p^v\equiv L^0_{v_x}=\delta/2-hv_x+3v_x^2+u_x\equiv\mathscr{B}, \tag{90}$$

and construct the Hamiltonian [compare to Eq. (31)],

$$H^0\equiv u_xp^u+v_xp^v-L^0=(\mathscr{B}-\delta/2)v_x+(h/2)v_x^2-v_x^3-u^2/2\equiv\mathscr{A}. \tag{91}$$

Now, we define new slow objects $B\equiv-\mathscr{B}+\delta/2$ and $A\equiv-\mathscr{A}$, so the last equation becomes

$$u^2/2+Bv_x-(h/2)v_x^2+v_x^3=A, \tag{92}$$

or, by returning to the original field variable,

$$\eta_x^2/2+B\eta-(h/2)\eta^2+\eta^3=A. \tag{93}$$

Thus, A can be viewed as the ''energy'' of a quasiparticle performing spatial oscillations in an effective potential well

$$V_{\rm eff}(\eta)=\eta^3-(h/2)\eta^2+B\eta. \tag{94}$$

Finally, on using (91), we express $L^0=u_xp^u+v_xp^v-\mathscr{A}=u_xv_x+(\delta/2-B)v_x+A$ and take the average

$$\mathscr{L}=\langle L^0\rangle=\langle u_xv_x\rangle+(\delta/2-B)\langle v_x\rangle+A. \tag{95}$$

Here $\langle v_x\rangle=\langle\eta\rangle=\beta$,

$$\langle u_xv_x\rangle=\langle u_x\eta\rangle=k(2\pi)^{-1}\int_0^{2\pi}u_\theta\eta d\theta=-k(2\pi)^{-1}\int_0^{2\pi}u\eta_\theta d\theta=-k(2\pi)^{-1}\oint\eta_x^*d\eta, \tag{96}$$

and

$$(\eta_x^*)^2=2A-2B\eta+h\eta^2-2\eta^3 \tag{97}$$

is the solution for η_x^2 from (93). Thus,

$$\mathscr{L}=-kW(A,B,h)+(\delta/2-B)\beta+A=\mathscr{L}(A,B,\beta,\gamma), \tag{98}$$

where

$$W(A,B,h)\equiv(2\pi)^{-1}\oint\sqrt{2A-2B\eta+h\eta^2-2\eta^3}\,d\eta \tag{99}$$

is the canonical action of the abovementioned quasiparticle [the analog of the action integral (35) in the general theory] with the integration in (99) performed over one oscillation of the quasiparticle in the effective potential.

Expression (98) can now be used in the perturbed averaged Lagrangian (85) for studying the autoresonance in our system. The desired set of the slow evolution equations is obtained by substituting (85) in the averaged variational principle (25), taking variations with respect to A, B, Φ, and ξ (recall that $\beta=\xi_x$ and $\gamma=-\xi_t$), and adding the consistency relation [compare to Eqs. (47), (50), (51) and (53)],

$$\mathscr{L}_A=1-kW_A=0, \tag{100}$$

$$\mathscr{L}_B=-\beta-kW_B=0, \tag{101}$$

$$-\mathscr{L}_{ax}+\mathscr{L}_{kx}=-\epsilon a\sin\Phi, \tag{102}$$

$$-\mathscr{L}_{\gamma t}+\mathscr{L}_{\beta x}=(\beta/2)_t+(\beta\omega/k-\gamma/2-B)_x=0, \tag{103}$$

$$\gamma_x+\beta_t=0. \tag{104}$$

Here, being interested in the nonlinear evolution beyond the linear resonance line, we have neglected the terms $\epsilon\alpha_{A,B}\cos\Phi$ in Eqs. (100) and (101). We shall discuss the initial wave excitation stage later within the weakly nonlinear theory. Note, that the partial derivatives W_A and W_B in Eqs. (100) and (101) have simple meanings. Indeed,

$$W_A=(2\pi)^{-1}\oint(\eta_x^*)^{-1}d\eta=(2\pi)^{-1}\oint dx=(2\pi)^{-1}\lambda(A,B,h), \tag{105}$$

$$W_B=-(2\pi)^{-1}\oint(\eta_x^*)^{-1}\eta d\eta=-(2\pi)^{-1}\oint\eta dx=-(2\pi)^{-1}\lambda\beta(A,B,h), \tag{106}$$

where $\lambda(A,B,h)$ is the ''period'' (wavelength) of the spatial oscillations of the quasiparticle in the potential (94), while $\beta(A,B,h)$ is the average value of η associated with these oscillations. Thus, our system of evolution equations can be rewritten as

$$\lambda(A,B,h)=2\pi/k, \tag{107}$$

$$\beta(A,B,h)=\beta, \tag{108}$$

$$B=(\omega/k)\beta-\gamma, \tag{109}$$

$$(k\mathscr{L}_\omega+\beta\mathscr{L}_\gamma)_t+(\mathscr{L}-k\mathscr{L}_k-\beta\mathscr{L}_\beta)_x=k(\mathscr{L}_{\omega t}-\mathscr{L}_{kx})=\epsilon k\alpha\sin\Phi, \tag{110}$$

$$\gamma_x+\beta_t=0, \tag{111}$$

where Eqs. (107), (108) replaced Eqs. (100), (101); Eq. (109) [the analog of Eq. (52)] was written by combining Eqs. (103) and (104), while Eq. (110) is an equivalent form (the momentum equation[5]) of (102). Finally, after simple algebra, in Eq. (110),

$$k\mathscr{L}_\omega+\beta\mathscr{L}_\gamma=-kW_h;\quad \mathscr{L}-k\mathscr{L}_k-\beta\mathscr{L}_\beta=-\omega W_h+A, \tag{112}$$

where, by definition,

$$W_h=(2\pi)^{-1}\oint(\eta_x^*)^{-1}(\eta^2/2)d\eta$$
$$=-(2\pi)^{-1}\oint(\eta^2/2)dx$$
$$=-(2\pi)^{-1}\lambda P(A,B,h) \tag{113}$$

and $P(A,B,h)$ is the average value of $\eta^2/2$ associated with the oscillations in the potential (94). Finally, the averaging in Eq. (90) yields $P(A,B,h)=(1/6)[h\beta(A,B,h)-B]$ and, therefore, (113) becomes

$$W_h=(12\pi)^{-1}\lambda(A,B,h)[h\beta(A,B,h)-B]. \tag{114}$$

At this stage, we shall discuss the smooth autoresonant solutions in the autoresonant region. We shall return to the full solution of the slow equations later, when studying applications to purely temporal or spatial modulations. The smooth autoresonant solutions $\overline{A,B,\beta,\gamma}$ in our theory are described by a set of Eqs. (54)–(57), obtained similarly to the complete set of the slow evolution equations, but by neglecting all the first order corrections in the averaged Lagrangian and by using the frequency and the wave vector ω_0,k_0 of the pump wave instead of ω,k. Thus, in our case, we have the following complete set of the smooth equations [compare to Eqs. (107)–(109) and (111)] beyond the linear resonance line:

$$\lambda(\overline{A,B},h_0)=2\pi/k_0, \tag{115}$$

$$\beta(\overline{A,B},h_0)=\overline{\beta}, \tag{116}$$

$$\overline{B}=\overline{\beta}\omega_0/k_0-\overline{\gamma}, \tag{117}$$

$$\overline{\gamma}_x+\overline{\beta}_t=0. \tag{118}$$

The linear resonance line is viewed as the boundary for solving (115–118). Note that according to Eqs. (115) and (116), at the smooth autoresonant values $\overline{A,B,\beta,\gamma}$, the ''period'' and the averaged value of the spatial oscillations of the quasiparticle in the potential (94) are the same as the local wavelength of the pump and the averaged value of the driven nonlinear wave, respectively. Finally, we observe that in two situations the problem of finding smooth autoresonant solutions requires dealing with a pair of algebraic equations only. These are the cases of purely time or space modulations. For instance, in the time dependent case, $k_0=\text{const}$, $\omega_0=\omega_0(t)$, while $\overline{A}$, $\overline{B}$, $\overline{\gamma}$, and $\overline{\beta}$, are viewed as functions of t only. Then, Eq. (118) yields $\overline{\beta}=0$ beyond the resonance line, and Eqs. (115) and (116),

$$\lambda(\overline{A,B},h_0)=2\pi/k_0;\ \ \beta(\overline{A,B},h_0)=0, \tag{119}$$

are sufficient for finding $\overline{A}(t)$, $\overline{B}(t)$, while Eq. (117) yields $\overline{\gamma}=-\overline{B}$ [note, however, that γ is not needed in finding η, since it does not enter the quasipotential (94)]. Similarly, for purely space-dependent modulations, $k_0=k_0(x)$, $\omega_0=\text{const}$ and one seeks solutions for $\overline{A}$, $\overline{B}$, $\overline{\gamma}$, and $\overline{\beta}$ as functions of x only. Then, from (118), $\overline{\gamma}=0$, while (117) yields $\overline{\beta}=k_0\overline{B}/\omega_0$ and, therefore, the algebraic system for $\overline{A}(t)$ and $\overline{B}(t)$ is

$$\lambda(\overline{A,B,h})=2\pi/k_0;\ \ \beta(\overline{A,B},h_0)=k_0\overline{B}/\omega_0. \tag{120}$$

Now, we return to the full system of slow evolution Eqs. (107)–(111) associated with the Lagrangian (85). Obviously, this system is quite complex and, therefore, we limit our discussion below to purely temporal or spatial modulations, discussed earlier in the context of the smooth autoresonant solutions.

A. Purely temporal modulations

Here $k_0=\text{const}$, $\omega_0=\omega_0(t)$, while A, B, γ, β, and Φ are viewed as functions of t only. Then, $k=k_0$, Eq. (111) yields $\beta=0$ and our system reduces to

$$(2\pi)^{-1}k_0\lambda(A,B,h)=1, \tag{121}$$

$$\beta(A,B,h)=0, \tag{122}$$

$$\gamma=-B, \tag{123}$$

$$W_{ht}=-\epsilon\alpha\sin\Phi, \tag{124}$$

where $W_h=-(12\pi)^{-1}B\lambda(A,B,h)$ [see Eq. (114)]. Formally, one can view the algebraic Eq. (122) as defining $B=B(A,h)$, which upon the use in (121) allows one to find the relation $\omega=\Omega(A)$, i.e., an ordinary differential equation (ODE) for Φ of the form,

$$\Phi_t=\omega_0(t)-\Omega(A). \tag{125}$$

Obviously, $\Omega(A)$ is the frequency of the uniform wave train solutions associated with the quasiparticle in the potential (94). Thus, our problem reduces to the solution of a system of just two ODE's (124) and (125) for A and Φ. Remarkably, these equations are similar in form to the system of evolution equations describing the autoresonance in driven dynamical systems with one degree of freedom.[6] Thus, formally, the problem reduces to that studied previously.

At this stage, for numerical applications, we adopt the conventional representation[7] of the oscillations of η in the quasipotential in terms of the amplitude

$$a\equiv(\eta_3-\eta_2)/2 \tag{126}$$

and nonlinearity parameter

$$m\equiv(\eta_3-\eta_2)/(\eta_3-\eta_1), \tag{127}$$

where $\eta_1\leqslant\eta_2\leqslant\eta_3$ are the roots of the cubic equation

$$V_{\text{eff}}(\eta)-A=\eta^3-(h/2)\eta^2+B\eta-A=0. \tag{128}$$

In other words, we shall use new dependent variables m and $s\equiv 2a/m$ instead of the original variables A and B. Note that $0<m<1$, where the lower and upper bounds correspond to the linear and solitary wave limits respectively. By definition,

$$\begin{aligned}\eta_1&=h/6-(s/3)(2-m),\\ \eta_2&=h/6-(s/3)(2m-1),\\ \eta_3&=h/6+(s/3)(m+1),\end{aligned} \tag{129}$$

while

$$A=\eta_1\eta_2\eta_3,$$

$$B=\eta_1\eta_2+\eta_1\eta_3+\eta_2\eta_3$$
$$=(1/2)h^2-(1/3)s^2(m^2-m+1).$$

Furthermore,[7]

$$\lambda(A,B,h)=2^{3/2}K(m)s^{-1/2}, \tag{130}$$

$$\beta(A,B,h)=\eta_1+sE(m)/K(m), \tag{131}$$

where $E(m)$ and $K(m)$ are the complete elliptic integrals in standard notation. Finally, we write the coupling coefficient α in terms of s and m,[4]

$$\alpha\equiv k^{-1}\langle\eta\cos\theta\rangle=(s/k)[\pi/K(m)]^2q/(1-q^2), \tag{132}$$

where $q\equiv\exp[-\pi K(1-m)/K(m)]$.

Now, Eqs. (121) allows us to express

$$s=2[k_0K(m)/\pi]^2 \tag{133}$$

and eliminate this variable from the problem. Then, on using (122) in (131) we have

$$h=[2k_0/\pi]^2[(2-m)K^2-3EK], \tag{134}$$

or

$$\Phi_t=\omega_0(t)-k_0+(2/\pi)^2k_0^3[3EK+(m-2)K^2]\}. \tag{135}$$

Similarly, by elimination of s, we have

$$W_h=-(2k_0^3/3\pi^4)[3(EK)^2-2(2-m)EK^3$$
$$+(1-m)K^4]. \tag{136}$$

Therefore, upon substitution into Eq. (124), we obtain

$$m_t=-\epsilon(\alpha/W_{hm})\sin\Phi. \tag{137}$$

Equations (135) and (137) comprise the desired system of ODE's for m and Φ. The same system of equations was obtained in Ref. 4 by using the conventional perturbation theory, which provides an additional test of the variational approach. Our nonlinear evolution equations are valid beyond the linear resonance point. The evolution of the system in the initial excitation region can be conveniently treated by using the weakly nonlinear theory (see Appendix). This theory yields the following equations for the wave amplitude a and mismatch Φ in the time dependent case [see Eqs. (A11)];

$$a_t=-\epsilon\sin\Phi,$$
$$\Phi_t=D-4.5(k_0a)^2[D+3k_0^3]^{-1}-(\epsilon/a)\cos\Phi, \tag{138}$$

where $D=\omega_0-k_0+k_0^3$. It is the presence of the singular term with a^{-1} in the second equation in (136) which guarantees (see Ref. 1) the trapping into the resonance as the pump wave approaches the linear resonance point ($D\rightarrow 0$), provided the excitation starts out of resonance and $a\approx 0$ at the initial integration point. Finally, we check that the transition from the weakly nonlinear system (138) to fully nonlinear Eqs. (135) and (137) is smooth. We use the expansions of the elliptic integrals[8]

$$K=(\pi/2)(1+m/4+9m^2/64+\cdots),$$

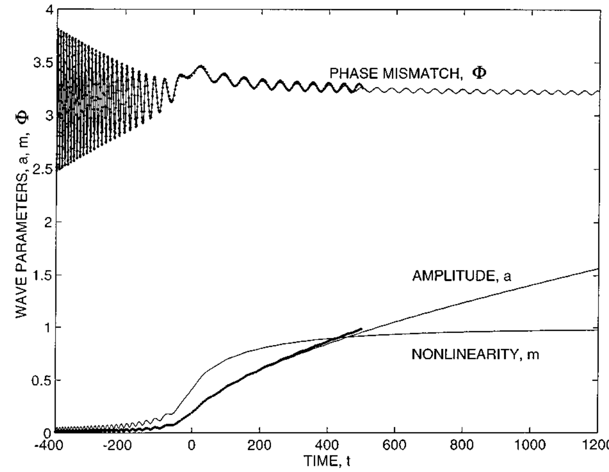

FIG. 1. Temporal autoresonance in the driven KdV system. The frequency of the pump wave is slowly increasing in time, while the wave vector is constant. The daughter wave amplitude a, nonlinearity parameter m, and the phase mismatch Φ are shown vs time. The dotted lines are numerical solutions of the weakly nonlinear system, while the solid lines represent fully nonlinear calculations. Emergence of a solitary wave solution is seen as $m\rightarrow 1$.

$$E=(\pi/2)(1-m/4-3m^2/64+\cdots).$$

Then, to $O(m^2)$, Eq. (135) yields

$$\Phi_t=\omega_0(t)-k_0+k_0^3-3m^2/32. \tag{139}$$

On the other hand, (133) gives $m\approx 4ak_0^{-2}$ and therefore the RHS of (139) coincides with the RHS of the second equation in (138) at the linear resonance point ($D=0$), if the interaction term in (138) is negligible. Similarly, Eqs. (136) and (132) yield

$$W_h=k_0^3m/32,\ \alpha=(2k_0)^{-1}a, \tag{140}$$

in the small amplitude limits, which after substitution into (137) gives the first equation in (138). This proves the smooth transition from the weakly nonlinear to fully nonlinear theory at the resonance. Remarkably, the above analysis also shows that instead of switching between the weakly nonlinear and fully nonlinear theories at the linear resonance point we can include the initial wave excitation stage in the fully nonlinear theory by adding the term $-(\varepsilon/a)\cos\Phi$ [see Eq. (138)] to the fully nonlinear evolution Eq. (135) for the phase mismatch,

$$\Phi_t=\omega_0(t)-k_0+(2/\pi)^2k_0^3[3EK+(m-2)K^2]\}$$
$$-(\varepsilon/a)\cos\Phi. \tag{141}$$

The interaction term in (141) is of $O(\varepsilon)$ in the nonlinear region and, therefore, is negligible, while for small values of the amplitude, as shown above, Eq. (141) coincides with the second equation in (138), where one can neglect the nonlinear term. Consequently, to the desired accuracy, Eq. (141) in combination with Eq. (137) are valid uniformly, for all times, and comprise our final system of equations for m and Φ in studying the temporal autoresonance in the driver KdV problem.

Now we present a numerical example. Figure 1 shows (thin solid lines) the evolution of the nonlinearity parameter

m amplitude a and phase mismatch Φ characterizing a resonantly excited KdV wave by using a pump wave with constant $k_0=1.2$ and $\omega_0=k_0-k_0^3+0.002t$. The local resonance between the pump wave and the linear KdV wave (i.e., when $D=\omega_0-k_0+k_0^3=0$) in this example takes place at $t=0$. We used $\varepsilon=0.01$ and initial conditions (at $t_0=-400$) $a=0.0125$ and $\Phi=0.8\pi$ in these calculations. One can see the initial excitation and phase trapping stage in the figure for $t<0$ followed by the autoresonant evolution beyond the linear resonance point. In the autoresonant region one can also see the stable autoresonant oscillations of the solution around the slowly evolving average. Finally, one observes the approach of m to unity at large times. This indicates the autoresonant transition to the solitary solution. Note that in the $m\rightarrow 1$ limit the wave amplitude scales as [see (133)] $a=[k_0K(m\rightarrow 1)/\pi]^2$. On the other hand, since, in the same limit, $E\rightarrow 1$, Eq. (135) yields $K^2-3K\rightarrow(\pi/2)^2k_0^{-3}[\omega_0(t)-k_0]$ and, therefore, we have the approximate limiting autoresonant relation $a-3(k_0/\pi)a^{1/2}\approx 0.25[\omega_0(t)/k_0-1]$, which has been verified in our calculations. We also compared our fully nonlinear calculations with the predictions of the weakly nonlinear theory [Eqs. (138)]. The weakly nonlinear results are shown by the thick dotted lines in Fig. 1. We discontinue the weakly nonlinear calculations at $t=500$. Nevertheless, one can see in the figure that the weakly nonlinear theory gives reasonable predictions even at the values of the nonlinearity parameter as large as $m=0.8$. Finally, note that only an increasing frequency of the pump wave yields the transition to the desired autoresonant evolution. Indeed, the weakly nonlinear phase mismatch Eq. (138) gives the autoresonant relation $a^2\approx(2/3)k_0D$ beyond the linear resonance point (i.e., for $a\gg\varepsilon$). Therefore, for positive values of k_0, D must be positive beyond the linear resonance point and, consequently, the frequency must increase in time.

B. Purely spatial modulations

In this case $\omega_0=\text{const}$, $k_0=k_0(x)$, while A, B, γ and β are viewed as functions of x only. Then, Eq. (111) yields $\gamma=0$ and Eqs. (107)–(110) reduce to

$$\lambda(A,B,h)=2\pi/k, \tag{142}$$

$$B=(\omega_0/k)\beta, \tag{143}$$

$$\beta=\beta(A,B,h), \tag{144}$$

$$M_x=\varepsilon k\alpha\sin\Phi, \tag{145}$$

where $h=\omega_0/k-1$, $M\equiv B/6+A$ and we have used $W_h=-B/(6\omega_0)$ [see Eq. (114)]. Then, formally, the first three algebraic equations allow to eliminate B and β from the problem and express $k=k(A)$, i.e., obtain an ODE for Φ,

$$\Phi_x=k(A)-k_0(x), \tag{146}$$

while Eq. (145) is viewed as the spatial evolution equation for A. Similar to the time dependent case, these equations again are of the form describing the autoresonance in driven dynamical systems with one degree of freedom.[6] Now, we again use the representation via dependent variables m and $s\equiv 2a/m$ defined above instead of the original variables A and B. Then, Eqs. (142) and (143) yield $s=2[kK(m)/\pi]^2$ and $(k/\omega_0)[h^2-4s^2(m^2-m+1)]=2h-4s(2-m-3E/K)$, respectively. By combining these two expressions, we obtain the following algebraic relation:

$$N(k,\omega_0,m)\equiv(k/\omega_0)[h^2-16(kK/\pi)^4(m^2-m+1)]-2h+8(k/\pi)^2[(2-m)K^2-3EK]=0. \tag{147}$$

Next, we substitute $k=k_0(x)+\Phi_x$ and expand around k_0 to first order in Φ_x, yielding

$$N_{k_0}\Phi_x+N(k_0,\omega_0,m)=0. \tag{148}$$

We can also view (147) as defining $k=k(m)$, and, therefore, Eq. (145) can be written as $(M_kk_m+M_m)m_x=\varepsilon k\alpha\sin\Phi$, where, by differentiation in (147), $k_m=-N_m/N_k$. Thus, to lowest order, Eq. (145) yields

$$(M_{0m}-M_{k_0}N_{0m}/N_{k_0})m_x=\varepsilon k\alpha\sin\Phi, \tag{149}$$

where the subscript 0 means evaluation at ω_0, k_0. The last equation, in combination with (148), comprise a system of first order ODEs for the pair m and Φ. The solution of the system (148), (149) proceeds at the linear resonance point. The boundary values for m and Φ, at this point, are obtained by using the weakly nonlinear theory presented in the Appendix. For purely spatial modulations, this theory yields the following set of the slow amplitude and phase mismatch equations [see Eqs. (A12)],

$$(1-3k_0^2)a_x=3k_0k_{0_x}a-\varepsilon\sin\Phi,$$

$$(1-3k_0^2)\Phi_x=D+9k_0a^2-4.5(k_0a)^2(D+3k_0^3)^{-1}-(\varepsilon/a)\cos\Phi, \tag{150}$$

where, as before, D is the linear dispersion function. Similarly to the time dependent case, the singular term with a^{-1} in the second equation in (150) is responsible for the phase trapping in the initial wave excitation stage. Finally, we check the smoothness of transition from the weakly nonlinear limit at the resonance point to fully nonlinear theory. We take the small amplitude limit of (147) near the linear resonance point

$$N_0\equiv N(k_0,\omega_0,m)=-(k_0\omega_0)^{-1}D(D+2k_0)+(3/16)k_0^2\omega_0^{-1}(\omega_0-5k_0^3)m^2. \tag{151}$$

Then, at $D=0$, $N_0=(3/16)k_0^3\omega_0^{-1}(1-6k_0^2)m^2$, while $N_{k_0}=2\omega_0^{-1}(1-3k_0^2)+O(m^2)$ and, therefore, upon substitution of $m\approx 4ak_0^{-2}$, Eq. (148) reduces to the second equation in (150), if one neglects the interaction term. The latter can be included, at this stage, as a small correction in the phase mismatch Eq. (148) in the vicinity of the linear resonance point. The result is [compare (148) with the second equation in (150)]

$$\Phi_x=-N_0/N_{k_0}-(1-3k_0^3)^{-1}(\varepsilon/a)\cos\Phi. \tag{152}$$

Finally, we verify the smooth transition between Eq. (145) and the first equation in (150). First, we use the definitions of A and B in terms of η_i [see the expressions below Eq. (129)]

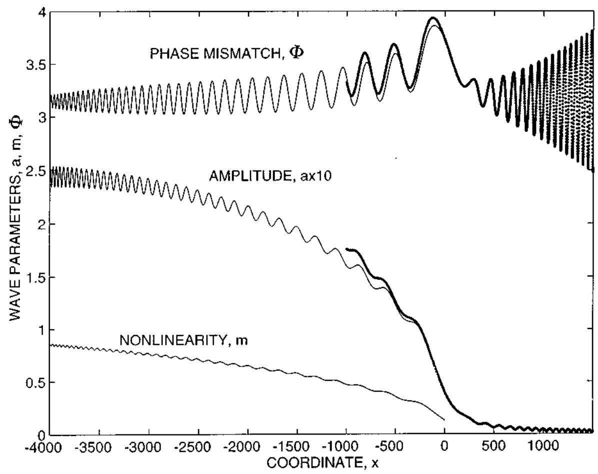

FIG. 2. Spatial autoresonance in the driven KdV system. The wave vector of the pump wave is slowly varying in space, while the frequency is constant. The daughter wave amplitude a, nonlinearity parameter m, and the phase mismatch Φ are shown vs coordinate x. The dotted lines are numerical solutions of the weakly nonlinear system, while the solid lines represent fully nonlinear calculations. The weakly nonlinear solution at the linear resonance point $x=0$ serves as initial data for solving fully nonlinear equations as the nonlinear wave propagates in the negative x-direction.

and calculate the small amplitude limit of the object $M \equiv -\omega_0 W_h + A = B/6 + A$ in the LHS of (145),

$$M \approx (1/72)[h^2 - k^3\omega_0 + h^3/3 - (2/3)k^6] + (m^2k^4/384)(12k^2 - 5). \tag{153}$$

Next, we expand (153) around k_0 (recall $k = k_0 + \Phi_x$) and evaluate it at the linear resonance point, i.e., for $\omega_0 = k_0 + k_0^3$,

$$M \approx -(k_0^4/64)(1-3k_0^2)\Phi_x + (m^2k_0^4/384)(12k_0^2 - 5). \tag{154}$$

Now, we use (148) to find

$$\Phi_x = -N_0/N_{k_0} \approx -(3/32)[(1-6k_0^2)/(1-3k_0^2)]k_0^3 m^2 \tag{155}$$

at the resonance, and substitute this expression into (154), yielding

$$M \approx -(k_0^4/64)(1-3k_0^3)m^2. \tag{156}$$

Finally, we use (156) in (145), substitute $m \approx 4ak_0^{-2}$ and $\alpha \approx (2k_0)^{-1}a$ and, to lowest order, recover the first equation in (150). This completes the proof of the smooth transition between the weakly nonlinear evolution Eq. (150) and the fully nonlinear system (149), (152), applicable beyond the linear resonance point.

Now, we present a numerical example. A typical case is shown in Fig. 2, where the dotted lines are the numerical results obtained by solving the weakly nonlinear system (150) for the case $\omega = \omega_0 = 0.231$, $k = -1.1 - 0.0001x$, $\varepsilon = 0.001$ and initial conditions (at $x = 1500$) $a = 0.0021$ and $\Phi = 0.8\pi$. In this case, the linear resonance point is at $x = 0$, while the linear KdV wave group velocity, $\omega_k = 1 - 3k_0^2$, is negative, so we set the initial conditions on the wave at the positive value of x and solve the equations into the negative x direction. The figure illustrates the phase trapping stage prior the linear resonance point, which is followed by weakly nonlinear autoresonant evolution. We discontinue the weakly nonlinear calculations at $x = -1000$ and use the intermediate results at $x = 0$ as the initial conditions for solving the fully nonlinear system (149), (152) in the interval $-4000 < x < 0$. These results are shown in Fig. 2 by the solid lines. Note that, similarly to the temporal modulations, we obtain a significant increase of the nonlinearity parameter m at large negative values of x. Nevertheless, one does not reach the solitary wave limit $m \to 1$ in this example, because the autoresonant wave approaches the ''turning'' point shortly beyond $x = -4000$, where $m \approx 0.86$, while the coefficient $(M_{0m} - M_{k_0}N_{0m}/N_{k_0})$ multiplying m_x in Eq. (149) vanishes [note that $(M_{0m} - M_{k_0}N_{0m}/N_{k_0})$ is proportional to the wave group velocity in the linear limit], so this equation becomes singular, violating the ordering assumptions of our theory. The existence of a singular point is only one of the complications in the purely x-dependent case, as compared to the time dependent situation discussed earlier. Another difference is the multiplicity of linearly resonant values of the wave vector k_0 for ω_0 in the range $|\omega_0| < 2/(3\sqrt{3}) \approx 0.385$. The case shown in Fig. 2 belongs to this range of multiple solutions. In addition to $k_0 = -1.1$ for the same value of $\omega_0 = 0.231$, at the linear resonance point, there exist two more resonant values $k_0 = 0.246$, and 0.854. This branching of the resonant values of the wave vector at the linear resonance point indicates the possibility of having multiple autoresonant solutions beyond the linear resonance for the same driving frequency. Indeed, calculations, similar to those in Fig. 2, show that the autoresonance can be reached on all these additional branches of solution.

VI. CONCLUSIONS

We have developed the theory of autoresonance for a class of driven two-component nonlinear waves given by Lagrangian (10). The autoresonant solutions in such systems comprise quasiuniform wavetrains of the unperturbed (undriven) problem, which at the same time are globally phase locked with the pump (driving) wave in an extended region of space–time.

The present work is a generalization of the previous theory[1] of the autoresonance of scalar driven nonlinear waves and is also based on the averaged variational principle.[5] We relate the multifield autoresonance problem to that in a two degrees of freedom dynamical problem and develop a perturbation expansion approach to the wave autoresonance based on the integrability of the associated dynamics. The wave problems studied in this work are characterized by unperturbed Lagrangians of the form $L(u_t, u_x, u; v_t, v_x; q)$ involving a potential field variable v. The potential field corresponds to a cyclic variable in the associated two degrees of freedom autonomous dynamical problem, which is integrable. This dynamical integrability is the main feature which allowed the development of our variational approach to the wave autoresonance.

We have shown that, similarly to the single-field autoresonance, the autoresonant excitation of two-component driven waves proceeds at the boundary of the region of interest, where one launches a quasiuniform pump wave towards the region where it resonates with (initially) linear two-component daughter wave. Typically, this region is a three-dimensional surface in four-dimensional space–time (a line in the x,t dependent case studied in detail). At the linear resonance surface the multicomponent daughter wave is excited efficiently and, under certain conditions, the system enters the autoresonant interaction stage. Beyond the resonant surface, in the autoresonance, the pump and the daughter waves are globally phase locked and the driven wave automatically adjusts its parameters (the amplitude, its averaged value etc.) to preserve the nonlinear resonance condition. The autoresonance needs the adiabaticity and sufficient nonlinearity of the system [inequalities (71) and (83)].

We have illustrated our theory by examples of the temporal and spatial autoresonance in the driven KdV system. The predictions of our variational theory in the temporal autoresonance example agreed with previous results based on the conventional perturbation expansion method in application to the driven KdV case.[4] The spatial autoresonance in the KdV system was studied for the first time. It involved complications associated with (a) the possible branching of the autoresonant solutions, and (b) the existence of singular turning points in the problem.

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APPENDIX: WEAKLY NONLINEAR LIMIT OF DRIVEN KDV PROBLEM

In this Appendix we shall derive the slow evolution system for the driven KdV equation in the weakly nonlinear limit. This system is much simpler than its fully nonlinear counterpart and, consequently, the initial nonlinear wave excitation stage involving the trapping into the resonance and the entrance into the autoresonant stage are most easily analyzed in the weakly nonlinear limit. We assume a sufficiently small amplitude solution η of the driven KdV Eq. (8), express this solution in terms of the auxiliary field v (i.e., $\eta \equiv v_x$), and introduce the usual harmonic decomposition [compare to Eq. (14)],

$$v \approx \xi + (1/2)[a_1 \exp(i\theta) + a_2 \exp(i2\theta)] + \text{c.c.}, \qquad \text{(A1)}$$

where $\theta = \psi + \Delta$, $|a_1| \ll 1$ and $|a_2|$ is assumed to scale as $|a_1|^2$. Then

$$\eta = \beta + (1/2)[\alpha_1 \exp(i\psi) + \alpha_2 \exp(2i\psi)] + \text{c.c.}, \qquad \text{(A2)}$$

where $\alpha_1 \equiv (ik_0 a_1 + i\Delta_x a_1 + a_{1x})\exp(i\Delta)$ and $\alpha_2 \equiv (2ik_0 a_2 + 2i\Delta_x a_2 + a_{2x})\exp(2i\Delta)$. Next, we substitute (A2) into (8) and separate different harmonic components in the resulting equation. The zero (averaged) and the first harmonic components yield, to leading order in the wave amplitude,

$$\beta_t + (\beta + 1.5|\alpha_1|^2)_x = 0, \qquad \text{(A3)}$$

$$iD\alpha_1 + D_{k_0}\alpha_{1x} - D_{\omega_0}\alpha_{1t} + 0.5 D_{k_0 x}\alpha_1 - 6ik_0\beta\alpha_1 - 3ik_0\alpha_1^*\alpha_2 = -\epsilon, \qquad \text{(A4)}$$

where, as previously, $D \equiv \omega_0 - k_0 + k_0^3$ and we set the amplitude of the pump wave $b=1$, for simplicity. Similarly, by making the nonresonant approximation for the second harmonic component, i.e., by dropping the x,t derivatives in the slow equation for the second harmonic component we obtain

$$\alpha_2 = 1.5 k_0 \alpha_1^2 / (\omega_0 - k_0 + 4k_0^3). \qquad \text{(A5)}$$

Finally, we use (A5) in (A4), and introduce the real amplitude and the phase mismatch of the wave, i.e., write $\alpha_1 \equiv ia \exp(i\Phi)$, where $\text{Im}(a,\Phi)=0$, separate the real and imaginary parts in Eq. (A4) and arrive at the following system of slow evolution equations for the real variables a, Φ, and β:

$$a_t + (1-3k_0^2)a_x = 3k_0 k_{0x} a - \varepsilon \sin\Phi, \qquad \text{(A6)}$$

$$\Phi_t + (1-3k_0^2)\Phi_x = D - 6k_0\beta - 4.5(k_0 a)^2 (D + 3k_0^3)^{-1} - (\varepsilon/a)\cos\Phi, \qquad \text{(A7)}$$

$$\beta_t + \beta_x + 3a a_x = 0. \qquad \text{(A8)}$$

Now we discuss the evolution predicted by Eqs. (A6)–(A8). We start at the boundary where the amplitude a is negligible and the linear wave is out of resonance with the pump wave (i.e., D is large). Then, before the pump wave reaches the linear resonance line in the x,t plane, where $D=0$, we can neglect all the quadratic objects in the evolution equations (i.e., a^2 and β), yielding

$$da/d\tau \equiv a_t + (1-3k_0^2)a_x = 3k_0 k_{0x} a - \varepsilon \sin\Phi, \qquad \text{(A9)}$$

$$d\Phi/d\tau \equiv \Phi_t + (1-3k_0^2)\Phi_x = D - (\varepsilon/a)\cos\Phi, \qquad \text{(A10)}$$

which is a system of first order ODE's along, generally curved characteristics $dx/d\tau = (1-3k_0^2)$, $dt/d\tau = 1$, in the x, t-plane. A similar set of evolution equations is familiar in the context of one-dimensional mode conversion.[9] Thus, we can use the results of this theory, which predicts the phase trapping $\Phi \to 0 (\text{mod } \pi)$ as the pump wave approaches the linear resonance line $D \to 0$. This phase trapping is a direct consequence of the presence of the singular term with ε/a in the phase equation. In addition to the phase trapping, an efficient excitation of the driven wave takes place in the vicinity of the linear resonance line so that ε/a becomes small as $D \to 0$ and one needs to include the quadratic terms in the theory, i.e., to consider the full system (A6)–(A8) of evolution equations.

The simplest form of these equations is obtained for purely temporal or spatial modulations. Indeed, in the former case, $k_0 = \text{const}$, $\omega_0 = \omega_0(t)$, while a, Φ, and β are viewed as functions of t only. Then, Eq. (A8) yields $\beta \equiv 0$ for the entire time, since β vanishes initially. As the result, our evolution system (A6) and (A7) becomes

$$a_t = -\varepsilon \sin\Phi,$$

$$\Phi_t = D - 4.5(k_0 a)^2 [D + 3k_0^3]^{-1} - (\varepsilon/a)\cos\Phi. \qquad \text{(A11)}$$

We use this system in Sec. V for comparison with the predictions of the fully nonlinear variational theory. Similarly, for purely x-dependent modulations, $\omega_0=\text{const}$, $k_0=k_0(x)$, while a, Φ, and β are viewed as functions of x only. Then, Eq. (A8) yields

$$\beta=-1.5a^2,$$

which allows us to write (A6) and (A7) in this case as

$$(1-3k_0^2)a_x=3k_0k_{0x}a-\varepsilon\ \sin\Phi,$$

$$(1-3k_0^2)\Phi_x=D+9k_0a^2-4.5(k_0a)^2(D+3k_0^3)^{-1} -(\varepsilon/a)\cos\Phi. \tag{A12}$$

Numerical solutions of this system are presented in Sec. V. Furthermore, in the same section, these solutions are used as boundary conditions at the linear resonance point, for solving the fully nonlinear system of slow evolution equations.

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