Multidimensional autoresonant mode conversion

L. Friedland
Racah Institute of Physics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel

(Received 24 October 1994; accepted 26 January 1995)

It is shown that the autoresonance effect is characteristic of wave interactions in slowly varying weakly nonlinear multidimensional media. The theory of the phenomenon is presented for the mode conversion case and illustrated in two dimensions. It is demonstrated that multidimensional autoresonance is stable with respect to weak damping and transverse nonuniformity of the incident wave.© *1995 American Institute of Physics.*

I. INTRODUCTION

Autoresonance is a unique nonlinear phase locking phenomenon characteristic of resonantly perturbed nonlinear oscillating systems with adiabatically varying parameters. The simplest model problem exhibiting autoresonance is that of a nonlinear oscillator driven by a resonant quasiharmonic perturbation with a slowly varying frequency $\omega(t)$. The isolated resonance Hamiltonian[1] $H(I,\theta)=H_0(I)+\epsilon g(I)\cos[\int\omega(t)dt-\theta]$ for this system expressed in terms of the action-angle variables of the unperturbed oscillator (ϵ being a small dimensionless parameter measuring the strength of the perturbation) yields the following set of evolution equations for the action and the phase shift $\Phi=\int\omega(t)dt-\theta$:

$$\frac{dI}{dt}=\epsilon g\sin\Phi,\quad \frac{d\Phi}{dt}=\omega(t)-\Omega(I)+\epsilon\left(\frac{dg}{dI}\right)\cos\Phi. \tag{1}$$

Here $\Omega=dH_0/dI$ is the frequency of the unperturbed oscillator and system (1) differs from that describing the classical nonlinear resonance only by a slow variation of ω with time. The essence of the autoresonance phenomenon is that despite the time variation of ω, under certain conditions (see below), Eqs. (1) preserve [to $O(\sqrt{\epsilon})$] the approximate resonance relation $\omega(t)\approx\Omega[I(t)]$ provided this relation is satisfied initially.

The idea of autoresonance was proposed by McMillan[2] and Veksler,[3] and developed at early stages by Bohm and Foldy[4] in applications to relativistic particle accelerators. The term *phase stability principle* was used to describe autoresonance in these studies. The synchrotron, synchrocyclotron,[5] and later the gyroresonance cyclic accelerator (GYRAC)[6] and the spatial autoresonance cyclotron (SAC)[7] accelerator are based on the autoresonance idea. Recently, similar ideas were applied in atomic physics,[8] intense plasma wave excitation by chirped laser pulses,[9] nonlinear dynamics,[10–12] mode conversion,[13] and resonant three-wave interactions.[14]

A common feature of all previous studies of autoresonance was the possibility of reducing the problem to the aforementioned driven nonlinear oscillator model. There exists, however, an important difference between autoresonance associated with particle dynamics and that in resonant wave interactions. While in the former case the system evolves in *time*, the waves may propagate in *four-dimensional* space–time. Nevertheless, in the past, the autoresonant wave interactions were studied in media with one-dimensional nonuniformities and the coordinate in the direction of the nonuniformity played the role of time in treating the wave evolution similarly to the dynamics of the driven nonlinear oscillator. Thus, there remains the question of the possibility of autoresonant wave interactions in *multidimensional* media, where the reduction to the one-dimensional oscillator model seems to be inapplicable. In this work this question is addressed for the first time. For simplicity, we shall limit the discussion to the mode conversion case (two-wave resonant interaction) in a three-dimensional, but stationary medium.

II. WEAKLY NONLINEAR COUPLED MODE PROBLEM

Consider the following system of wave transport equations:[13]

$$\begin{aligned} &V_i^a\partial_{x_i}A_a+\Gamma_aA_a+iC_a|A_a|^2A_a\\ &\quad=-iHp_aA_b\exp[i(\Psi_b-\Psi_a)],\\ &V_i^b\partial_{x_i}A_b+\Gamma_bA_b+iC_b|A_b|^2A_b\\ &\quad=-iH^*p_bA_a\exp[i(\Psi_a-\Psi_b)], \end{aligned} \tag{2}$$

describing the spatial evolution of the amplitudes of linearly coupled eikonal-type waves $Z_{a,b}(\mathbf{r},t)=A_{a,b}(\mathbf{r})\times\exp\{i[\Psi_{a,b}(\mathbf{r})-\omega t]\}$ in a three-dimensional weakly nonuniform medium. The vectors $\mathbf{V}^{a,b}$ are the group velocities, the factors $\Gamma_{a,b}=\tilde{\Gamma}_{a,b}+(1/2)\partial_{x_i}V_i^{a,b}$ include the linear damping coefficients ($\tilde{\Gamma}_{a,b}$) and the divergence effect of the geometric optics rays associated with the waves, H is the complex coupling coefficient, $C_{a,b}|A_{a,b}|^2$ are the lowest order nonlinear corrections to the wave dispersion functions, and $p_{a,b}$ are the wave energy signs. The phases $\Psi_{a,b}$ are defined as the solutions of the corresponding *linear* problem (i.e., when $C_{a,b}=0$). We also assume that all the terms involving spatial derivatives in (2) are of $O(\delta)$, where δ is a small dimensionless nonuniformity parameter. The coupling coefficient H is also of $O(\delta)$, since, otherwise, the problem reduces to a *single* first-order transport equation.[13] Now, we must specify the boundary conditions. We shall limit the discussion to the case when only one of the modes (say mode a) is launched *externally* (i.e., this mode is known on some boundary of the volume of interest containing the resonance region), while mode b, is excited *internally* as the result of the mode coupling. Because of generally rapid spatial variation of the phases $\Psi_{a,b}$ in the right-hand side (RHS) of Eqs. (2), the excitation of mode b takes place in the vicinity of the *resonant region* in space, where $\partial_{\mathbf{r}}(\Psi_b-\Psi_a) \equiv \mathbf{k}_b(\mathbf{r})-\mathbf{k}_a(\mathbf{r})=0$, i.e., where the two modes have the same wave vectors. One of the important consequences of the aforementioned choice of *only one* of the modes to be launched *externally* is its effect on the *geometry* of the resonant region in the problem. Let us discuss this effect in detail.

First, we observe that the phase mismatch $\Psi_b-\Psi_a$ in the RHS of Eqs. (2) is associated with the corresponding linear problem. Thus, in discussing the geometry of the resonance region, we proceed from the linear limit of Eqs. (2), which describes the three-dimensional *linear* mode conversion case.[15] Far from the resonance we neglect the coupling between the modes and treat their linear evolution independently within the geometric optics approximation. We characterize the modes by their dispersion functions $D_{a,b}(\mathbf{k},\mathbf{r})$ and solve the wave propagation problem along the geometric optics rays originating on the boundaries or source points. For each mode, these rays lie on the five-dimensional dispersion surfaces $D_{a,b}(\mathbf{k},\mathbf{r})=0$ in the six-dimensional phase space $(\mathbf{k},\mathbf{r})$. In our case, only mode a is excited on the boundary. Then, one starts on this boundary and follows the rays of mode a by solving their Hamilton equations. This set of rays generates the three-dimensional Lagrangian manifold $\mathbf{k}_a(\mathbf{r})$. In addition, when *each* ray of mode a intersects the dispersion surface $D_b(\mathbf{k},\mathbf{r})=0$ of mode b, it excites a ray of mode b on this surface with the same wave vector.[15] By continuing these new rays on their dispersion surface one finds $\mathbf{k}_b(\mathbf{r})$. Hence, the condition for the exact resonance is $D_b[\mathbf{k}_a(\mathbf{r}),\mathbf{r}]=0$, which describes a *surface* in the configuration space. The local form of the equation of the resonance surface can be found by choosing a point $\mathbf{r}_0$ (set to be zero, for convenience) on the resonance surface and expanding $D_b[\mathbf{k}_a(\mathbf{r}),\mathbf{r}]=0$ around $\mathbf{r}_0$, i.e., $(\partial D_b/\partial k_i)_{\mathbf{r}_0,\mathbf{k}_0} \times(\partial^2\Psi_a/\partial x_i\partial x_j)_{\mathbf{r}_0}x_j+(\partial D_b/\partial x_j)_{\mathbf{r}_0,\mathbf{k}_0}x_j=0$, where $\mathbf{k}_0=\mathbf{k}_a(\mathbf{r}_0)=\mathbf{k}_b(\mathbf{r}_0)$. On the other hand, for mode b, $D_b[\mathbf{k}_b(\mathbf{r}),\mathbf{r}]\equiv 0$, or, again, upon expansion, $(\partial D_b/\partial k_i)_{\mathbf{r}_0,\mathbf{k}_0}(\partial^2\Psi_b/\partial x_i\partial x_j)_{\mathbf{r}_0}x_j+(\partial D_b/\partial x_j)_{\mathbf{r}_0,\mathbf{k}_0}x_j=0$. Since this relation is valid for all $\mathbf{r}$ in the vicinity of $\mathbf{r}_0$, we have $(\partial D_b/\partial x_j)_{\mathbf{r}_0,\mathbf{k}_0}=-(\partial D_b/\partial k_i)_{\mathbf{r}_0,\mathbf{k}_0}(\partial^2\Psi_b/\partial x_i\partial x_j)_{\mathbf{r}_0}$. Thus, if $\kappa_{ij}\equiv(\partial^2\Psi/\partial x_i\partial x_j)|_{\mathbf{r}=\mathbf{r}_0}$, then, locally, the linear resonance surface is the plane $(V_i^b)_{\mathbf{r}_0,\mathbf{k}_0}\kappa_{ij}x_j=0$. Mode b is excited on this plane and the amplitude of mode a changes accordingly.[15]

Another interesting consequence of the presence of the resonance surface in the linear mode conversion problem is the existence of constraints on the elements of matrix κ_{ij}. Indeed, since the resonance relation $[\mathbf{k}_b(\mathbf{r})-\mathbf{k}_a(\mathbf{r})]_i=\kappa_{ij}x_j=0$ is satisfied on the *plane* $V_i^b\kappa_{ij}x_j=0$ the matrix κ_{ij} must have *two* zero eigenvalues. Then, we can write $\kappa_{ij}=\kappa q_i q_j$, where κ is the nonzero eigenvalue of κ_{ij} and vector $\mathbf{q}$ is the normalized $(\Sigma q_i^2=1)$ eigenvector corresponding to this eigenvalue. Thus, we see that κ_{ij} can be described by just *three* parameters (κ and, say, two of the components of $\mathbf{q}$) and, therefore, only three of the six elements of this real symmetric matrix can be viewed as independent. We also observe that the zero eigenvalues of κ_{ij} correspond to its eigenvectors tangent to the resonance surface, while $\mathbf{q}$ is normal to this surface. This completes the discussion of the geometry of the linear resonance and we return to our nonlinear problem.

We shall be solving a *nonlinear* boundary value problem in a given volume U of the medium containing the linear resonance surface as defined above. We shall again assume that mode a (the incident wave) is launched *externally* from a uniform plane source, so that $A_a=A_0=$const on a plane surface S_a normal to $\mathbf{V}^a$ and located sufficiently far away from the resonance surface, so the two modes are decoupled on S_a because of the large phase mismatch $\Psi_b-\Psi_a$. In contrast to mode a, mode b is assumed to be excited *internally* through the mode conversion process. Thus, there exists another distant (with respect to the resonance surface) plane surface S_b, where mode b is asymptotically small. We shall limit our discussion to the case when, due to the weak nonuniformity, to $O(\delta)$, one can treat the group velocities $\mathbf{V}^{a,b}$ and the coefficients $\Gamma_{a,b}$, $C_{a,b}$, and H in Eqs. (2) as constant in our volume of interest. Similarly, we shall neglect the possibility of a finite curvature of the resonance surface, i.e., assume that the nonzero eigenvalue κ and the eigenvector $\mathbf{q}$ of κ_{ij} are constant and use the local expansion of the phase mismatch near the resonance plane, $\Psi_b-\Psi_a\approx\kappa_{ij}x_ix_j/2=\kappa(q_ix_i)^2/2$, in the RHS of Eqs. (2).

Now, it is convenient to transform $A_{a,b}$ to $A_a'=|\mathbf{V}^a|^{1/2}A_a$ ($|A_a'|^2$ is the wave-action flux) and $A_b'=|\mathbf{V}^b|^{1/2}A_b\exp[i\kappa(q_ix_i)^2/2+i\,\mathrm{Arg}(H)]$. Then (2) becomes

$$u_i^a\partial_{x_i'}A_a'+\gamma_aA_a'+ic_a|A_a'|^2A_a'=-i\eta p_aA_b',$$
$$u_i^b\partial_{x_i'}A_b'+(\gamma_b-i\xi)A_b'+ic_b|A_b'|^2A_b'=-i\eta p_bA_a', \tag{3}$$

where $\mathbf{u}^{a,b}=\mathbf{V}^{a,b}/|\mathbf{V}^{a,b}|$ are normalized velocities, $\xi\equiv q_ix_i'$, and we use the dimensionless coordinates $x_i'=K^{1/2}x_i$ ($K\equiv\kappa q_iu_i^b$ and the direction of $\mathbf{q}$ is chosen so that $K>0$) and parameters $\gamma_{a,b}=(|\mathbf{V}^{a,b}|K^{1/2})^{-1}\Gamma_{a,b}$, $c_{a,b}=(|\mathbf{V}^{a,b}|^2K^{1/2})^{-1}C_{a,b}$, $\eta=(|\mathbf{V}^a||\mathbf{V}^b|K)^{-1/2}|H|$.

Next, we observe that the equation $\xi=0$ describes the linear resonance plane (see the discussion above), while $\xi+d=q_ix_i'+d=0$ is another plane parallel to the resonance plane. Furthermore, the eigenvector $\mathbf{q}$ is normal to the resonance plane and, since $\Sigma q_i^2=1$, d is the distance between these two planes. We shall see later that for a certain value of $d=d_0$ (similar to the linear mode conversion case, where $d_0=0$) the plane $\xi+d_0=0$ plays an important role in our mode conversion problem. In particular, the efficient excitation of mode b proceeds near this plane, so the term *excitation*, or *e-plane* will be used for this surface in the following. The e-plane allows us to complete our boundary conditions regarding the location of the boundary surfaces $S_{a,b}$ in volume U. This plane divides the volume into two parts. We shall limit our discussion to the case when both boundary surfaces $S_{a,b}$ are located in the same part of U denoted as $U_{\rm in}$. This corresponds to the situation when the group velocities of the waves are both directed from $S_{a,b}$ toward the e-plane. We shall see that, under these conditions, mode b remains small inside $U_{\rm in}$ except near the e-plane and the waves enter the multidimensional autoresonant interaction phase as they propagate beyond the e-plane in the second part of U (denoted by $U_{\rm out}$).

At this stage, we proceed to the solution of (3). First, we consider the limit $\gamma_{a,b}=0$ and solve the problem in the asymptotic part of $U_{\rm in}$ located far away from the e-plane. Here, $|A_b/A_a| \ll 1$ (the assumption to be checked later) and, consequently, we neglect the RHS in the first equation in (3). The resulting equation

$$u_i^a \partial_{x_i'} A_a' + i c_a |A_a'|^2 A_a' \approx 0 \tag{4}$$

yields the solution $A_a'(\mathbf{r}') = A_0' \exp[-ic_a|A_0'|^2 s(\mathbf{r}')]$, where $s(\mathbf{r}')$ is the distance from point $\mathbf{r}'$ to the boundary surface S_a. By substituting this solution into the second equation in (3), neglecting the small nonlinear term in this equation, and replacing A_b' by $A_b'' = A_b' \exp[ic_a|A_0'|^2 s(\mathbf{r}')]$, we obtain

$$u_i^b \partial_{x_i'} A_b'' - i(\xi + d_0) A_b'' = -i \eta p_b A_0', \tag{5}$$

where $d_0 \equiv c_a |A_0'|^2 u_i^b \partial_{x_i'} s = (\mathbf{u}^a \cdot \mathbf{u}^b) c_a |A_0'|^2$. The surface $\xi + d_0 = 0$ is the e-plane mentioned above. The asymptotic solution of Eq. (5), vanishing at $|\xi| \to \infty$, is

$$A_b'' = \eta p_b A_0' \xi^{-1} + O(\xi^{-2}). \tag{6}$$

We see that this solution is a function of ξ only and, therefore, A_b' is uniform on any asymptotic plane in $U_{\rm in}$ parallel to the e-plane. We shall use this important fact later.

After solving (3) in the asymptotic region in $U_{\rm in}$, we proceed to the solution in the rest of the volume (still in the $\gamma_{a,b}=0$ limit). For convenience, we choose our coordinate system so that x' and y' axes are in the plane $\xi=0$ (i.e., $z' \equiv \xi$) and rewrite (3) as a system of real equations for the absolute values $B_{a,b} \equiv |A_{a,b}'|$

$$u_\xi^a \frac{\partial_\xi B_a}{\partial \xi} = R_a + \eta p_a B_b \sin \phi,$$
$$u_\xi^b \frac{\partial_\xi B_b}{\partial \xi} = R_b - \eta p_b B_a \sin \phi \tag{7}$$

and phases $\phi_{a,b} \equiv \operatorname{Arg}(A_{a,b}')$

$$u_\xi^a \left(\frac{\partial \phi_a}{\partial \xi} \right) = Q_a - c_a B_a^2 - \eta p_a \left(\frac{B_b}{B_a} \right) \cos \phi,$$
$$u_\xi^b \left(\frac{\partial \phi_b}{\partial \xi} \right) = Q_b + \xi - c_b B_b^2 - \eta p_b \left(\frac{B_a}{B_b} \right) \cos \phi, \tag{8}$$

where $\phi = \phi_b - \phi_a$ and the derivatives with respect to x' and y' are combined in the terms $R_{a,b} \equiv -u_{x'}^{a,b} \partial_{x'} B_{a,b} - u_{y'}^{a,b} \partial_{y'} B_{a,b}$ and $Q_{a,b} \equiv -u_{x'}^{a,b} \partial_{x'} \phi_{a,b} - u_{y'}^{a,b} \partial_{y'} \phi_{a,b}$. Finally, we use (8) in deriving the equation for ϕ

$$u_\xi^a u_\xi^b \partial_\xi \phi = F + u_\xi^a (\xi - c_b B_b^2) + u_\xi^b c_a B_a^2 + \eta G \cos \phi, \tag{9}$$

where $F \equiv u_\xi^a Q_b - u_\xi^b Q_a$ and $G \equiv p_a u_\xi^b (B_b/B_a) - p_b u_\xi^a (B_a/B_b)$.

Now consider a plane $\xi = \xi_0$ in the asymptotic part of $U_{\rm in}$, where the solution of our system is already known. We can view this plane as a new boundary surface ($\tilde{S}$) in the problem. We have seen that both $B_a \approx |A_0'|$ and B_b [see Eq. (6)] are *constant* on this boundary. Therefore, the terms $R_{a,b}$ in (7) *vanish* on $\tilde{S}$. Furthermore, $\phi_a = c_a |A_0'|^2 s$ and $\phi_b = \phi_a$ [see Eq. (6)] on this boundary. Then, $\phi \approx 0$ on $\tilde{S}$ and simple calculation yields $F = u_\xi^a d_0 - u_\xi^b c_a |A_0'|^2$, which is also constant on $\tilde{S}$. The uniformity of $B_{a,b}$ and ϕ on $\tilde{S}$ combined with the fact that ξ is the only coordinate entering Eqs. (7) and (9) explicitly, allow us to conclude that $B_{a,b}$ and ϕ are functions of ξ only in the entire volume U, while F and $R_{a,b}=0$ remain constant everywhere. Thus, the problem is reduced to the one-dimensional mode conversion case described by the closed system of equations

$$u_\xi^a \, \partial_\xi B_a = \eta p_a B_b \sin \phi,$$
$$u_\xi^b \, \partial_\xi B_b = -\eta p_b B_a \sin \phi, \tag{10}$$
$$u_\xi^a u_\xi^b \partial_\xi \phi = F + u_\xi^a (\xi - c_b B_b^2) + u_\xi^b c_a B_a^2 + \eta G \cos \phi.$$

The solution of this one-dimensional problem is known[13] and we shall not repeat all its details here. Nonetheless, we observe that Eqs. (10) yield the conservation law (the Manley–Rowe condition)

$$\kappa_{ij} u_j^b (p_a u_i^a B_a^2 + p_b u_i^b B_b^2) = 0 \tag{11}$$

and, thus, one can express B_b in terms of B_a. Then the first and the third equation in (10) comprise a complete system of equations for B_a and ϕ which has a form similar to (1), i.e., our problem reduces to that of adiabatic nonlinear resonance. Therefore, in both cases, one may have autoresonance in the system. In particular, *spatial* autoresonance may be excited in the mode conversion case as the resonance relation $F + u_\xi^a(\xi - c_b B_b^2) - u_\xi^b c_a B_a^2 \approx 0$ is preserved continuously *beyond* the e-plane. Then, in the simplest case, $c_a = 0$, we have $B_b^2 \approx (\xi + d_0)/c_b$, $\xi > -d_0$, i.e., B_b^2 and B_a^2 (because of the Manley–Rowe condition) are linear functions of (x,y,z) in $U_{\rm out}$.

Before testing our theory numerically, we should mention that the relation $\Omega(I) - \omega(t) \approx 0$ for the driven nonlinear oscillator [or $F + u_\xi^a(\xi - c_b B_b^2) - u_\xi^b c_a B_a^2 \approx 0$ in the mode conversion case] in the autoresonant regime is preserved in time (or space) only approximately. The actual autoresonant evolution (see, for example, Ref. 8) of the action I can be written as $I = I_0 + \delta I$, where $I_0(t)$ is a slowly varying function of time satisfying the exact nonlinear resonance relation $\omega(t) = \Omega[I_0(t)]$, while δI is small [$\delta I/I_0 \sim O(\sqrt{\epsilon})$] and oscillating with frequency $\omega_{\rm nl} \sim O[(\epsilon g d\Omega/dI)^{1/2}]$. Two conditions must be satisfied in autoresonance,[8] i.e., the moderate nonlinearity condition[1] $\epsilon \ll (I/\Omega) d\Omega/dI \ll 1/\epsilon$, and the adiabaticity condition $(d\omega/dt)\omega_{\rm nl}^{-2} \ll 1$. A similar evolution (spatial oscillations of the amplitudes around a slowly varying average component) and necessary conditions characterize one-dimensional autoresonant mode conversion[13] and thus, also our reduced multidimensional problem.

III. NUMERICAL EXAMPLES

At this stage we proceed to examples illustrating our theory. We consider the following two-dimensional positive–negative energy ($p_a = -p_b = 1$) coupled mode system:

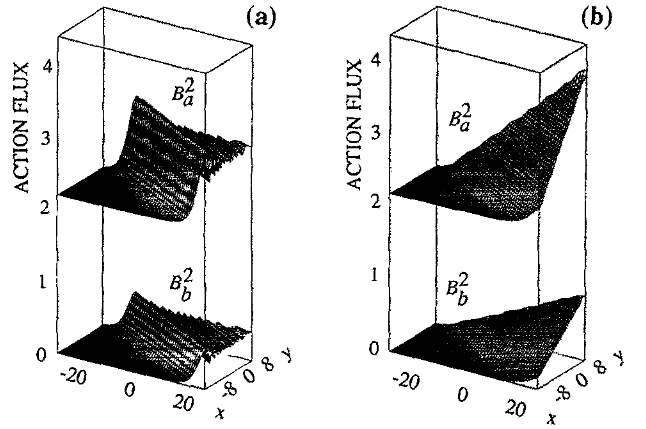

FIG. 1. Two-dimensional mode conversion: (a) linear case; (b) autoresonant interaction.

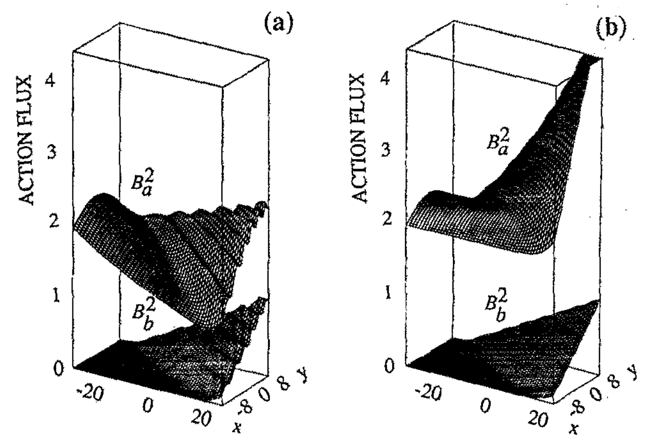

FIG. 2. Autoresonance in broken symmetry cases: (a) damped mode a; (b) damped mode b.

$$\frac{\partial A_a}{\partial x}+ic_a|A_a|^2A_a=-i\eta A_b,$$
$$\frac{\partial A_b}{\partial y}+i(c_b|A_b|^2-q_xx-q_yy)A_b=+i\eta A_a, \tag{12}$$

where the same normalized dependent and dimensionless independent variables as in (3) are used, but the primes are omitted for simplicity. The parameters in (12) are $q_x=-0.45$, $q_y=-0.89$, $\eta=0.15$ and we solve the problem in the rectangle $|x|\leq 20$, $-20\leq y\leq 8$ subject to the boundary conditions $A_a=A_0=1.5$ on $S_a:(x=-20,-20\leq y\leq 8)$ and $A_b=A_0\exp(-ic_aA_0^2x)/(q_xx+q_yy)$ [see Eq. (6)] on $S_b:(|x|\leq 20, y=-20)$. A fourth-order predictor–corrector scheme[16] was used in the calculations and the results were tested by verifying the Manley–Rowe condition (11). Accuracy better than 0.1% was generally achieved in these tests. Figure 1(a) shows our results for the *linear* mode conversion case ($c_{a,b}=0$). One can see in the figure that, indeed, as predicted, mode b is excited near the line $\xi\equiv q_xx+q_yy=0$ in the x-y plane (the e-line). The solution is a function only of the distance ξ from the e-line, and beyond the e-line one rapidly reaches saturation due to the increasing phase mismatch between the modes. Next, we include the nonlinearity. Figure 1(b) shows the numerical results for the same parameters as in Fig. 1(a), but $c_a=0.5$, $c_b=-15$. We see in the figure that, again, the solution is a function only of ξ, and that the initial excitation stage in this case is similar to that in the linear problem. However, beyond the e-line, we observe the continuously growing autoresonant solution $B_b^2\approx(q_xx+q_yy)/c_b$ (the plane originating on the e-line). One can also see, in the figure, the characteristic spatial oscillations of the amplitudes around the exact resonance solution.

Now we include other effects in the problem. The results of the calculations in two cases, with all the parameters as in Fig. 1(b), but $\gamma_a=0.01$, $\gamma_b=0$, and $\gamma_a=0$, $\gamma_b=0.03$, respectively, are shown in Figs. 2(a) and 2(b). Besides $\gamma_{a,b}\neq 0$ in these examples, we also used a nonuniform boundary condition $A_a=A_0\exp(-y^2/\sigma^2)$ (with $\sigma=30$) for the incident wave on S_a. One can see that the main new effect in the Fig. 2 is the breaking of purely ξ dependence of B_a^2. The problem becomes intrinsically two dimensional. Nevertheless, the character of the evolution of mode b beyond the e-line is similar to that in Fig. 1(b), indicating the stability of autoresonance with respect to the plane symmetry breaking factors. To understand this feature of autoresonance, we return to the original equations (7) and (9) for nonvanishing $\gamma_{a,b}$,

$$u_\xi^a\,\partial_\xi B_a=R_a'+\eta p_aB_b\sin\phi,$$
$$u_\xi^b\,\partial_\xi B_b=R_b'-\eta p_bB_a\sin\phi, \tag{13}$$
$$u_\xi^au_\xi^b\partial_\xi\phi=F+u_\xi^a(\xi-c_bB_b^2)-u_\xi^bc_aB_a^2+\eta G\cos\phi,$$

where $R_{a,b}'=R_{a,b}-\gamma_{a,b}A_{a,b}$. If the factors $R_{a,b}'$ and F in (13) would be known, one could solve this system along the characteristics (in the direction of ξ). Then, for a characteristic originating at a given point on S_a, Eqs. (13) would comprise a one-dimensional system similar to Eqs. (10). The only differences would be the *nonvanishing* factors $R_{a,b}'$, *varying* F, and *nonuniform* boundary conditions. However, for sufficiently small $\gamma_{a,b}$ and large σ (the case of interest here) the factors $R_{a,b}'$ are *small*, F is *slowly* varying, and the nonuniformity of the boundary conditions is *weak*. It is the stability[12] of the one-dimensional autoresonance with respect to (a) the addition of small adiabatic factors in the first two equations in (13) and (b) the slow variation of F in the third equation, which leads to the preservation of spatial autoresonance in the broken symmetry cases in Fig. 2.

IV. CONCLUSIONS

In conclusion: (i) We have presented the evidence for existence and a first theory of *multidimensional* autoresonance in mode conversion in a nonuniform medium. (ii) It was shown that the phenomenon is due to the self-sustained *balance* between the spatial and nonlinear dispersion effects in the medium. (iii) The autoresonance may enhance the efficiency of the wave interaction as the spatial width of the nonlinear resonance region increases significantly. (iv) The theory was illustrated numerically in two dimensions. (v) It was also demonstrated that the multidimensional autoresonance is stable with respect to weak damping and transverse nonuniformities of the incident wave.

[1]B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[2]E. M. McMillan, Phys. Rev. **68**, 143 (1945).
[3]V. Veksler, J. Phys. USSR **9**, 153 (1945).
[4]D. Bohm and L. Foldy, Phys. Rev. **70**, 249 (1946).
[5]M. S. Livingston, *High-Energy Accelerators* (Interscience, New York, 1954).
[6]K. S. Golovanivski, IEEE Trans. Plasma Sci. **PS-11**, 28 (1983).
[7]L. Friedland, Phys. Plasmas **1**, 421 (1994).
[8]B. Meerson and L. Friedland, Phys. Rev. A **41**, 5233 (1990).
[9]M. Deutsch, J. E. Golub, and B. Meerson, Phys. Fluids B **3**, 1773 (1991).
[10]B. Meerson and S. Yariv, Phys. Rev. A **44**, 3570 (1991).
[11]G. Cohen and B. Meerson, Phys. Rev. E **2**, 967 (1993).
[12]S. Yariv and L. Friedland, Phys. Rev. E **48**, 3072 (1993).
[13]L. Friedland, Phys. Fluids B **4**, 3199 (1992).
[14]L. Friedland, Phys. Rev. Lett. **69**, 1749 (1992).
[15]L. Friedland, G. Goldner, and A. N. Kaufman, Phys. Rev. Lett. **58**, 1392 (1987).
[16]W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes* (Cambridge University Press, New York, 1986), p. 571.