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AUTORESONANT WAVE INTERACTIONS IN NONUNIFORM PLASMAS

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The theory of the autoresonance is outlined for the case of resonant two-wave interactions (mode conversion) in a nonuniform, weakly nonlinear plasma. The autoresonance is encountered when the incident wave passes the region, where it resonates with another wave supported by the medium. The new wave is then excited and the incident wave amplitude changes accordingly. The autoresonance proceeds from the characteristic trapping into the resonance in the vicinity of the linear resonance point followed by the self-preservation of the approximate nonlinear resonance relation in an extended plasma region. The broadening of the resonant region may significantly enhance the mode conversion efficiency as compared to that in the linear mode conversion case. The effect of the dissipation on the autoresonance and a multidimensional generalization are discussed.

I. INTRODUCTION

The autoresonance is a fascinating nonlinear phase locking phenomenon characteristic of resonantly perturbed nonlinear oscillating systems with adiabatically varying parameters. The simplest model problem exhibiting the autoresonance is that of the nonlinear oscillator driven by a resonant quasi-harmonic perturbation with a slowly varying frequency $\omega(t)$. The isolated resonance Hamiltonian [1] $H(I,\theta)=H_0(I)+\varepsilon g(I)\cos[\theta-\int\omega(t)dt]$ for this system expressed in terms of the action-angle variables (I,θ) of the unperturbed oscillator (ε being a small dimensionless parameter measuring the strength of the perturbation) yields the following set of the evolution equations for the action and the phase shift $\Phi=\theta-\int\omega(t)dt$:

$$dI/dt=\varepsilon g\sin\Phi,\quad d\Phi/dt=\Omega(I)-\omega(t)+\varepsilon(dg/dI)\cos\Phi. \qquad (1)$$

Here $\Omega=dH_0/dI$ is the frequency of the unperturbed oscillator and system (1) differs from that describing the classical nonlinear resonance [1] only by a slow variation of ω with time. The essence of the autoresonance phenomenon is that despite the time variation of ω, under certain conditions, Eqs.(1) preserve [to $O(\sqrt{\varepsilon})$] the approximate resonance relation $\omega(t)\approx\Omega[I(t)]$, if this relation is satisfied initially.

The idea of the autoresonance was proposed by McMillan [2] and Veksler [3], and developed at early stages by Bohm and Foldy [4] in applications to relativistic particle accelerators. The term *phase stability principle* was used to describe the autoresonance in these studies. The synchrotron, synchrocyclotron [5] and later the GYRAC [6] and the SAC [7] accelerator are based on the autoresonance idea. Recently, similar ideas were applied in atomic physics [8], intense plasma wave excitation by chirped laser pulses [9], nonlinear dynamics [10-12], mode conversion [13] and resonant 3-wave interactions [14].

This work describes and further develops the concept of the autoresonance in the mode conversion representing the simplest case of wave interactions in a nonuniform plasma.

II. DYNAMIC AUTORESONANCE

The autoresonance in the aforementioned driven nonlinear oscillator problem (the term *dynamic autoresonance* or *DAR* was suggested for this case [8]) serves as the prototype for the theory of autoresonant wave interactions. Thus, we shall start by a brief review of the theory of the DAR. Consider an oscillator described by system (1), where $g = const$ and $H_0 \equiv MI^2/2$. Then, by defining $I_0(t) \equiv \omega(t)/M$ and introducing a new action variable $\delta I \equiv I - I_0$, one obtains the following evolution equations

$$d(\delta I)/dt = \varepsilon g \sin\Phi - M^{-1} d\omega/dt, \quad d\Phi/dt = M\delta I. \qquad (2)$$

This system can be viewed as generated by the effective Hamiltonian

$$H_{eff} = (M/2)(\delta I)^2 + \varepsilon g \cos\Phi + M^{-1}(d\omega/dt)\Phi, \qquad (3)$$

describing (in the $d\omega/dt = const$ case) the motion of a "quasi-particle" of "mass" M in a *stationary* potential of form $V_{eff}(\Phi) = \varepsilon g \cos\Phi + M^{-1}(d\omega/dt)\Phi$. Then, if

$$|(\varepsilon g M)^{-1}(d\omega/dt)| < 1, \qquad (4)$$

there exist regions of trapped trajectories in the $(\delta I, \Phi)$ phase space, in which the "quasi-particle" performs nonlinear oscillations around stable equilibrium points $(\delta I_e = 0, \Phi_e)$, where $\sin\Phi_e \equiv (\varepsilon g M)^{-1}(d\omega/dt)$, while $\varepsilon g \cos\Phi_e < 0$. The characteristic width of such a region is small ($\Delta I < |\varepsilon g/M|^{1/2}$). Therefore, to $O(\varepsilon^{1/2})$, the solution of the original problem is $I \approx I_0(t)$. This result can be interpreted as a continuing, automatic preservation of the nonlinear resonance relation $\omega(t) \approx \Omega[I(t)] \approx MI_0(t)$ in our system despite the time variation of the driving frequency, provided the "quasi-particle" starts in a trapped region of the phase space, i.e., satisfies the approximate resonance condition initially. This is the simplest example of the DAR and (4) can be viewed as the necessary (adiabaticity) condition for the DAR in the system. This condition must be supplemented by the *moderate nonlinearity* conditions $\Delta I/I << 1$, and $\Delta\Omega/\Omega = (d\Omega/dI)\Delta I/\Omega << 1$ of the theory of the nonlinear resonance [1], or, in our case,

$$|\varepsilon g/MI_0^2|^{1/2} << 1, \qquad (5)$$

so I_0 must be sufficiently large in the autoresonance. We have seen that the DAR is accompanied by oscillations around the exact nonlinear resonance. The frequency of these oscillations is small and of order $\nu_{eff} \sim O(|\varepsilon g M|^{1/2})$. Finally, note that the general DAR problem described by Eqs.(1) is similar to that discussed above and is characterized by the Hamiltonian of form (3), where M and g are replaced by $d\Omega/dI_0$ and $g(I_0)$ respectively and, therefore, vary (slowly) in time. Since trapped trajectories of the oscillator remain trapped under the *adiabatic* variation of the parameters, all the general features of the autoresonance are preserved in the general case, but the "quasi-particle" will perform nonlinear oscillations around the exact resonance with an *adiabatically varying* amplitude. This amplitude variation can be found from the theory of adiabatic invariants.

III. SPATIAL MODE CONVERSION AND TRAPPING INTO THE RESONANCE

In this Section we shall exploit the similarity between the DAR equations and those governing the spatial mode conversion in a nonuniform plasma. The weakly nonlinear mode conversion is a simplest two-wave resonant process described (in a one-dimensional case) by the following system of coupled, conservative, slow amplitude transport equations [13]:

$$V^a dA_a / dx + \Gamma_a A_a + i\delta D_a A_a = i\mathrm{H} p_a A_b \exp[i\kappa x^2/2],$$
$$V^b dA_b / dx + \Gamma_b A_b + i\delta D_b A_b = i\mathrm{H}^* p_b A_a \exp[-i\kappa x^2/2]. \qquad (6)$$

Here $V^{a,b}$ are the components of the group velocities of the two waves in the direction of the nonuniformity (x-direction), $\Gamma_{a,b} = (1/2)dV^{a,b}/dx$, H is the complex linear coupling coefficient, $\delta D_{a,b} = C_{a,b}|A_{a,b}|^2$ are the lowest order nonlinear corrections to the wave dispersion functions and $p_{a,b}$ are the wave energy signs. The phases in the rhs of Eqs.(6) represent the nonuniformity of the medium and comprise the expansion of the fast phase mismatch between the waves near the resonance point (x=0), where the two modes have the same frequencies and wave vectors. Note that (6) conserves the total action flux (the Manley-Rowe relation):

$$J = p_a V^a |A_a|^2 + p_b V^b |A_b|^2 = const. \qquad (7)$$

At this point, we assume $\kappa, J, V_{a,b} > 0$ and transform to dimensionless amplitudes $A'_a = (V^a/J)^{1/2} A_a$ and $A'_b = (V^b/J)^{1/2} A_b \exp[i\kappa x^2/2 + iArg(H)]$. Then (6) yields

$$dA'_a / d\xi + ic_a |A'_a|^2 A'_a = i\eta p_a A'_b,$$
$$dA'_b / d\xi - i(\xi - c_b |A'_b|^2) A'_b = i\eta p_b A'_a, \qquad (8)$$

where we defined the dimensionless coordinate $\xi = \kappa^{1/2} x$ and parameters $\eta = (V^a V^b \kappa)^{-1/2} |\mathrm{H}|$ and $c_{a,b} = [(V^{a,b})^2 \kappa^{1/2}]^{-1} C_{a,b}$. Next, we write $A'_{a,b} = B_{a,b} \exp(i\Phi_{a,b})$, where $\mathrm{Im} B_{a,b} = 0$ and use (8) in deriving three *real* equations for $B_{a,b}$ and the phase difference $\Phi \equiv \Phi_b - \Phi_a$:

$$dB_a / d\xi = -\eta p_a B_b \sin\Phi,$$
$$dB_b / d\xi = +\eta p_b B_a \sin\Phi, \qquad (9)$$
$$d\Phi / d\xi = \xi - c_b B_b^2 + c_a B_a^2 - \eta(p_a B_b / B_a - p_b B_a / B_b)\cos\Phi.$$

The Manley-Rowe relation now becomes $p_a B_a^2 + p_b B_b^2 = 1$. Note that because of this relation we have only two independent equations in (9). For example, one can express B_a in terms of B_b and view the second and the third equation in (9) as a complete set. The similarity of this set with system (1) for the driven oscillator is obvious and one can expect the *spatial* autoresonance (or SAR) in the mode conversion problem. The approximate nonlinear resonance relation

$$\xi - c_b B_b^2 + c_a B_a^2 \approx 0 \qquad (10)$$

will be preserved in this case to $O(\eta^{1/2})$, if the following necessary autoresonance conditions are fulfilled [compare to Eqs.(4) and (5)]:

$$|2\eta B_a B_b (c_b p_a + c_a p_b)|^{-1} < 1, \qquad (11)$$

$$(\eta B_{b,a} / B_{a,b})^{1/2} (c_b p_a + c_a p_b)^{-1/2} << 1. \qquad (12)$$

We observe that the SAR yields simple approximate spatial dependencies of the wave amplitudes. Indeed, from the Manley-Rowe relation and Eq.(10) we find:

$$B_a^2 \approx (c_b - p_b \xi) / (c_a p_b + c_b p_a), B_b^2 \approx (c_a + p_a \xi) / (c_a p_b + c_b p_a). \qquad (13)$$

Thus, as functions of ξ, $B_{a,b}^2$ are straight lines and the slopes of these lines may differ only by signs (when the energy signs of the waves are different). This is a much simpler dependence than that in the *linear* mode conversion problem, where the waves are described by the parabolic cylinder functions [15]. Therefore, the autoresonance comprises a rare example, where the addition of the nonlinearity simplifies the solution significantly. Finally, we observe that, as always in the autoresonance, the wave amplitudes in the SAR regime exhibit spatial oscillations around the exact nonlinear resonance solutions (13). These oscillations are described by the effective Hamiltonian of form (3) and their characteristic frequency is of order

$$\nu_{eff} \sim (2\eta |c_b p_a + c_a p_b| B_a B_b)^{1/2}. \qquad (14)$$

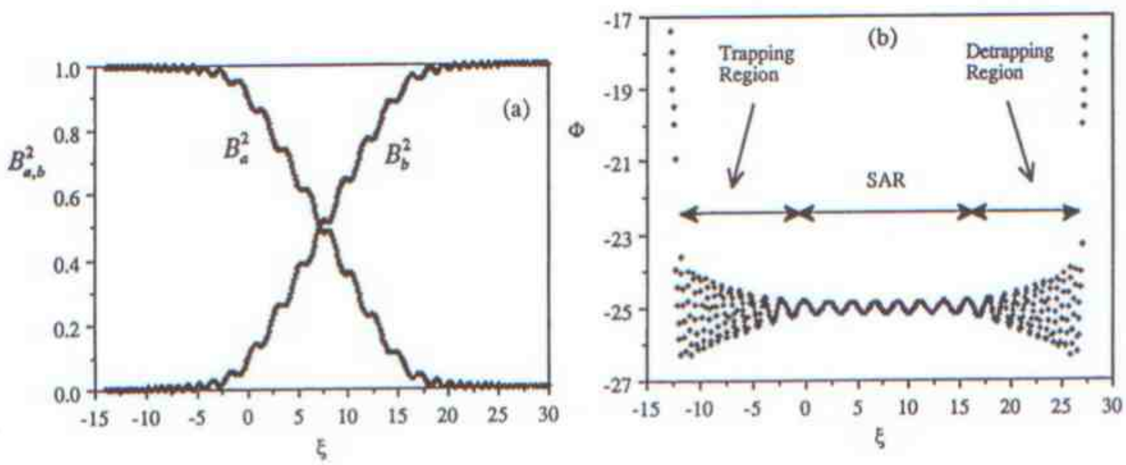

Fig.1 The spatial autoresonance (SAR) and the trapping into the resonance in mode conversion: (a) wave action fluxes vs. ξ; (b) phase difference Φ vs. ξ.

Now we proceed to the illustration of the SAR in the mode conversion. Fig.1a shows the numerical solution [13] of Eqs.(9) for the fluxes $B_{a,b}^2$ in the positive energy mode interaction case, subject to initial conditions $B_b = 0.077$ ($B_a = 0.997$), $\Phi = 0$ at $\xi = -14$ and the parameters $\eta = 0.5, c_a = 0, c_b = 15$. Note that we do not start in resonance initially. Nevertheless, one finds the autoresonant solutions (13) (the linear averaged dependence of $B_{a,b}^2$ on ξ) in the region $0 < \xi < 16$ in Fig.1a. This surprising result, seemingly contradicting our previous requirement of starting in resonance, is explained as follows. While starting off resonance, we have assigned, at the same time, a relatively *small* initial value to one of the amplitudes (mode b). This was also in violation of the autoresonance conditions (11) and (12). Nevertheless, in this case, the term with $\cos\Phi$ in the third equation in (9) becomes important despite the smallness of η and our previous ordering assumptions in (9) may be violated. It was shown in Ref.[13] that this singularity in the third equation in (9) leads to a new and important effect of the *trapping* into the resonance as one approaches the resonant region. It was shown that the smaller the initial value of one of the amplitudes, the stronger is the inequality (12) after the trapping. We illustrate this trapping effect in Fig.1b, showing the relative phase between the waves in our example. We see in the Figure that after the transition period, the relative phase is trapped at $\Phi \approx -8\pi$, while B_b becomes large enough to satisfy the autoresonance conditions (11) and (12) at $\xi \approx 0$. Beyond this point the SAR sets in and continues until the initial action flux is almost completely transferred to mode b. The amplitude of mode a then becomes small, leading to the gradual phase detrapping as the autoresonance conditions are violated.

The trapping into the resonance is a very important phenomenon. It allows to conveniently enter the autoresonant interaction stage even by starting outside the resonance. This finding [13] may be of great help in experiments (accelerators [6,7] being a good example) since (a) due the small $[O(\eta^{1/2})]$ width of the nonlinear resonance it may be difficult to tune the experimental conditions to the resonance initially and (b) the strong trapping effect removes the system from the separatrix region, where the phase detrapping may destroy the autoresonant interaction.

IV. NON-CONSERVATIVE SPATIAL AUTORESONANCE

Now, we shall consider the following non-conservative generalization of (9):

$$\begin{aligned} dB_a^2/d\xi &= -2\gamma_a B_a^2 + 2\eta p_a B_a B_b \sin\Phi, \\ dB_b^2/d\xi &= -2\gamma_b B_b^2 - 2\eta p_b B_b B_a \sin\Phi, \\ d\Phi/d\xi &= \xi - c_b B_b^2 + c_a B_a^2, \end{aligned} \tag{15}$$

where $\gamma_{a,b}$ are the linear spatial dissipation (or growth) rates and we have neglected the $O(\eta)$ term with $\cos\Phi$ in the third equation. It is also assumed that, initially, one starts in the vicinity of the resonance, i.e., $\xi - c_b B_b^2 + c_a B_a^2 \approx 0$. Now one must solve three and not just two equations as in the conservative mode conversion case. Nevertheless, for sufficiently small $\gamma_{a,b}$, we can proceed as follows. We differentiate the third equation in (15) and use the first two equations to obtain

$$d^2\Phi/d\xi^2 = 1 + 2(\gamma_b c_b B_b^2 - \gamma_a c_a B_a^2) + 2\eta(p_a c_a + p_b c_b) B_a B_b \sin\Phi. \tag{16}$$

Next, we seek quasi-equilibrium solutions $B_{a,b0}, \Phi_0$, such that

$$\xi - c_b B_{b0}^2 + c_a B_{a0}^2 \approx 0, \tag{17}$$

i.e., $d^2\Phi_0/d\xi^2 \approx 0$ continuously. Then, from (16),

$$2\eta B_{a0} B_{b0} \sin\Phi_0 \approx -[1 + 2(\gamma_b c_b B_b^2 - \gamma_a c_a B_a^2)](p_b c_b + p_a c_a)^{-1} \tag{18}$$

and, after substituting (18) into the first two equations in (15), we obtain

$$\begin{aligned} dB_{a0}^2/d\xi &= -2c_b(c_a p_a + c_b p_b)^{-1}(\gamma_a p_b B_{a0}^2 + \gamma_b p_a B_{b0}^2) - p_a(c_a p_a + c_b p_b)^{-1}, \\ dB_{b0}^2/d\xi &= -2c_a(c_a p_a + c_b p_b)^{-1}(\gamma_a p_b B_{a0}^2 + \gamma_b p_a B_{b0}^2) + p_b(c_a p_a + c_b p_b)^{-1}. \end{aligned} \tag{19}$$

Finally, if one defines $Z \equiv \gamma_a p_b B_{a0}^2 + \gamma_b p_a B_{b0}^2$, Eq.(19) yields

$$dZ / d\xi = \alpha - \beta Z, \tag{20}$$

where $\alpha = p_a p_b(\gamma_b - \gamma_a)(c_a p_a + c_b p_b)^{-1}$ and $\beta = 2(p_b c_b \gamma_a + c_a p_a \gamma_b)(c_a p_a + c_b p_b)^{-1}$. Thus,

$$Z(\xi) = (\alpha / \beta) + [Z(0) - \alpha / \beta]\exp(-\beta\xi), \tag{21}$$

where $Z(0)$ is the value at the initial integration point (assumed to be at 0). By using this result and Eq.(17) we find that the fluxes $B_{a,b0}^2$ in the non-conservative mode conversion case comprise a nontrivial combination of linear and exponential functions of ξ. The study of the stability of these quasi-equilibrium [autoresonant, see Eq.(17)] solutions will be reported elsewhere. Here, we shall illustrate this stability by presenting a numerical example. Fig.2 shows the same case as Fig.1, but with $\gamma_{a,b} = -0.015$. Therefore, $\alpha = 0, \beta = 2\gamma_a$ and, as before, $B_{b0}^2 = \xi / c_b$. However, the competition between the linear and exponential spatial dependencies is seen in the evolution of B_{a0}^2 in Fig.2a. The trapping stage and stable oscillations around $B_{a,b0}^2$ in our example are seen in Fig.2b. Finally, we observe that the autoresonance continues indefinitely in this case.

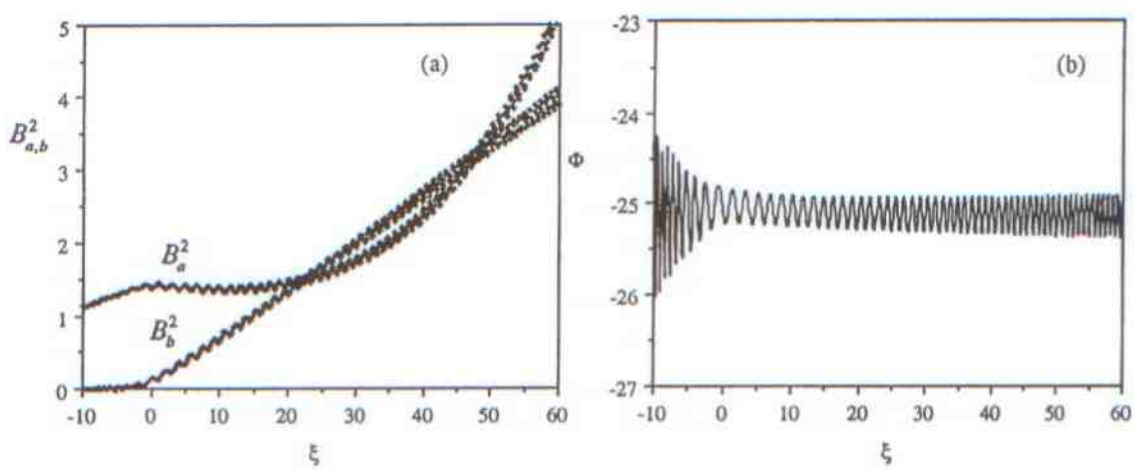

Fig.2 Non-conservative autoresonance: (a) (a) $B_{a,b}^2$ vs. ξ; (b) Φ vs. ξ.

V. MULTIDIMENSIONAL SPATIAL AUTORESONANCE

The one-dimensional theory of the autoresonant mode conversion (Sections III, IV) was based on the possibility of reducing the problem to the aforementioned driven nonlinear oscillator model. There exists, however, an important difference between the autoresonance associated with the oscillator dynamics and that in resonant wave interactions. While in the former case the system evolves in *time*, the waves may propagate in the *four-dimensional* space-time. Nevertheless, until recently, the autoresonant wave interactions were studied in media with a one-dimensional nonuniformity and, the coordinate in the direction of the nonuniformity played the role of time in treating the wave evolution similarly to the dynamics of the driven nonlinear oscillator.

Very recently [16] we have performed the first study of the multidimensional *mode conversion* problem described by the following generalization of (6):

$$V_i^a \partial_{x_i} A_a + \Gamma_a A_a + i\delta D_a A_a = i\mathrm{H} p_a A_b \exp[i\kappa_{ij} x_i x_j / 2],$$
$$V_i^b \partial_{x_i} A_b + \Gamma_b A_b + i\delta D_b A_b = i\mathrm{H}^* p_b A_a \exp[-i\kappa_{ij} x_i x_j / 2]. \qquad (22)$$

We have shown that when one of the waves is not excited externally, similarly to the one-dimensional case, the phase trapping phenomenon takes place as the second wave (the incident mode) passes the resonance region and the system enters the autoresonant stage of interaction despite the multidimensionality of the problem. An example of such a multidimensional autoresonance is given in Fig.3a, showing the fluxes $B_{a,b}^2 \equiv |A_{a,b}|^2$ in a two-dimensional, positive-negative energy ($p_b = -p_a = 1$) mode conversion problem [15], described by [compare to (8)]:

$$\partial A'_a / \partial x + i c_a |A'_a|^2 A'_a = -i\eta p_a A'_b,$$
$$\partial A'_b / \partial y - i(q_x x + q_y y - c_b |A'_b|^2) A'_b = i\eta p_b A'_a. \qquad (23)$$

Mode a *only* is launched from the left boundary in the Figure, the boundary conditions are uniform, $q_x = -0.45, q_y = -0.89, \eta = 0.15, c_a = 0.5$ and $c_b = -15$.

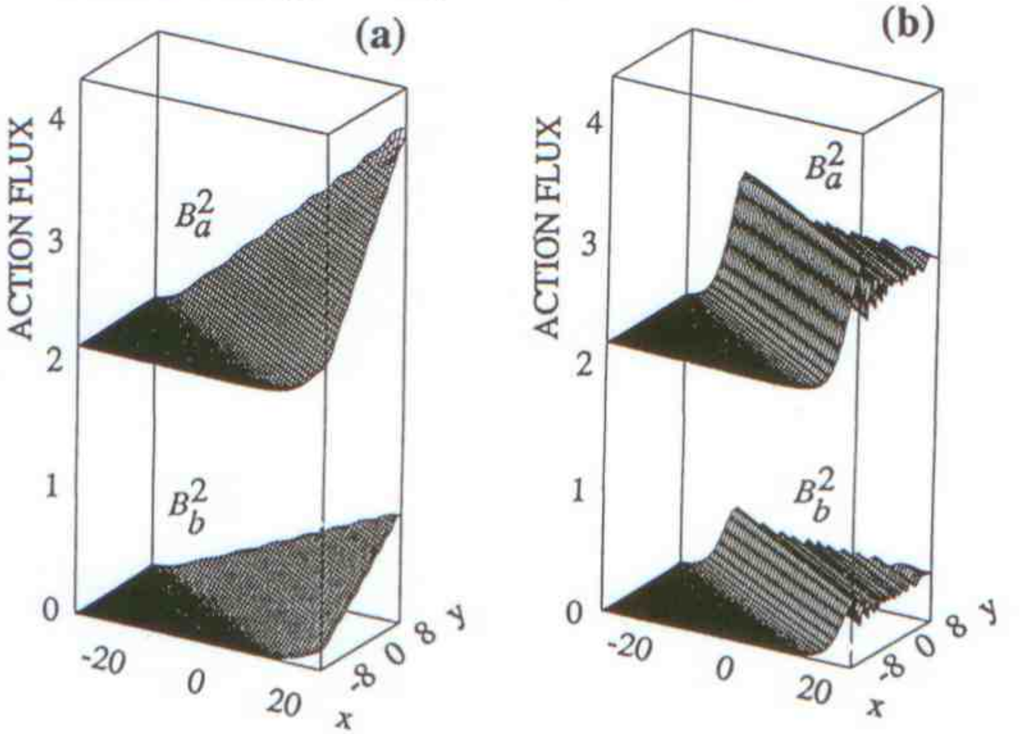

Fig.3 Two-dimensional mode conversion: (a) autoresonant interaction (b) linear mode conversion

We can see in Fig.1a that mode b is excited beyond a certain line in the x-y plane and the system enters the spatial autoresonance regime in which both $B_{a,b}^2$ are nearly linear functions of x and y, as the spatial and nonlinear dispersions are in balance (the waves are in the nonlinear resonance) in an extended plasma region. The interaction is accompanied by the characteristic of the autoresonance spatial oscillations around the exact nonlinear resonance. For comparison, Fig.1b shows the same example, but for the linear mode conversion case ($c_{a,b}=0$) where, in contrast to the autoresonance, the interaction takes place only in a narrow region in space and saturates rapidly due to the luck of the nonlinear phase shifts capable of balancing the growing spatial phase mismatch beyond the excitation line. Thus, presently, we have an evidence and a first theory [16] of the multidimensional autoresonant mode conversion.

VI. CONCLUSIONS

In conclusion: (i) We have presented and illustrated the theory of the autoresonant spatial mode conversion in a nonuniform medium. The autoresonance phenomenon is the manifestation of the self-preserved balance between the nonlinear and spatial dispersion effects. This balance increases the width of the resonant interaction region and may enhance the wave interaction efficiency. (ii) Our theory was based on the analog to the dynamic autoresonance effect in a resonantly perturbed nonlinear oscillator with adiabatically varying parameters. (iii) We have discussed the necessary conditions for the spatial autoresonance and described the important possibility of the trapping into the resonance. (iv) The effect of the dissipation or/and growth on the autoresonant mode conversion was treated in detail, and (v) recent results on the multidimensional mode conversion were discussed.

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