Autoresonant excitation and evolution of nonlinear waves: The variational approach

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It is shown that a large class of multidimensional nonlinear waves can be excited and controlled in adiabatically varying systems driven by an externally launched pump wave. The excitation proceeds via the trapping into the resonance, while later the nonlinear wave evolves by being phase locked with the pump wave in an extended region of space and/or time despite the variation of system's parameters. This automatic phase locking (autoresonance) yields a possibility of shaping the parameters of the nonlinear wave by varying the nonuniformity and/or time dependence of the parameters of the system. The multidimensional theory of the autoresonance for driven nonlinear waves is developed on the basis of the averaged variational principle and is illustrated by an example of a driven sine-Gordon equation. [S1063-651X(97)04202-5]

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I. INTRODUCTION

Autoresonance is best known as a persisting phase locking between resonantly driven nonlinear oscillators and driving oscillations when the parameters of the system vary adiabatically. The essence of the phenomenon in this case is that, under certain conditions, if started in resonance, the nonlinear oscillator *automatically* adjusts the amplitude of oscillations (and therefore also its frequency) and continuously remains in an approximate resonance with the driver despite the variation of system's parameters. In early studies, the autoresonance idea was used in relativistic particle accelerators [1–4]. More recently, many other applications of the autoresonance can be found in the literature. Those include controlled excitation of atoms [5] and dissociation of molecules [6], applications in nonlinear dynamics [7–9] and unconventional particle accelerators [10,11], autoresonant excitation of intense plasma waves [12] and solitons [13], and, finally, autoresonant mode conversion [14] and three-wave interactions [15]. Despite the variety, all these studies were essentially reduced to the aforementioned one-dimensional driven nonlinear oscillator problem. In autoresonant wave interaction problems [12–15] such a reduction was possible by assuming either only time or one spatial dependence of the parameters of the system. Nevertheless, recently, it was shown that the one-dimensionality assumption is not always necessary and, in some cases, the autoresonance is characteristic of multidimensional wave interactions [16]. This work demonstrated that if an adiabatically varying medium can support a number of waves and one of the waves is excited externally (the pump wave) and launched towards the region where it can resonate with another wave (the daughter wave), then, under certain conditions, the daughter wave is excited in the medium and remains in autoresonance with the pump wave despite the multidimensionality of the problem. However, Ref. [16] was limited to the case of *weakly* nonlinear waves.

In the present work we shall develop a theory of the autoresonant excitation and evolution of a class of multidimensional nonlinear waves $u(\vec{r},t)$ described by the variational principle. An example is the driven Klein-Gordon equation

$$u_{tt}-c^2u_{xx}-f(u,q)=\varepsilon b\ \cos\psi, \tag{1}$$

where the right-hand side (RHS) represents an *externally* launched *linear* eikonal pump wave, and $\varepsilon \ll 1$ is a small dimensionless coupling parameter. We assume that the amplitude $b(\vec{r},t)$, wave vector $\vec{k}(\vec{r},t)=D\psi$, and frequency $\omega(\vec{r},t)=-\psi_t$ of the pump wave, as well as the velocity c and the parameter q on the left-hand side (LHS) of Eq. (1), all are *given slow* functions of space-time. (We shall also assume the existence of an additional small dimensionless adiabaticity parameter $\mu \ll 1$ in our problem characterizing this slow variation.) Our goal is to investigate the possibility of the *autoresonance* in the system described by Eq. (1), i.e., the regime when the nonlinear wave represented by the LHS of Eq. (1) is phase locked with the pump wave. We shall not limit ourselves to Eq. (1) only, but consider the autoresonance in more general nonlinear wave evolution problems described by the variational principle $\delta_u(\int\int\mathcal{L}\ d\vec{r}\ dt)=0$, where the integration is over a *finite* region of space-time and

$$\mathcal{L}=L(u_t,u_x,u_y,u_z,u,q)+\varepsilon bu\ \cos\psi. \tag{2}$$

This Lagrangian yields the variational evolution equation

$$\partial_t(L_{u_t})+\partial_x(L_{u_x})+\partial_y(L_{u_y})+\partial_z(L_{u_z})-L_u=\varepsilon b\ \cos\psi. \tag{3}$$

[The Klein-Gordon equation case (1) corresponds to $L=\frac{1}{2}(u_t^2-c^2u_x^2)+\int f(u,q)du$.]

The purpose of this work is to construct the *averaged variational principle* for studying autoresonance effects associated with Eq. (3). The idea of the averaged variational principle was developed by Whitham [17] in the theory of modulations of nonlinear waves. The slowness of modulations is an essential assumption of Whitham's averaging method. The possibility of applying a similar approach to autoresonant wave interactions is based on the conjecture of the existence and slowness of the evolution of *autoresonant* nonlinear waves. Finding the conditions for entering and sustaining such solutions of Eq. (3) comprises the main goal of the present work. The variational approach guarantees the generality of our theory and allows the use of many ingredients of Whitham's theory of adiabatic modulations.

Our presentation will be as follows. In Sec. II, as a preliminary step, we shall construct the averaged variational principle for a driven adiabatic oscillator. We shall present the main steps of the averaging procedure in this one-dimensional application and compare the results of the averaged variational approach with those based on the Hamiltonian formalism and used previously in studying the autoresonance in nonlinear dynamics. In Sec. III we shall discuss the trapping into the resonance and the autoresonance conditions in the driven nonlinear oscillator case and illustrate the theory in the case of an adiabatically driven nonlinear pendulum. In Sec. IV we shall generalize the theory for applications to multidimensional nonlinear waves. In the same section we shall also present a numerical example of a driven sine-Gordon equation. Section V gives our conclusions.

II. AVERAGED VARIATIONAL PRINCIPLE FOR RESONANTLY DRIVEN DYNAMICAL SYSTEMS

In this section we shall consider the dynamics of adiabatic driven nonlinear oscillations $u(t)$ characterized by the Lagrangian of form $\mathcal{L}=L[u_t(t),u(t),q(t)]+\varepsilon u(t)b(t)\cos[\psi+\psi_0]$, where $\psi=\int_0^t\omega(t)dt$ and $\psi_0(t)$ is a given *slow* phase modulation of the driver. Thus we study solutions of the one-dimensional evolution equation [see Eq. (3)]

$$d_t(\mathcal{L}_{u_t})-\mathcal{L}_u=\varepsilon b\cos(\psi+\psi_0). \tag{4}$$

We have defined two small parameters in our problem, i.e., the adiabaticity parameter μ and the coupling constant ε. Although, in principle, ε and μ are independent, we shall see in Sec. III that in the autoresonance ε must be the *larger* of the two parameters. Consequently, the following theory is $O(\varepsilon)$ perturbation analysis. It is known that the autoresonance phenomenon in the one-dimensional problem described by Eq. (4) can be dealt with by using the conventional Hamiltonian approach (see, for example, Ref. [5]). Nevertheless, this method is not suitable for applications to multidimensional driven nonlinear wave systems. Therefore, in this section we shall develop an alternative approach based on the averaged variational principle and apply it first to driven dynamical systems and later (Secs. IV and V) to autoresonant wave evolution problems.

We proceed by introducing a two-scale representation of the solution, i.e., write $u(t)=U(\theta,T)$. Here $T\equiv\varepsilon t$ is a slow variable representing the effects of the presence of the driver and of the adiabatic variation of q, b, ψ_0, and ω, which all are assumed to be functions of T. On the other hand, the *angle variable* θ represents the phase of the nonlinear oscillations and we shall assume that it can be written as $\theta=\psi(t)+\Delta(T)$. In other words, on the fast scale, the phase is locked on that of the driver, but we also add a slow modulation Δ in θ. Consequently, the frequency of the nonlinear oscillations $\Omega=d\theta/dt=\omega(T)+\varepsilon\Delta_T$, to leading order, is the same as the frequency of the driver. Conditions for the validity of the phase-locking assumption will be outlined in Sec. III.

By differentiation,

$$u_t=\omega U_\theta+\varepsilon U_T',\tag{5}$$

where $(\cdots)_T'=(\cdots)_T+(\cdots)_\theta\Delta_T$, which after the substitution into Eq. (4) yields

$$\omega L_{1\theta}-L_0+\varepsilon L_{1T}'=\varepsilon b\cos(\psi+\psi_0).\tag{6}$$

Here $L_0\equiv L_u$, $L_1\equiv L_{u_t}$, and the arguments in $L_{0,1}$ are $(\omega U_\theta+\varepsilon U_T',U,q)$. At this stage, following Whitham [17] we start treating the variables θ and T in Eq. (6) as *independent*. Then, since, originally, $u(t)=U(\theta,T)$ is a function of a single variable, we have the freedom of adding an additional equation to our problem. One must choose this equation so that the solution $u(t)=U(\theta,T)$ has the desired overall time dependence. The convenient method for achieving this goal is by assuming 2π *periodicity* of U with respect to θ for fixed T. The reasoning behind this assumption will be given below, when we shall identify θ as the canonical angle variable of the corresponding unperturbed oscillator problem.

In order to exploit the periodicity assumption, we multiply Eq. (6) by U_θ and rewrite its LHS in the form of a conservation law

$$(\omega U_\theta L_1-L)_\theta+\varepsilon(U_\theta L_1)_T'=\varepsilon bU_\theta\cos(\theta-\Delta+\psi_0),\tag{7}$$

where (see above) $\Delta(T)=\theta-\psi$ is the slow phase mismatch. Now we average Eq. (7) with respect to θ over the period of 2π, holding T fixed. This yields

$$dI_1/dT=b\langle U_\theta(\theta,T)\cos[\theta-\Delta+\psi_0]\rangle,\tag{8}$$

where $I_1(T)=\langle U_\theta L_1\rangle$ and $\langle\cdots\rangle=(1/2\pi)\int_0^{2\pi}(\cdots)d\theta$. The periodicity of U also allows us to expand in the Fourier series $U(\theta,T)=\Sigma_n a_n(T)\exp(in\theta)$, $a_{-n}=a_n^*$, and complete the averaging on the RHS of Eq. (8),

$$dI_1/dT=-ab\sin(\Delta+\theta_0-\psi_0),\tag{9}$$

where $a(T)=|a_1|$ and $\theta_0(T)=\arg(a_1)$. Equations (7) and (9) comprise a complete set for the two unknown functions $U(\theta,T)$ and $\Delta(T)$ in our problem.

Now we further exploit the idea of using variables θ and T independently and observe that Eq. (6) can be also obtained from the variational principle

$$\delta_U\int\frac{1}{2\pi}\int_0^{2\pi}[L(\omega U_\theta+\varepsilon U_T',U,q)+\varepsilon bU\cos(\theta-\Delta+\psi_0)]d\theta\,dT=0\tag{10}$$

or

$$\delta_U\int\langle\mathcal{L}\rangle dT=0,\tag{11}$$

where

$$\langle\mathcal{L}\rangle=\langle L\rangle+\varepsilon ab\cos(\Delta+\theta_0-\psi_0).\tag{12}$$

Equation (11) is the exact form of Whitham's [17] averaged variational principle for applications to resonantly driven phase-locked oscillations. It reduces the problem to variational equations associated with the averaged Lagrangian $\langle\mathcal{L}\rangle$ depending on slow variables only. The variational principle

(10) is exact and involves variation of the functional within a class of functions $U(\theta,T)$ periodic with respect to θ. Further progress in the theory can be made by restricting this class of functions via the following perturbation analysis.

As a first step, we observe that our problem is simplified significantly if one neglects the *small* term $\varepsilon U_T'$ in the first argument of L in Eq. (10), i.e., considers the problem given by the *approximate* variational principle

$$\delta_U \int \frac{1}{2\pi}\int_0^{2\pi}[L(\omega U_\theta,U,q) + \varepsilon b U \cos(\theta-\Delta+\psi_0)]d\theta\, dT=0. \tag{13}$$

The variational evolution equation for this problem is

$$\omega L^0_{1\theta} - L^0_U = \varepsilon b \cos(\theta-\Delta+\psi_0), \tag{14}$$

where $L^0 \equiv L(\omega U_\theta,U,q)$. Later, solutions of Eq. (14) will serve as a zeroth-order approximation in the exact variational principle (10). Note that this is not a usual perturbation scheme in terms of ε since we have left the driving term in Eq. (13). If the oscillator is not excited initially, this term is important and dominates during the initial evolution stage (see Sec. III).

In contrast to (6), Eq. (14) is a second-order *ordinary* differential equation for $U(\theta,T)$ as a function of θ, while T plays the role of a *fixed* parameter and enters via q, b, ω, ψ_0, and Δ evaluated at their values at time T in our exact problem. A convenient way of finding solutions of Eq. (14) having the desired periodicity properties is by using the Hamiltonian formalism. We introduce the notation $\dot{U} \equiv \partial U/\partial t = \omega U_\theta$, define the usual generalized momentum $P \equiv L^0_1(\dot{U},U,q)$, use this definition to express $\dot{U}=F(P,U,q)$, and construct the Hamiltonian

$$\begin{aligned} H(P,U,T,t) &= \omega U_\theta L^0_1 - \mathcal{L}^0 \\ &= FP - L^0(F,P,q) - \varepsilon b U \cos[\theta(t)-\Delta+\psi_0]. \end{aligned} \tag{15}$$

Finally, we transform from P,U in Eq. (15) to the canonical action-angle variables I,θ of the unperturbed oscillator. Note that we identify θ with the cyclic variable in our variational principle. Thus we write $U=U(\theta,I,q)$, $P=P(\theta,I,q)$, where [18] $I(A,q)=(2\pi)^{-1}\oint P^* dU$ and $P^*=P^*(U,A,q)$ is the solution of

$$FP - L(F,U,q) = A. \tag{16}$$

Obviously, A is the energy of the unperturbed oscillator and, using the definition of I, we can also write $A=A(I,q)$. In terms of the canonical variables, the Hamiltonian (15) becomes

$$H(\theta,I,q,T) = A(I,q) - \varepsilon b U(\theta,I,q)\cos[\theta-\Delta+\psi_0]. \tag{17}$$

The next step is to expand $U(\theta,I,q)=\Sigma_n a_n(I,q)\exp(in\theta)$ and leave only the resonant term $n=1$ in the interaction part of the Hamiltonian. Thus we consider the dynamics given by

$$\widetilde{H} \approx A(I,q) - \varepsilon a b \cos(\Delta+\theta_0-\psi_0), \tag{18}$$

where $a(I,q)=|a_1|$ and $\theta_0(I,q)=\arg(a_1)$. This step is the usual single resonance approximation of the theory of the nonlinear resonance [19], which assumes that the neglected rapidly oscillating terms in the Hamiltonian have a negligible effect on the dynamics. The Hamiltonian (18) yields the evolution equations

$$\begin{aligned} \dot{I} &= -\partial_\theta \widetilde{H} = -\epsilon a b \sin(\Delta+\theta_0\psi_0), \\ \dot{\theta} &= \omega = \partial_I \widetilde{H} = \Omega_0(I,q) - \varepsilon a_I b \cos(\Delta+\theta_0-\psi_0), \end{aligned} \tag{19}$$

where $\Omega_0 = \partial_I A(I,q)$ is the frequency of the unperturbed oscillator. One can see that, within the zeroth-order approximation, $I=I(T)$ and, consequently, the zeroth-order solution $U=U(\theta,I,q)$ has the desired overall time dependence, i.e., U is periodic with respect to the fast variable θ and has a parametric dependence on the slow time via ω, I, and q.

At this stage, we return to our exact variational principle (10), where we choose U to be in the vicinity of the zeroth-order approximation, i.e.,

$$U(\theta,T) = U^0[\theta,I(T),q(T)] + \varepsilon U^1(\theta,T) + O(\varepsilon^2), \tag{20}$$

U^0 representing the zeroth-order solution discussed above. Now one can show that, to $O(\varepsilon)$, the averaged Lagrangian $\langle\mathcal{L}\rangle$ in our original problem depends on U^0 only. Indeed, to $O(\varepsilon)$,

$$\begin{aligned} \mathcal{L} &= L(\omega U^0_\theta + \varepsilon U_T'^0 + \varepsilon \omega U^1_\theta, U^0+\varepsilon U^1, q) \\ &\quad + \varepsilon b U^0 \cos(\theta-\Delta+\psi_0) \end{aligned} \tag{21}$$

or, by expansion,

$$\mathcal{L} = L^0 + \varepsilon[(U_T'^0 + \omega U^1_\theta) L^0_1 + U^1 L^0_0 + b U^0 \cos(\theta-\Delta+\psi_0)]. \tag{22}$$

Then, on averaging over θ, we find

$$\begin{aligned} \langle\mathcal{L}\rangle &= \omega I - A + \varepsilon[\langle U_T'^0 L^0_1\rangle + \langle \omega U^1_\theta L^0_1 + U^1 L^0_0\rangle \\ &\quad + a^0 b \cos(\Delta+\theta^0_0-\psi_0)], \end{aligned} \tag{23}$$

where we used

$$\langle L^0\rangle = \omega I - A, \tag{24}$$

obtained by averaging in Eq. (16). Next we calculate

$$\begin{aligned} \langle \omega U^1_\theta L^0_1 + U^1 L^0_0\rangle &= \omega(2\pi)^{-1}\int_0^{2\pi} U^1_\theta L^0_1 d\theta + \langle U^1 L^0_0\rangle \\ &= \langle U^1(-\omega L^0_{1\theta} + L^0_0)\rangle. \end{aligned} \tag{25}$$

Thus, because of Eq. (14), $\langle \omega U^1_\theta L^0_1 + U^1 L^0_0\rangle \sim O(\varepsilon)$ and indeed the effect of U^1 in $\langle\mathcal{L}\rangle$ is of $O(\varepsilon^2)$. Finally, instead of the action $I=I(A,q)$, we now view the energy A as the dependent variable and write $\langle U_T'^0 L^0_1\rangle = \Delta_T I + A_T\langle U^0_A L^0_1\rangle + \alpha$, where $\alpha = q_T\langle U^0_q L^0_1\rangle$. Then, to $O(\varepsilon)$, and by omitting (for simplicity) the superscripts denoting the zeroth-order solution, we have

$$\begin{aligned} \langle\mathcal{L}\rangle &= (\omega+\varepsilon\Delta_T)I - A + \varepsilon[A_T\langle U_A L_1\rangle + \alpha \\ &\quad + ab \cos(\Delta+\theta_0-\psi_0)]. \end{aligned} \tag{26}$$

Expression (26) is the main result at this stage of the theory. We see that, to $O(\varepsilon)$, the averaged Lagrangian is a function of $A(T)$, $\Delta(T)$, and other slow (but known) parameters of the problem, i.e., $\langle\mathcal{L}\rangle = L[A(T),\Delta(T),T]$. Thus the averaged variational principle (11) becomes

$$\delta\int L[A(T),\Delta(T),T]dT=0. \tag{27}$$

This equation must be supplemented by Eq. (9), where, to the desired order, one can replace I_1 by I, yielding

$$I_A A_T = -I_q q_T - ab\ \sin(\Delta+\theta_0-\psi_0). \tag{28}$$

One can view this equation as defining the slow phase mismatch $\Delta(T)$ and therefore $A(T)$ remains the only free function in the functional in Eq. (27). By taking the variation in Eq. (27) with respect to A, defining $\Phi\equiv\Delta+\theta_0-\psi_0$, and using the identity

$$\alpha_A + A_T\langle U_A L_1\rangle_A - \langle U_A L_1\rangle_T = q_T\{U,L_1\}, \tag{29}$$

where the averaged Poisson brackets are $\{U,L_1\}\equiv\langle U_q L_{1A} - U_A L_{1q}\rangle$, we obtain

$$(\omega+\varepsilon\Delta_T)I_A - 1 + \varepsilon[q_T\{U,L_1\} + ba_A\cos\Phi - ba\,\theta_{0A}\sin\Phi] = 0. \tag{30}$$

Now Eq. (30) can be interpreted as the evolution equation for the slow phase mismatch $\Delta(T)$, while Eq. (28) becomes the evolution equation for A. Remarkably, one can also obtain Eq. (28) by using the same variational principle (27), but taking the variation with respect to Δ. Thus the evolution equations (28) and (30) are unified by a *single* variational principle. Finally, we return to the original time variable $t=T/\varepsilon$ and rewrite Eqs. (28) and (30) as

$$I_A A_t = \beta - \varepsilon ab\ \sin\Phi, \tag{31}$$

$$I_A \Phi_t = 1 - \omega I_A + \gamma - \varepsilon a_A b\ \cos\Phi, \tag{32}$$

where

$$\beta = -q_t I_q,$$

$$\gamma = I_A(q_t\theta_{0q} - \psi_{0t}) - q_t\{U,L_1\}. \tag{33}$$

Equations (31) and (32) comprise a complete set of evolution equations for studying the dynamic autoresonance in the system. These evolution equations can be obtained directly also from the conventional Hamiltonian formalism. In fact, they were used in previous studies of the autoresonance in nonlinear dynamics (see, for example, Refs. [5–9]). Nevertheless, here we have derived the evolution equations via the averaged variational principle and all the steps in this derivation are generalizable to driven multidimensional wave problems, the study of which is the ultimate goal of the present work. We shall postpone this generalization until Sec. IV and devote the next section to the summary of our present understanding of the trapping into resonance followed by the autoresonant evolution in driven dynamical systems. Such a summary is necessary not only for completeness, but also because many results of this theory can be used in studying the autoresonant excitation and evolution of nonlinear waves (see Sec. IV).

III. PHASE LOCKING AND DYNAMIC AUTORESONANCE

The phase-locking phenomenon in the dynamical system considered in the preceding section corresponds to the situation when the phase Φ [see Eqs. (31) and (32)] varies slowly and remains bounded despite the time variation of the parameters (q,ω,b). One possibility for such a phase locking corresponds to the case when one can neglect the interaction term and the small factor γ on the RHS of Eq. (32) and, at the same time, the difference $1-\omega I_A = I_A(\Omega_0-\omega)$ is small during the interaction, i.e., the system automatically adjusts its nonlinear frequency to remain in the approximate resonance continuously. We shall refer to this situation as to the *dynamic autoresonance* [5] (DAR) in the following. In addition to the DAR, there also exists another important situation, when the difference $1-\omega I_A$ in Eq. (32) is of $O(1)$, but, despite the smallness of ε, the interaction term on the RHS of Eq. 10 (32) is sufficiently large to nearly cancel $1-\omega I_A$. One finds [14] that this is a generic situation if, initially, the system starts out of resonance, i.e., $\Omega_0-\omega\sim O(\Omega_0)$, but the oscillator is not excited significantly ($u\ll 1$). The initial interaction stage, in this case, can be treated within a *linear theory* and the function a on the RHS of Eq. (32) is the amplitude of these linear oscillations. Then a scales as $A^{1/2}$ and therefore $a_A\sim A^{-1/2}$ is large during the initial excitation stage. It was found in Ref. [14] that this large factor in Eq. (32) leads to the phase locking in the system followed by the automatic cancellation of the term $1-\omega I_A$ on the RHS in Eq. (32). The cancellation continues until, due the variation of the driving frequency, the system approaches the resonance and $1-\omega I_A = I_A(\Omega_0-\omega)$ becomes small at some time moment t_0. Beyond t_0, the energy A of the oscillator is sufficiently large, so the interaction term in Eq. (32) becomes unimportant and the system enters the DAR stage. Another important result [14] is that, if the above-mentioned trapping into resonance starts sufficiently far from the resonance, it leads to a *strong* phase locking in the initial DAR stage. In other words, at the beginning of the DAR, Φ *oscillates* around 0 or $\pi(\mathrm{mod}\,2\pi)$, depending on whether $a_A^{-1}(1-\omega I_A)\rightarrow +0$ or -0 as one approaches the linear resonance, while the amplitude $\Delta\Phi$ of these oscillations is relatively small $\Delta\Phi\ll\pi$. This strong phase-locking effect is described in Ref. [14] so we shall not present its details here and proceed directly to the DAR.

Assume that, at $t=t_0$ (the initial stage of the DAR), the system is strongly trapped in the resonance in the vicinity of, say, $\Phi(\mathrm{mod}\,2\pi)\approx\pi$, while $1-\omega I_A\ll 1$ ($\Omega_0\approx\omega$). Then, under certain conditions, one finds that, for $t>t_0$, the system of (31) and (32) evolves so that A and Φ perform small oscillations around slowly varying averages such that the difference $1-\omega I_A$ remains small continuously, i.e., the system remains in the DAR regime. Indeed, we seek solutions of Eqs. (31) and (32) in the form

$$A(t)=\overline{A(t)}+\delta A,$$
$$\Phi(t)=\pi+\overline{\Phi}(t)+\delta\Phi, \tag{34}$$

where δA and ψ are the assumed small $(|\delta A/A|,|\delta\Phi|/\pi\ll 1)$ oscillating parts of the solutions, while A and $\pi+\Phi$ represent the slowly varying averages. We also assume that $\Phi\ll\pi$. Then we can linearize Eqs. (31) and (32) and write the following systems of equations for the averaged and oscillating components:

$$\overline{I_A A_t}=\overline{\beta}+\varepsilon\overline{ab}\,\overline{\Phi},$$
$$\overline{I_A\Phi_t}=1-\omega\overline{I_A}+\overline{\gamma}+\varepsilon\overline{a_A}b \tag{35}$$

and

$$\overline{I_A}(\delta A)_t=\varepsilon\overline{ab}\,\delta\Phi,$$
$$\overline{I_A}(\delta\Phi)_t=-\omega\overline{I_{AA}}\delta A, \tag{36}$$

where $\overline{I_A},\overline{I_{AA}},\overline{a,a_A},\overline{\beta},\overline{\gamma}$ all are evaluated at $\overline{A}$.

Equations (36) are Hamilton's equations associated with the Hamiltonian

$$H(\delta A,\delta\Phi,t)=-(2\overline{I_A})^{-1}[\varepsilon\overline{ab}(\delta\Phi)^2+\omega\overline{I_{AA}}(\delta A)^2]. \tag{37}$$

Define

$$\nu^2=\varepsilon\omega\overline{ab}\,\overline{I_{AA}}(\overline{I_A})^{-2}=\varepsilon\sigma p\omega\overline{\Omega}_0, \tag{38}$$

where $p=\overline{ab}/\overline{A}$ and $\sigma=\overline{A}\ \overline{I}_{AA}/\overline{I_A}=-\overline{A}\ \overline{\Omega}_{0A}/\overline{\Omega}_0$. If $\nu^2>0$ (i.e., $I_{AA}>0$), Eqs. (36) describe stable *adiabatic* oscillations with slowly varying angular frequency ν, provided all time-dependent parameters of the problem, say ω, satisfy the adiabaticity condition

$$|\omega_t|/\omega\ll\nu. \tag{39}$$

Note that the dimensionless parameter σ in Eq. (38) measures the degree of the nonlinearity and vanishes in the linear case, in which Ω_0 is independent of A. Therefore, the inequality (39) requires a sufficient nonlinearity. In addition to the adiabaticity condition (39), we must also add the following two conditions imposed by the assumed smallness of $\delta A/A$. The Hamiltonian (37) shows that the amplitudes of the oscillations of A and Φ are related, i.e., $\Delta A\approx|\varepsilon\overline{ab}/\omega I_{AA}|^{1/2}\Delta\Phi$. Thus, since in the autoresonance $\omega\approx\Omega_0=1/I_A$, we obtain the condition

$$\Delta A/\overline{A}\sim|\varepsilon p/\sigma|^{1/2}\Delta\Phi\ll 1. \tag{40}$$

Furthermore, the expansion of I_A in powers of δA on the RHS in Eq. (32) assumed $|\Delta\overline{I_A}/\overline{I_A}|\approx\Delta A|\overline{I_{AA}}/\overline{I_A}|\ll 1$ or

$$|\varepsilon p\sigma|^{1/2}\Delta\Phi\ll 1. \tag{41}$$

Satisfaction of this condition also justifies the single resonance approximation used earlier in Sec. II. The two inequalities (40) and (41) can be rewritten as

$$|\varepsilon p(\Delta\Phi)^2|\ll|\sigma|\ll|\varepsilon p(\Delta\Phi)^2|^{-1}, \tag{42}$$

which, in the case $\Delta\Phi\sim O(1)$, can be identified as the *moderate nonlinearity* conditions of the theory of the nonlinear resonance [19]. The smaller the amplitude $\Delta\Phi$ of the oscillations, the easier it is to satisfy Eq. (42), which in combination with Eq. (39) comprises the set of necessary conditions for the validity of the theory. Finally, as the average quantities characterizing the problem evolve in time (see below), the amplitudes $\Delta\Phi$ and ΔA also change, preserving, at the same time, the corresponding adiabatic invariant

$$\Delta A\Delta\Phi\approx\text{const.} \tag{43}$$

Now, assuming the satisfaction of Eqs. (39) and (42), we return to Eqs. (35) for the averaged quantities. We write $\overline{A}=A_0(t)+d$, where A_0 is the value of $\overline{A}$ for which the RHS of the second of Eqs. (35) vanishes at all times, i.e.,

$$(1-\omega I_A+\gamma+\varepsilon a_A b)_{A=A_0}\equiv 0 \tag{44}$$

and we assume that $|d/A_0|\ll 1$. Then, to lowest order in d, Eqs. (35) become

$$I_{A0}A_{0t}=\beta_0+\varepsilon a_0 b\overline{\Phi},$$
$$I_{A0}\overline{\Phi}_t=-(\omega I_{AA}-\gamma_A-\varepsilon a_{AA}b)_0 d\approx-\omega I_{AA0}d, \tag{45}$$

where the subscript zero indicates the evaluation at A_0. The first of Eqs. (45) yields

$$\overline{\Phi}=(\varepsilon a_0 b)^{-1}(I_{A0}A_{0t}-\beta_0). \tag{46}$$

Now, in orders of magnitude, the differentiation of Eq. (44) with respect to t yields $|A_{0t}/A_0|\sim O(\mu\omega/|\sigma|)$, while $|\beta_{0t}/\beta_0|\sim O(\mu\omega)$ by definition. Therefore, $\Phi\sim\mu/\varepsilon p\sigma$ and the assumed smallness of Φ requires

$$\mu/|\varepsilon p\sigma|\approx\mu(\omega/\nu)^2\ll 1, \tag{47}$$

which will be assumed to be satisfied in the following. If, for simplicity, we set σ, $p\sim 0(1)$ then $\Phi_1\sim O(\omega\mu^2/\varepsilon)$ and the second of Eqs. (45) guarantees the relative smallness of d during the interaction. Note that Eq. (39) can be written also as $\mu\omega/\nu\ll 1$ and therefore the satisfaction of the stronger inequality (47) guarantees our adiabaticity condition. Finally, we observe that the only effects of the small terms β and γ on the dynamics in the DAR regime are additional small shifts of the average values A and Φ, while the oscillating parts δA and $\delta\Phi$ of the solution remain unchanged. Thus, if one neglects these small shifts, one can also omit β and γ in solving the evolution equations.

In conclusion, strong initial trapping and satisfaction of the moderate nonlinearity and adiabaticity conditions, (42) and (47), are the necessary and sufficient conditions for sustaining the DAR in the driven dynamical system, i.e., preserving the resonance $[1-\omega I_{1A}\sim O(\varepsilon)$; see Eq. (44)] between the driven and driving oscillations.

Before generalizing the averaged variational principle to the autoresonance problem for multidimensional waves, we demonstrate the DAR phenomenon in the case of the driven nonlinear pendulum described by

$$u_{tt}+\omega_0^2\sin u=\varepsilon a\cos\left(\int\omega(t)dt\right). \tag{48}$$

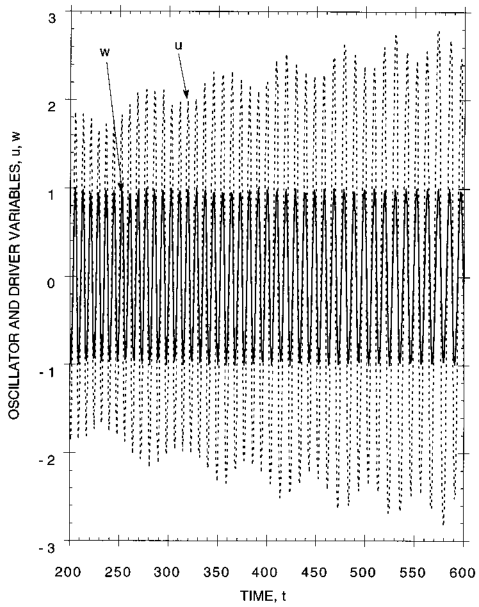

FIG. 1. Variable $u(t)$ of the driven pendulum (dotted line) and the driving oscillation $w=\cos[\psi+\pi]$ (solid line) vs time for $\psi(t)=\alpha^{-1}[1-\exp(-\alpha t)]$ and $\alpha=0.001$. One observes phase locking between the driven and driving oscillations despite the decrease of the driving frequency by a factor of ~ 1.5 for t between 200 and 600.

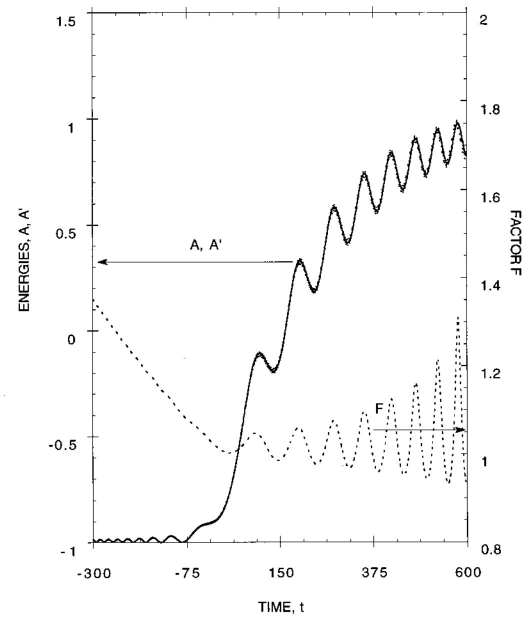

FIG. 2. Evolution of the slow energy A (solid line) and of the factor $F=\omega I_A$ (dashed line) for the a nonlinear pendulum driven by an oscillation with a slowly varying frequency. The dots represents $A'=\frac{1}{2}u_t^2-\omega_0^2\cos u$ found by solving the exact evolution equation for u. Oscillations of ωI_A around unity indicate a persisting autoresonance in the system beyond $t=0$.

This application will allow us to illustrate our theory and make the necessary preparations for treating an example of a driven sine-Gordon equation in Sec. IV.

The action I and the frequency Ω_0 of the unperturbed nonlinear pendulum of energy $A=(1/2)u_t^2-\omega_0^2\cos u$ are [20]

$$I=\left(\frac{8\omega_0}{\pi}\right)[E(\pi/2;\kappa)-(1-\kappa^2)F(\pi/2;\kappa)],$$
$$1/\Omega_0=I_A=2(\pi\omega_0)^{-1}F(\pi/2;\kappa), \qquad (49)$$

where $\kappa=\frac{1}{2}(1+A/\omega_0^2)$ ($\kappa<1$ in the case of interest), while F and E are elliptic integrals of the first and the second kind, respectively. Furthermore, the function a in the evolution equations (31) and (32) is [20]

$$a=4g^{1/2}(1+g)^{-1}, \qquad (50)$$

where $g=\exp[-(\pi F'/F)]$ and $F'=F(\pi/2;1-\kappa)$, while $\theta_0=-\pi/2$ (recall that $\Phi=\Delta+\theta_0$ and Δ is the phase mismatch of the driven and driving oscillations) and $\beta=\gamma=0$.

Now consider the case in which initially, at $t=t_1$, the oscillator is weakly excited ($A\approx-1$), $\omega(t_1)>\omega_0$, and $\omega(t)$ is a slowly *decreasing* function of time. Suppose also that, at some time moment $t=t_0$, the driving frequency passes the linear resonance point, i.e., $\omega(t_0)=\omega_0$. Then, if the variation of $\omega(t)$ is slow enough, the system will be strongly trapped in the resonance [14] in the vicinity of $t=t_0$ (since $1-\omega I_A\rightarrow-0$ as $t\rightarrow t_0$, we have $\Phi\rightarrow\approx\pi$ in this case) and the aforementioned theory predicts the DAR-type evolution of the oscillator for $t>t_0$, provided the autoresonance conditions are satisfied. In the DAR regime ($t>t_0$), $\Omega_0(t)\approx\omega(t)$ [or $1-\omega I_A\ll 1$; see Eq. (44)], i.e.,

$$2\omega(t)(\pi\omega_0)^{-1}F(\pi/2;\kappa)\approx 1. \qquad (51)$$

This autoresonance relation defines $\kappa=\kappa(t)$, which, in turn, yields the evolution of the lowest-order energy A of the system. Since the frequency Ω_0, in our case, is a decreasing function of A, the energy of the system increases with a decrease of ω. The increase of A continues as long as the autoresonance conditions (42) and (47) are satisfied. For instance, the adiabaticity condition requires $d\omega/dt\rightarrow 0$ as the energy of the oscillator increases and approaches the separatrix of the unperturbed oscillations, i.e., $A\rightarrow\omega_0^2$ (or $\kappa\rightarrow 1$). Otherwise, the autoresonant evolution discontinues and the effect of the coupling becomes small due to the growing phase mismatch between the driven and driving oscillations. It should be mentioned that the dynamics of the departure of the system from the autoresonance in the vicinity of the separatrix involves crossing of resonances and may be rather complex. The discussion of these effects can be found in Ref. [7].

Now we proceed to our numerical example. Figure 1 shows (dotted line) the results of the numerical solution of Eq. (48) for $u(t)$ in the case $\omega_0=1$, $\omega(t)=\exp(-\alpha t)$ ($\alpha=0.001$), $\varepsilon=0.03$, and subject to the initial conditions $u=0$ and $u_t=0.1673$ ($A=-0.986$) at $t_1=-300$. We set the phase of the driving oscillation to be $\psi=\alpha^{-1}[1-\exp(-\alpha t)]$. In addition to $u(t)$ in Fig. 1, we show (solid line) the function $w=\cos(\psi+\pi)$ representing the phase shifted (by π) driving oscillation. One can see that, indeed, during the time interval shown in the figure the oscillator and the driver are phase locked despite the decrease of the frequency of the driver by a factor of ~ 1.5. The slow oscillatory modulation of the amplitude, characteristic of the autoresonance, is also seen in the figure. More details on the initial excitation stage of the pendulum followed by the DAR are given in Fig. 2. This figure shows the results of the numerical solution of the system of slow evolution equations (31) and (32) for A (the solid line) for the same set of parameters and initial conditions ($A=-0.986$ and $\Phi=-\psi-\pi/2$ at $t_1=-300$) as in Fig. 1. In the same figure we also show (dots) the evolution of $A'=(1/2)u_t^2-\omega_0^2\cos u$ found by solving the original equation (48). One can see a very good agreement between A and A', demonstrating the satisfaction of all the conditions of the averaging procedure. Finally, Fig. 2 also shows the factor $F=\omega I_A$ (dashed line) on the RHS of the equation (32) for the slow phase. This factor must oscillate around unity if the system is in the DAR regime, which is the case in the figure beyond the linear resonance point $t=0$. Note also that the oscillations of ωI_A (representing the oscillations of I_A) become relatively large as the system approaches the separatrix ($A=1$). The left inequality in Eq. (42) is then violated and, at $t=600$, we stop the calculations based on the slow evolution equations. Shortly beyond this time the autoresonance in the system discontinues and the effect of the coupling with the driver becomes negligible because of the growing phase mismatch.

IV. MULTIDIMENSIONAL AUTORESONANCE IN DRIVEN NONLINEAR WAVE SYSTEMS

In this section we shall generalize the theory for applications to driven nonlinear waves described by Lagrangians of form (2). It is sufficient to consider the time and one spatial dimension case, i.e., $u=u(x,t)$, $L=L[u_t,u_x,u,q(x,t)]$, and the evolution equation

$$\partial_t(L_{u_t})+\partial_x(L_{u_x})-L_u=\varepsilon b(x,t)\cos\psi(x,t), \tag{52}$$

since the case of higher dimensionality can be treated similarly. The slowly varying amplitude $b(x,t)$, frequency $\omega(x,t)=-\partial_t\psi$, and wave vector $k(x,t)=\partial_x\psi$ of the pump wave on the RHS in Eq. (52) are assumed to be known throughout the region of interest. We shall be solving the initial-value problem describing *internal* excitation and subsequent evolution of the daughter wave via the resonant interaction with the pump and assume that u, u_t, and u_x are negligibly small for all values of x of interest at the initial time $t=t_1$. We shall see below that these initial conditions, under certain restrictions, may lead to the autoresonance in the system, i.e., to the adiabatic evolution of the excited nonlinear wave propagating in resonance with the pump wave in an extended region of space-time.

In view of the assumed adiabaticity of the problem, and similarly to the DAR case, we introduce the two-scale representation of the solution, i.e., write $u(x,t)=U(\theta,X,T)$, where $X=\varepsilon x$ and $T=\varepsilon t$ are the slow variables and θ is the fast *angle* variable, which we shall identify later with the canonical angle variable of a certain dynamical system. We shall also assume that the daughter and pump waves are phase locked on the fast space-time scale, i.e., $\theta=\psi(t,x)+\Delta(T,X)$. Then the frequency and wave vector of the nonlinear wave $-\theta_t=\omega-\varepsilon\Delta_T$ and $\theta_x=k+\varepsilon\Delta_x$ vary on the slow scale and, to leading order, are as those of the pump. As for the driven nonlinear oscillator, we assume the 2π *periodicity* of U with respect to θ and rewrite our variational principle in the form [compare with Eqs. (10) and (11)]

$$\delta_U\int\int\langle\mathcal{L}\rangle dX\,dT=0, \tag{53}$$

where

$$\langle\mathcal{L}\rangle=\frac{1}{2\pi}\int_0^{2\pi}[L(-\omega U_\theta+\varepsilon U_T',kU_\theta+\varepsilon U_x',U,q)+\varepsilon bU\cos(\theta-\Delta+\psi_0)]d\theta \tag{54}$$

and $(\cdots)'_{T,X}=(\cdots)_{T,X}+U_\theta\Delta_{T,X}$.

At this stage, to lowest order, we neglect the terms $\varepsilon U'_{T,X}$ in the arguments of L in Eq. (54), i.e., consider the variational evolution equation [compare to Eq. (14)]

$$-\omega L^0_{1\theta}+kL^0_{2\theta}-L^0_0=\varepsilon b\cos(\theta-\Delta+\psi_0), \tag{55}$$

where $L^0\equiv L(-\omega U_\theta,kU_\theta,U,q)$ and $L^0_{0,1,2}\equiv L_{u,u_t,u_x}(-\omega U_\theta,kU_\theta,U,q)$. Equation (55) is an *ordinary* differential equation (with respect to θ) in which T and X enter as fixed parameters. Therefore, we treat this zeroth-order approximation via the Hamiltonian formalism. We introduce $\dot{U}\equiv-\omega U_\theta$, write $L^0=L(\dot{U},-k\dot{U}/\omega,U,q)$, define the generalized momentum

$$P\equiv\partial L^0/\partial\dot{U}=L^0_1-(k/\omega)L^0_2, \tag{56}$$

use Eq. (56) to express $\dot{U}=F(P,U,k/\omega,q)$, and construct the Hamiltonian

$$H(P,U,X,T,\theta)=\dot{U}P-\mathcal{L}^0=FP-L^0-\varepsilon bU\cos[\theta-\Delta+\psi_0]. \tag{57}$$

Note that, in contrast to the nonlinear oscillator [see Eq. (15)], in addition to q we have another slow parameter k/ω in the Hamiltonian. In order to shorten the notation, we shall denote the set $\{q,k/\omega\}$ by a single letter Q. Next, we transform from P and U to the action-angle variables I and θ of the unperturbed problem described by Eq. (57) with $\varepsilon=0$, i.e., write $P=P(\theta,I,Q)$ and $U=U(\theta,I,Q)$, where

$$I(A,Q)=(2\pi)^{-1}\oint P^*dU=I_1-(k/\omega)I_2. \tag{58}$$

Here $P^*=P^*(U,A,Q)$ is the solution of [compare with Eq. (16) in Sec. III]

$$FP-L(F,-kF/\omega,U,q)=A \tag{59}$$

and

$$I_{1,2}(A,Q)=(2\pi)^{-1}\oint L^0_{1,2}(F,-kF/\omega,U,q)dU, \tag{60}$$

where F is evaluated at $P=P^*$. Note that, as in Sec. III, we identify the fast angle variable θ in our problem with the canonical angle variable of the unperturbed case with fixed Q. Finally, Eq. (58) yields $A=A(I,Q)$, so, in terms of the action-angle variables, the Hamiltonian (57) becomes [compare with Eq. (17) for the driven oscillator problem]

$$H(\theta,I,Q,X,T)=A(I,Q)-\varepsilon bU(\theta,I,Q)\cos[\theta-\Delta+\psi_0]. \quad (61)$$

Then, by making the usual single resonance approximation in Eq. (61), we find that it yields the desired form of $U=U(\theta,I,Q)$, which is periodic with respect to θ, but also includes the slow variables X,T via θ_t, θ_x, I, and Q. This completes our zeroth-order solution.

Now we return to the exact variational principle (53), where, in view of the above, we use trial functions of form [compare with Eq. (20)]

$$U(\theta,X,T)=U^0[\theta,I(X,T),Q(X,T)]+\varepsilon U^1(\theta,X,T)+O(\varepsilon^2), \quad (62)$$

with U^0 being the zeroth-order solution. Following the steps of Sec. III, one finds that, to first order in ε, $U^0(\theta,I,Q)$ is the only object necessary for calculating the averaged Lagrangian in the driven nonlinear wave problem. Then, to $O(\varepsilon)$, $L=L(-\omega U^0_\theta+\varepsilon U'^0_T,kU^0_\theta+\varepsilon U'^0_x,U^0,q)\approx L^0+\varepsilon(L^0_1U'^0_T+L^0_2U'^0_X)$ and, by averaging over θ, we have $\langle L\rangle=\langle L^0\rangle+\varepsilon(\Delta_T I^0_1+\Delta_X I^0_2+\langle U^0_T L^0_1\rangle+\langle U^0_X L^0_2\rangle)$. Furthermore, on averaging in Eq. (59), one finds $\langle L^0\rangle=-\omega I^0_1+kI^0_2-A$ and, by choosing $A(X,T)$ as the dependent variable instead of $I(X,T)$, one can write $\langle U^0_{T,X}L^0_{1,2}\rangle=A_{T,X}\langle U^0_A L^0_{1,2}\rangle+\Sigma_Q Q_{T,X}\langle U^0_Q L^0_{1,2}\rangle$. Combining all these results, one obtains the final expression for the averaged Lagrangian to $O(\varepsilon)$ in our problem [compare with Eq. (26)]:

$$\langle\mathcal{L}\rangle=\langle L+\varepsilon bU\cos(\theta-\Delta+\psi_0)\rangle$$
$$=(-\omega+\varepsilon\Delta_T)I_1+(k+\varepsilon\Delta_x)I_2-A+\varepsilon[A_T\langle U_A L_1\rangle$$
$$+A_X\langle U_A L_2\rangle+\alpha+ab\cos(\Delta+\theta_0-\psi_0)]. \quad (63)$$

Here the averages are taken with respect to θ between 0 and 2π and U^0 is used everywhere, but we omit the zero superscripts for simplicity. Also, in Eq. (63), $\alpha=\Sigma_Q[Q_T\langle U_Q L_1\rangle+Q_X\langle U_Q L_2\rangle]$ and, as before, a and θ_0 are the absolute value and the complex phase of the coefficient a_1 in the Fourier expansion $U^0(\theta,I,Q)=\Sigma_n a_n(I,Q)\exp(in\theta)$.

At this stage, we observe that one can write $\langle\mathcal{L}\rangle=L[A(X,T),\Delta(X,T),T,X]$, i.e., $u(x,t)$ in our original variational principle is now represented by the slow functions $A(X,T)$ and $\Delta(X,T)$ in the averaged variational principle (53), which becomes

$$\delta\int\int L[A(X,T),\Delta(X,T),X,T]dX\,dT=0. \quad (64)$$

By using the Lagrangian (63) in Eq. (64) and taking the variation with respect to A and Δ, we arrive at the system of evolution equations [compare with Eqs. (31) and (32)]

$$I_{1A}A_t+I_{2A}A_x=\beta-\varepsilon ab\sin\Phi, \quad (65)$$

$$I_{1A}\Phi_t+I_{2A}\Phi_x=1+\omega I_{1A}-kI_{2A}-\gamma-\varepsilon a_A b\cos\Phi, \quad (66)$$

where $\Phi=\Delta+\theta_0-\psi_0$ and, similarly to Eq. (33),

$$\beta=-\sum_Q(I_{1Q}Q_t+I_{2Q}Q_x),$$

$$\gamma=\sum_Q[I_{1A}(Q_t\theta_{0Q}-\psi_{0t})+I_{2A}(Q_x\theta_{0Q}-\psi_{0x})]$$
$$-\sum_0[Q_t\{U,L_1\}+Q_x\{U,L_2\}], \quad (67)$$

with the averaged Poisson brackets defined via $\{f,g\}\equiv\langle f_Q g_A-f_A g_Q\rangle$.

Equations (65) and (66) comprise a set of partial differential equations, which can be solved along the characteristics, originating on the boundary of the region of interest (i.e., on the x axis at $t=t_1$), and we recall that u, u_t, u_x, and therefore A are assumed to be small on this boundary. We define the characteristics via

$$dt/d\tau=1,\quad dx/d\tau=I_{2A}/I_{1A}, \quad (68)$$

subject to the initial conditions $t(\tau=0)=t_1$, and $x(\tau=0)=x_1$, where τ is the parameter along a characteristic and x_1 is an arbitrary position on the boundary. Then Eqs. (65) and (66) can be rewritten as

$$I_{1A}A_\tau=\beta-\varepsilon ab\sin\Phi, \quad (69)$$

$$I_{1A}\Phi_\tau=1+\omega I_{1A}-kI_{2A}-\gamma-\varepsilon a_A b\cos\Phi. \quad (70)$$

This is a system of ordinary differential equations for A and Φ in the region of the (x,t) plane accessible by the characteristics originating on the boundary (the *accessible region* in the following).

Now we observe that Eqs. (69) and (70) have the same form as the slow evolution equations (31) and (32) in the driven oscillator problem. Consequently, we can apply all the results of the theory of the dynamic autoresonance directly to the driven nonlinear wave problem. This observation leads to the following conclusions.

(a) Since, by assumption, the nonlinear wave is negligible on the boundary, its efficient excitation along a given characteristic takes place only in the vicinity of the point τ_0, where the function $D(\omega,k,q,A)\equiv1+\omega I_{1A}-kI_{2A}$ vanishes. On the other hand, in the absence of the pump and for fixed q; $D(\Omega,K,q,A)=0$ is the dispersion relation for the traveling-wave solution of the unperturbed, fixed parameter problem characterized by frequency Ω and wave vector K [17]. Therefore, the excitation of the daughter wave proceeds in the vicinity of the region in the (x,t) plane, where the wave resonates with the pump, i.e., $\Omega=\omega(x,t)$ and $K=k(x,t)$. Furthermore, assuming that the initial excitation stage is *linear*, the dispersion relation $D(\Omega,K,q,A)=0$ is independent of A and thus the wave excitation takes place in the vicinity of the resonance *line* $D[\omega(x,t),k(x,t),q(x,t)]=0$ in the (x,t) plane.

(b) In the initial excitation stage, as one approaches the resonance line, the wave becomes strongly trapped into the resonance. In other words, the phase mismatch Φ becomes *near* either 0 or $\pi(\mathrm{mod}\,2\pi)$ in the vicinity of the resonance line.

(c) If one moves further along the characteristics from the resonant line, under certain conditions (see below), the daughter wave enters the nonlinear autoresonant interaction stage in which

$$D(\omega,k,p,A)=1+\omega I_{1A}-kI_{2A}\approx 0. \tag{71}$$

Then $D(\omega,k,p,A)$ nearly vanishes in the accessible region beyond the resonance line, i.e., the wave is in an approximate resonance $\Omega\approx\omega(x,t)$ and $K\approx k(x,t)$ inside the accessible region. Note that Eq. (71) can be viewed as an *algebraic* equation for $A=A(x,t)$, while $\Phi\approx 0$ (or π). Thus we have obtained an approximate smooth solution for the daughter wave in the entire autoresonant region.

(d) The solutions for A and Φ described in (c) are only approximations to the autoresonant solutions (the analogs of $\bar{A}$ and $\bar{\Phi}$ in the DAR) and the characteristic oscillations can now be added in the autoresonant region. These small oscillations are found, similarly to the driven oscillator case, by solving the system [compare with Eqs. (36)]

$$I_{1A}(\delta A)_\tau=\varepsilon ab\,\delta\Phi,$$

$$I_{1A}(\delta\Phi)_\tau=(\omega I_{1AA}-kI_{2AA})\delta A \tag{72}$$

along the characteristics. One can see that if [compare with Eq. (38)]

$$\nu^2=\varepsilon ab(kI_{2AA}-\omega I_{1AA})(I_{1A})^{-2} \tag{73}$$

is positive, we have stable oscillations of Φ near $\pi(\text{mod } 2\pi)$; otherwise the oscillations are around $0(\text{mod } 2\pi)$, provided, of course, that the initial phase locking stage led to a proper value of Φ (i.e., 0 or π, respectively).

(e) Finally, the conditions guaranteeing the existence of the stable autoresonant evolution of the driven nonlinear wave, are the same moderate nonlinearity and adiabaticity inequalities (42) and (47), where now the nonlinearity parameter is $\sigma=A|kI_{2AA}-\omega I_{1AA}||\omega I_{1A}|^{-1}$.

We conclude this section by presenting an example of autoresonant excitation and evolution of the solution of the driven sine-Gordon equation

$$u_{tt}-c^2u_{xx}+\omega_0^2\sin u=\varepsilon b\ \cos\psi. \tag{74}$$

The unperturbed Lagrangian in this case is $L=\frac{1}{2}(u_t^2-c^2u_x^2)+\omega_0^2\cos u$. Equation (56) yields $P\equiv[1-(ck/\omega)^2]\dot{U}$ and Eq. (59) becomes

$$\tfrac{1}{2}P^2[1-(ck/\omega)^2]^{-1}-\omega_0^2\cos U=A. \tag{75}$$

This is the same as in the nonlinear pendulum case of Sec. III if one replaces $1-(ck/\omega)^2$ in Eq. (75) by unity. By using Eq. (75), we find $P^{*2}=[1-(ck/\omega)^2](2A+2\omega_0^2\cos U)$ and therefore [see the definition in Eq. (60)]

$$I_1=-\frac{\omega}{(\omega^2-c^2k^2)^{1/2}}\,J(\kappa), \tag{76}$$

where $J(\kappa)$ represents the action of the nonlinear pendulum of energy A [see the first of Eqs. (49)] and, as before, $\kappa=\frac{1}{2}(1+A/\omega_0^2)$. Similarly,

$$I_2=-\frac{c^2k}{(\omega^2-c^2k^2)^{1/2}}\,J(\kappa), \tag{77}$$

while [see the second of Eqs. (49)]

$$I_{1A}=-\frac{\omega}{(\omega^2-c^2k^2)^{1/2}}\,J_A(\kappa)=-\frac{2\omega F(\pi/2;\kappa)}{\pi\omega_0(\omega^2-c^2k^2)^{1/2}} \tag{78}$$

and

$$I_{2A}=-\frac{c^2k}{(\omega^2-c^2k^2)^{1/2}}\,J_A(\kappa)=-\frac{2c^2kF(\pi/2;\kappa)}{\pi\omega_0(\omega^2-c^2k^2)^{1/2}}. \tag{79}$$

Simple calculations show that function a in the evolution equations (69) and (70) for the driven sine-Gordon equation case is given by the same equation (50) as in Sec. III and, again, $\theta_0=-\pi/2$. However, now the factors β and γ do not vanish because of the presence of k/ω in their definitions. Nevertheless, as mentioned in Sec. III, these factors can only slightly shift the average components A and Φ along the characteristics [see Eqs. (45) and (46)], leaving, at the same time, the oscillatory parts δA and $\delta\Phi$ of the solution unchanged. The correction to A is relatively unimportant and, consequently, focusing on the slow evolution of the energy of the daughter wave, we shall neglect β and γ in Eqs. (69) and (70) and rewrite the full system of ordinary differential equations describing the driven Sine-Gordon problem as

$$\begin{aligned}
&dt/d\tau=1,\\
&dx/d\tau=c^2k/\omega,\\
&dA/d\tau=+\varepsilon ab\Omega_0(1-c^2k^2/\omega^2)^{1/2}\sin\Phi,\\
&d\Phi/d\tau=-(1-c^2k^2/\omega^2)^{1/2}[\Omega_0-(\omega^2-c^2k^2)^{1/2}]\\
&\qquad+\varepsilon a_Ab\Omega_0(1-c^2k^2/\omega^2)^{1/2}\cos\Phi,
\end{aligned} \tag{80}$$

where $\Omega_0(A)=1/J_A$ is the frequency of the unperturbed nonlinear pendulum of energy A.

At this point we proceed to our numerical example. We consider the initial-value problem in which the solution u of the driven sine-Gordon equation is a given, sufficiently small function of x at $t=t_1$ and study the evolution of u in the semiplane $(x,t>t_1)$ for the case when $\psi(x,t)=\tilde{k}[x-x^2/2X_0]-\tilde{\omega}[t-t^2/2T_0]$, where $\tilde{\omega},\tilde{k},T_0,X_0$ are constants. The frequency and the wave vector of the pump wave in this case are $\omega=\tilde{\omega}(1-t/T_0)$ and $k=\tilde{k}(1-x/X_0)$, respectively, and we shall use the values $\omega_0/\tilde{\omega}=1$, $c\tilde{k}/\tilde{\omega}=2^{-0.5}$, $T_0=10^3$, and $X_0=10^3$ in the calculations. Figure 3 shows the geometry and boundaries of our example in the (t,x) plane. The dotted lines in the figure are the linear resonance line (a hyperbola) on which

$$(\omega^2-c^2k^2)=\Omega_0^2|_{A\to-1}=\omega_0^2 \tag{81}$$

and the line $\omega=ck$, i.e., $t=T_0(1-c\tilde{k}/\tilde{\omega})+(c\tilde{k}/\tilde{\omega})T_0X_0^{-1}x$. Finally, we use $t_1=-400$ and show the characteristics (the solid lines in Fig. 3) starting at ten different values of x between -400 and 500. Since the autoresonance relation (71) in our case is

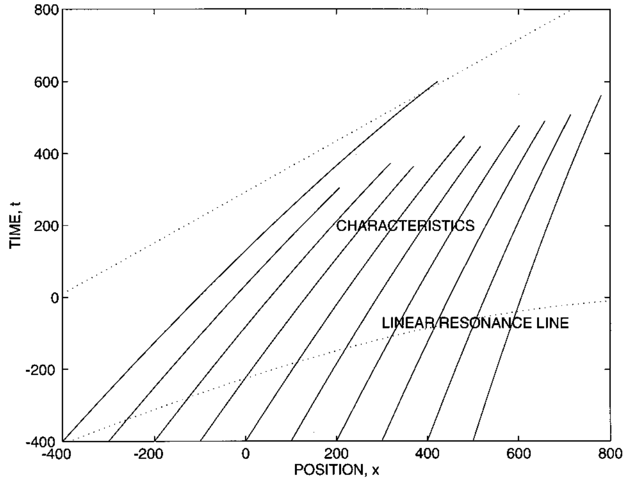

FIG. 3. Geometry and boundaries in the numerical example of the driven sine-Gordon equation. The linear resonance and $\tilde{\omega}=c\tilde{k}$ lines represented by dots. The solid lines show the characteristics in the (x,t) plane originating at $t=-400$ and ten different points on the x axis.

$$(\omega^2-c^2k^2)=\Omega_0^2(\overline{A}), \tag{82}$$

we expect the autoresonance to proceed via the trapping into the resonance near the resonant hyperbola (81) and continue beyond this line, but not past the $\tilde{\omega}=c\tilde{k}$ line in Fig. 3, where $A\rightarrow 1$ ($\Omega_0\rightarrow 0$) and the autoresonance conditions are violated. The departure from the autoresonance in our example took place at the end points of the characteristics shown in Fig. 3. The results of the solution of Eqs. (80) for the wave energy A are presented in Fig. 4. The figure shows the dependence of A along the aforementioned ten characteristics. We use $A=-0.986$ and $\Phi=-\psi-\pi/2$ at the initial integration points and the values $b=1$ and $\varepsilon=0.03$. The lines in the (t,x) plane in Fig. 4 are the characteristics themselves. One can see that, as expected, the efficient autoresonant excitation of the nonlinear wave proceeds in the vicinity of the linear resonance [the line passing trough the circles in the (t,x) plane in Fig. 4]. As the wave moves further into the autoresonant interaction region, the energy A performs characteristic autoresonant oscillations around a monotonically increasing average value A. We see that the overall dependence of A along the characteristics starting at $x\geqslant -300$ is similar to that in the DAR case (see Fig. 2). The automatic increase of A, similar to the DAR, guarantees the satisfaction of the nonlinear resonance condition (82). When A approaches unity, one expects to depart from the autoresonance. This departure is manifested via the phase detrapping, which, in our example, takes place at the end points of the characteristics in Figs. 3 and 4. We discontinue the calculations at these points. Note also that the characteristic starting at $t_1=-400$ and $x=-400$ in Fig. 4 does not cross the resonance line. Therefore, no trapping into the resonance and subsequent autoresonant interaction takes place along this characteristic, as can be seen in the figure. We conclude our illustration of autoresonant solutions of the sine-Gordon equation by presenting another numerical example with the same parameters as in Fig. 4, but the negative sign of $\tilde{k}$, i.e., for the pump wave propagating in the negative x direction. This case is shown in Fig. 5.

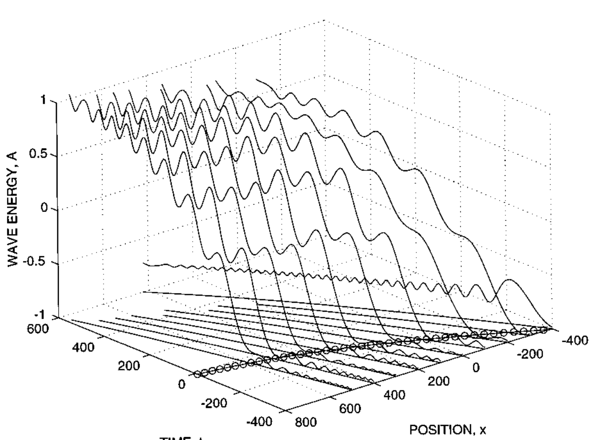

FIG. 4. Space-time dependence of the slow energy A of the daughter wave along the characteristics in Fig. 3. The characteristics themselves are also shown in the (x,t) plane. The circles indicate the linear resonance line. Note that the trapping into resonance and the subsequent autoresonant increase of the energy of the driven wave does not take place along the characteristic originating at $x=-400$ (i.e., slightly beyond the linear resonance line).

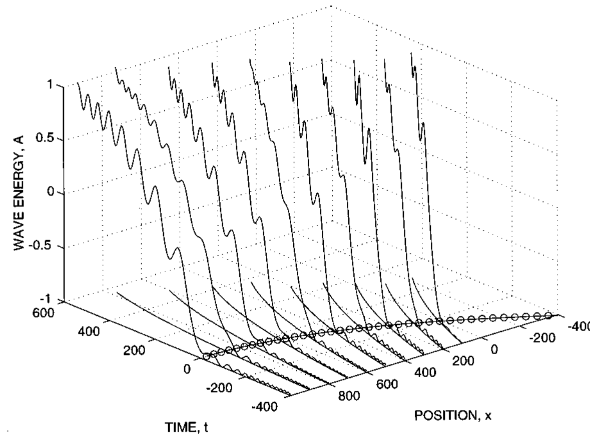

FIG. 5. Space-time dependence of the slow energy of the daughter wave in the example with the same parameters as in Fig. 4, but the pump wave propagating in the negative x direction.

V. CONCLUSION

We have developed a theory of multidimensional autoresonance of driven nonlinear waves in systems with adiabatically varying parameters. The theory is applicable to a broad class of resonantly perturbed nonlinear waves described by the variational principle.

Our theory is based on the analysis of the reduced system of slow evolution equations found from the averaged variational principle. As a starting point, we have developed the averaging procedure leading to the averaged variational principle in the dynamic autoresonance. The trapping into the resonance and the conditions for the DAR in the system were discussed and formed the bases for a generalization to nonlinear waves. The theory was illustrated by an example of a nonlinear pendulum perturbed by an oscillation with a slowly varying frequency.

Using the main ideas of the averaged variational approach developed in the DAR problem, we have also constructed the averaged variational principle for studying resonant excitation and subsequent evolution of nonlinear waves in slowly space-time varying media. We have considered the problem of autoresonance arising when a prescribed large-amplitude pump wave resonantly excites a nonlinear daughter wave inside the region of interests. The reduced system of slow evolution equations in this problem comprises a set of first-order partial differential equations that can be solved along characteristics originating on the boundary where the daughter wave is negligible. One finds that, along these characteristics, the system of slow equations describing the problem is similar to that of the DAR. This similarity allowed us to apply all the results of the DAR directly to the problem of the multidimensional autoresonance in nonlinear wave systems. The theory was illustrated by a two-dimensional numerical example of a driven sine-Gordon equation.

We conclude this section by making general remarks regarding the averaging method applied above. The method is based on the assumption of a resonant excitation of a quasiperiodic nonlinear wave such that its local wavelength is short compared to the scale length and frequency large compared to the time rate that characterize the variation of the macroscopic parameters of the system. A similar assumption is used in the eikonal approximation for linear wave propagation problems in adiabatically space-time varying media [21]. The aforementioned scale difference means the existence of a small parameter in the problem and the averaging method comprises a perturbation analysis in terms of this parameter. Remarkably, the theory reveals the underlying Hamiltonian structure of resonantly driven nonlinear traveling waves described by the variational principle. This structure is seen when, locally (in the zeroth-order of the perturbation scheme), one associates the wave problem with that of the evolution of a characteristic driven dynamical system. Then, in the first order, one obtains equations describing the space-time evolution of the slow variables of the zeroth-order dynamical problem such as the energy and the phase mismatch. As in Whitham's theory of free modulations [17], these slow evolution equations can be regarded as generalizations of Hamilton's equations in dynamics to the associated problem of evolution of the adiabatically varying nonlinear wave. The different ingredient in the present theory is the existence of the continuous phase locking between the pump and daughter waves. This intrinsic phase locking in the system has its origin in a similar dynamical problem (DAR) and allows one to generalize the averaging method to the autoresonant wave interactions. Finally, one major advantage of the averaged variational approach developed in this work belongs to numerical applications. As illustrated by our examples in Sec. IV, the theory allows one to calculate the characteristic parameters of a resonantly excited nonlinear wave in space-time regions large compared to its wavelength and period and thus to avoid numerical difficulties associated with the existence of the fast scales in the original system.

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