Hermitian description of interacting inhomogeneous electron beams

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Recently developed techniques for finding embedded pairwise mode couplings in high-order systems and calculating the transmission coefficient in linear mode conversion for passive, stable plasmas are extended to the treatment of coupling of positive and negative energy waves in nonuniform plasmas. In particular, the coupling of positive and negative energy waves in costreaming, nonuniform electron beams is treated in detail and the ensuing spatial amplification is determined.

I. INTRODUCTION

Many studies have recently dealt with the propagation of general N-component perturbations of form $Z_i(x) = A_i(x)\exp[i\psi(x)]$ $(i=1,2,...,N)$ defined on space-time $[x=(\mathbf{r},t)]$ in a weakly inhomogeneous and time-dependent plasma.[1] Often, the space-time evolution of the amplitude $\mathbf{A}(x)=(A_i, i=1,..., N)$ and of the generalized wave vector $k_\alpha(x)=\partial\psi/\partial x^\alpha$ of such perturbations is *slow* and is governed by the transport equation of the form[2]

$$-i\mathbf{D}\cdot\mathbf{A}=\frac{\partial \mathbf{D}}{\partial k_\alpha}\cdot\frac{\partial \mathbf{A}}{\partial x^\alpha}+\frac{1}{2}\left[\frac{d}{dx^\alpha}\left(\frac{\partial \mathbf{D}}{\partial k_\alpha}\right)\right]\cdot\mathbf{A}, \tag{1}$$

where $\mathbf{D}(k,x)$ is a *Hermitian* dispersion matrix and

$$\frac{d(\cdots)}{dx^\alpha}=\frac{\partial(\cdots)}{\partial x^\alpha}+\left(\frac{\partial(\cdots)}{\partial k_\beta}\right)\left(\frac{\partial k_\beta}{\partial x^\alpha}\right).$$

Following Ref. 3, we shall refer to these types of problems as "Hermitian problems." For example, cold, nonstreaming plasmas;[1] warm, Maxwellian-fluid plasmas;[4] and even the unreduced Maxwell–Vlasov system for a perturbed Maxwellian equilibrium,[5] all comprise a Hermitian problem.

Hermitian problems are of a special importance because they have several important properties. First, Eq. (1) for a Hermitian dispersion matrix $\mathbf{D}$ yields a conservation law, i.e.,

$$\frac{\partial J^\mu}{\partial x^\mu}=0,\quad J_\mu=-\mathbf{A}^\dagger\cdot\frac{\partial \mathbf{D}}{\partial k_\mu}\cdot\mathbf{A}. \tag{2}$$

Second, Eq. (1) allows a successive, systematic elimination of some components of $\mathbf{A}$, so that if, for example, one eliminates component A_1, the remaining *reduced* amplitude $\mathbf{A}'=(A_2, A_3,...,A_N)$ is again described by the equation of form (1), but with $\mathbf{D}$ replaced by a new *reduced*, Hermitian dispersion matrix $\mathbf{D}'(k,x)$ of dimension $N-1$. This is the essence of the *congruent reduction* technique.[6] The elimination of the amplitude components can usually be continued, one by one, until the final, irreducible system of order $M(M<N)$, characterized by a final reduced $M\times M$ dispersion matrix D^f, is obtained. All elements of this final matrix are small [of $O(\delta)$, where δ is a small, dimensionless inhomogeneity parameter]. The salient feature of the congruent reduction procedure is that not only the form (1) of the transport equation is preserved at each reduction step, but also the "action flux" J is both conserved

$$\frac{\partial J'_\mu}{\partial x^\mu}=0,\quad J'_\mu=-\mathbf{A}'^\dagger\cdot\frac{\partial \mathbf{D}'}{\partial k_\mu}\cdot\mathbf{A}' \tag{3}$$

and preserved, to $O(\delta)$,

$$\mathbf{J}'=\mathbf{J}+O(\delta) \tag{4}$$

throughout the reduction procedure.

Despite the above-mentioned advantages of dealing with Hermitian problems, one can notice that only systems with *no* free energy (i.e., passive, stable plasmas) were so far formulated in the Hermitian form. The present work is devoted to the Hermitian description of a different type of a problem, i.e., that of electrostatic perturbations in interacting inhomogeneous electron beams. The free energy in an electron beam leads to its small-amplitude perturbations exhibiting both positive and negative energy waves. We shall show that the presence of the free energy in this problem still allows the Hermitian description, but, nevertheless, leads to mode couplings different from those described previously in systems with no free energy. In particular, in contrast to embedded pairwise couplings of modes with the same energy sign, the positive–negative energy mode interactions in Hermitian systems may lead to a localized *amplification* of the modes (see the example in Sec. IV). The scope of the presentation will be as follows: In Sec. II, we shall formulate the simplest single beam, nonrelativistic problem in the Hermitian form. We shall apply the congruent reduction technique to this problem and generalize to the relativistic case in Sec. III. Finally, the Hermitian coupled beam problem will be considered in Sec. IV.

II. SINGLE, NONRELATIVISTIC BEAM PROBLEM

Consider a nonrelativistic electron beam propagating in the x direction in a neutralizing ion background. Assume the following time-independent one-dimensional equilibrium of the ions and electrons:

$$v_{0i}=0,\quad n_{0i}=n_{0i}(x),$$
$$v_{0e}=v_0(x),\quad n_{0e}=n_0(x), \tag{5}$$

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where

$$\frac{d}{dx}(n_0 v_0) = 0, \tag{6}$$

$$\frac{d}{dx}\left(\frac{m v_0^2}{2}\right) = -e[E_0(x) + E_{0\,\mathrm{ext}}(x)], \tag{7}$$

$$\epsilon_0 \frac{dE_0}{dx} = -e[n_0(x) - n_i(x)], \tag{8}$$

and $E_{0\,\mathrm{ext}}(x)$ is the externally generated electric field. Perturbing the electron equilibrium and leaving the ions stationary,

$$n_e(x,t) = n_0(x) + n_1(x,t),$$
$$v_e(x,t) = v_0(x) + v_1(x,t),$$
$$E(x,t) = E_0(x) + E_1(x,t), \tag{9}$$

we obtain the following complete set of linearized Maxwell, continuity, and momentum equations describing the perturbed quantities:

$$\epsilon_0 \frac{\partial E_1}{\partial t} - e\Gamma_1 = 0, \tag{10}$$

$$\frac{\partial n_1}{\partial t} + \frac{\partial \Gamma_1}{\partial x} = 0, \tag{11}$$

$$m\left(\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}(v_0 v_1)\right) + eE_1 = 0, \tag{12}$$

where

$$\Gamma_1 = n_0 v_1 + n_1 v_0 \tag{13}$$

is the linearized electron flux.

Equations (10)–(13) yield the small-amplitude conservation equation. Indeed, from Eq. (10) we obtain

$$\frac{\partial}{\partial t}\frac{\epsilon_0 E_1^2}{2} - eE_1\Gamma_1 = 0, \tag{14}$$

which on using (12) and (13) to express E_1 and Γ_1 becomes

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial x} = 0, \tag{15}$$

where

$$W = (\epsilon_0 E_1^2/2) + n_0(m v_1^2/2) + m v_0 v_1 n_1 \tag{16}$$

and

$$S = m v_0 v_1 \Gamma_1 \tag{17}$$

are the small-amplitude energy density and flux of the electron beam system.[7]

Our aim now is to cast Eqs. (10)–(12) into the Hermitian form (1). As a first step to this goal, we replace n_1 in (11) by the expression $(\Gamma_1 - n_0 v_1)/v_0$ obtained from Eq. (13). This substitution replaces (11) by

$$\frac{\partial \Gamma_1}{\partial t} + \left(v_0 \frac{\partial \Gamma_1}{\partial x}\right) - \left(n_0 \frac{\partial v_1}{\partial t}\right) = 0. \tag{18}$$

Now we introduce new dependent variables

$$E = \epsilon_0^{1/2} E_1,$$
$$U = (m n_0)^{1/2} v_1,$$
$$\Gamma = (m/n_0)^{1/2} \Gamma_1. \tag{19}$$

Then Eqs. (10), (18), and (12) become (on using $n_0 \times dv_0/dx = -v_0\, dn/dx$)

$$\frac{\partial E}{\partial t} - \omega_p \Gamma = 0, \tag{20}$$

$$\frac{\partial \Gamma}{\partial t} + v_0 \frac{\partial \Gamma}{\partial x} - \frac{1}{2}\frac{dv_0}{dx}\Gamma - \frac{\partial U}{\partial t} = 0, \tag{21}$$

$$\frac{\partial U}{\partial t} + v_0 \frac{\partial U}{\partial x} + \frac{3}{2}\frac{dv_0}{dx} U + \omega_p E = 0, \tag{22}$$

where we use the usual notation $\omega_p = (n_0 e^2/m\epsilon_0)^{1/2}$. Finally, we use the eikonal representation for the unknown three-component vector $\mathbf{Z} = (E,U,\Gamma)^T$, i.e.,

$$\mathbf{Z} = \begin{pmatrix} E \\ U \\ \Gamma \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} \tilde{E} \\ \tilde{U} \\ \tilde{\Gamma} \end{pmatrix} e^{-i\omega t + i\psi(x)}\right]$$
$$= \mathrm{Re}[\mathbf{A}(x) e^{-i\omega t + i\psi(x)}]. \tag{23}$$

Then, if (by definition)

$$k(x) = \frac{d\psi}{dx}, \tag{24}$$

we can rewrite Eqs. (20)–(22) in the standard form (1):

$$-i\mathbf{D}\cdot\mathbf{A} = \frac{\partial \mathbf{D}}{\partial k}\cdot\frac{d\mathbf{A}}{dx} + \frac{1}{2}\left[\frac{d}{dx}\left(\frac{\partial \mathbf{D}}{\partial k}\right)\right]\cdot\mathbf{A}, \tag{25}$$

where dispersion matrix $\mathbf{D}(k,x)$ is Hermitian and given by

$$\mathbf{D} = \begin{pmatrix} \omega & 0 & -i\omega_p \\ 0 & -\omega & \omega - k v_0 - i v_0' \\ i\omega_p & \omega - k v_0 + i v_0' & 0 \end{pmatrix}, \tag{26}$$

where $v_0' = dv_0/dx$. Thus, we have a Hermitian problem. Now, we shall make the geometric optics assumption regarding the slowness of variation of the equilibrium parameters and therefore, view $\tilde{E}$, $\tilde{U}$, $\tilde{\Gamma}$, and k as slowly varying quantities, in the sense that if ξ represents one of these quantities, then

$$\frac{1}{k}\left|\frac{d \ln \xi}{dx}\right| \sim O(\delta), \tag{27}$$

where $\delta \ll 1$ is a small, dimensionless parameter. Then, Eq. (25) can be solved perturbatively and yields the conventional *Hermitian transport equation* for the slow amplitude $A = (\tilde{E}, \tilde{U}, \tilde{\Gamma})^T$.

At this stage, we calculate the *conserved* action flux in the system:

$$J = -\mathbf{A}^\dagger\cdot\frac{\partial D}{\partial k}\cdot\mathbf{A} = v_0(U^*\Gamma + U\Gamma^*)$$
$$= 4 m v_0 \langle v_1 \Gamma_1 \rangle_{\mathrm{av}}, \tag{28}$$

where $\langle\cdots\rangle_{\mathrm{av}}$ means averaging over time. But, the action density is usually defined as

$$Q = \mathbf{A}^\dagger\cdot\frac{\partial \mathbf{D}}{\partial \omega}\cdot\mathbf{A}, \tag{29}$$

so that, in our case

$$Q = |\tilde{E}|^2 - |\tilde{U}|^2 + (\tilde{U}^*\tilde{\Gamma} + \tilde{U}\tilde{\Gamma}^*)$$
$$= 4\langle(\epsilon_0 E_1^2/2) + n_0(m v_1^2/2) + m v_0 n_1 v_1\rangle_{\mathrm{av}}. \tag{30}$$

It can be seen [compare to Eqs. (16) and (17)] that, to within a factor of 4, Q and J are the time-averaged small-amplitude energy density and flux in the system.

III. CONGRUENT REDUCTION AND RELATIVISTIC GENERALIZATION

Here, we shall apply the congruent reduction procedure[6] to the single, inhomogeneous beam problem described above. We consider the third-order system (25) characterized by dispersion matrix (26), view the right-hand side of Eq. (25) as small [of $O(\delta)$], and eliminate (reduce) component $\widetilde{E}$ of A from the problem. The remaining two component vectors $A^{(1)} = (\widetilde{U},\widetilde{\Gamma})^T$ after this first reduction step is again described by the transport equation of form (25) with D replaced by

$$\mathbf{D}^{(1)} = \begin{bmatrix} -\omega & \omega - kv_0 - iv_0' \\ \omega - kv_0 + iv_0' & -\omega_p^2/\omega \end{bmatrix}. \quad (31)$$

More generally, the elements of the reduced matrix [as (29), for example] are obtained from $\mathbf{D}$ by using the rule

$$D_{ij}^{(\text{reduced})} = D_{ij} - D_{ik}D_{kj}/D_{kk} + \Delta_{ij}, \quad i,j \neq k, \quad (32)$$

if component A_k associated with the diagonal element D_{kk} of $\mathbf{D}$ is eliminated. In (30), Δ_{ij} is an $O(\delta)$ correction given by

$$\frac{2i\Delta_{ij}}{D_{kk}} = \frac{\partial}{\partial k}\left(\frac{D_{ik}}{D_{kk}}\right)\frac{\partial}{\partial x}\left(\frac{D_{kj}}{D_{kk}}\right) - \frac{\partial}{\partial x}\left(\frac{D_{ik}}{D_{kk}}\right)\frac{\partial}{\partial k}\left(\frac{D_{kj}}{D_{kk}}\right), \quad i,j \neq k. \quad (33)$$

In the transition from (26) to (31) $\Delta_{ij} = 0$.

Our next (second) reduction step is the elimination of the component $\widetilde{U}$ of $\mathbf{A}^{(1)} = (\widetilde{U},\widetilde{\Gamma})^T$ [this is allowed since we view $-\omega(=D_{kk})$ as an object of $O(\delta)$]. Transformation (32) applied to matrix (31) then yields a scalar dispersion function

$$D^{(2)} = -\omega_p^2/\omega + [(\omega - kv_0)^2 + (v_0')^2]/\omega \quad (34)$$

associated with the remaining component $A^{(2)} = \widetilde{\Gamma}$. Finally, to $O(\delta)$ $[(dv_0/dx)^2 \sim O(\delta^2)]$

$$D^{(2)} = -[\omega_p^2 - (\omega - kv_0)^2]/\omega, \quad (35)$$

while $\widetilde{\Gamma}$ is described by the *scalar* transport equation of form (25) with $\mathbf{D}$ replaced by $D^{(2)}$, i.e.,

$$-iD^{(2)}\widetilde{\Gamma} = \frac{\partial D^{(2)}}{\partial k}\frac{d\widetilde{\Gamma}}{dx} + \frac{1}{2}\frac{d}{dx}\left(\frac{\partial D^{(2)}}{\partial k}\right)\widetilde{\Gamma}. \quad (36)$$

This equation can be easily solved by using the usual geometric optics approach,[1] i.e., by first defining $k(x)$ via

$$D^{(2)}[k(x),x] = 0 \Rightarrow k(x) = (\omega \pm \omega_p)/v_0. \quad (37)$$

Then, since $k(x)$ is aleady known, Eq. (36) becomes

$$\frac{\partial D^{(2)}}{\partial k}\frac{d\widetilde{\Gamma}}{dx} + \frac{1}{2}\frac{d}{dx}\left(\frac{\partial D^{(2)}}{\partial k}\right)\widetilde{\Gamma} = 0, \quad (38)$$

yielding the solution for the slow-amplitude component

$$\frac{\widetilde{\Gamma}(x)}{\widetilde{\Gamma}(0)} = \left(\left|\frac{\partial D^{(2)}(x)/\partial k}{\partial D^{(2)}(0)/\partial k}\right|_{D^{(2)}=0}\right)^{-1/2} = \left(\frac{v_0(x)\omega_p(x)}{v_0(0)\omega_p(0)}\right)^{-1/2} = \left(\frac{n_0(x)}{n_0(0)}\right)^{1/4} = \left(\frac{v_0(0)}{v_0(x)}\right)^{1/4}. \quad (39)$$

The two remaining components of the original amplitude $\mathbf{A}$ can now be found by reversing the congruent reduction procedure, i.e.,

$$\widetilde{U} = [(\omega - kv_0)/\omega]\widetilde{\Gamma} + O(\delta),$$
$$\widetilde{E} = i(\omega_p/\omega)\widetilde{\Gamma} + O(\delta). \quad (40)$$

Finally, we observe that Eq. (37) yields two possible modes in the system. These modes differ by their energy density sign. Indeed, to the lowest order in δ [see (29)]

$$Q = \mathbf{A}^\dagger \cdot \frac{\partial \mathbf{D}}{\partial \omega} \cdot \mathbf{A} = \frac{\partial D^{(2)}}{\partial \omega}|\widetilde{\Gamma}|^2 = 2\,\frac{\omega - kv_0}{\omega}\,|\widetilde{\Gamma}|^2 = \mp\frac{\omega_p}{\omega}|\widetilde{\Gamma}|^2, \quad (41)$$

where the upper sign corresponds to the positive sign in (37). We shall see in the next section that the possibility of having the negative energy mode is of crucial importance when mode coupling of two inhomogeneous interacting beams is considered.

The last topic being considered in this section is the relativistic generalization of the theory. In the one-dimensional relativistic case the exact momentum and energy equations are

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)(\gamma v) = -\frac{e}{m}E, \quad (42)$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\gamma = -\frac{e}{mc^2}Ev, \quad (43)$$

where the relativistic factor $\gamma = (1 - v^2/c^2)^{-1/2}$. We use Eqs. (43) and (42) and rewrite the latter as

$$\gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v - \frac{ev^2}{mc^2}E = -\frac{e}{m}E, \quad (44)$$

or

$$m\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)v + \frac{e}{\gamma^3}E = 0. \quad (45)$$

Linearization of (45) yields

$$m\left(\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}(v_0v_1)\right) + \frac{e}{\gamma_0^3}E_1 - \frac{3eE_0}{\gamma_0^4}\gamma_1 = 0, \quad (46)$$

where

$$1/\gamma_0^2 = 1 - (v_0^2/c^2), \quad (47)$$

$$\gamma_1 = v_0\gamma_0^3 v_1/c^2, \quad (48)$$

and

$$-\frac{ev_0E_0}{m\gamma_0c^2} = \frac{1}{\gamma_0}\frac{d\gamma_0}{dt} = \frac{v_0}{\gamma_0}\frac{d\gamma_0}{dx}. \quad (49)$$

On using (48) and (49) in Eq. (46) we obtain

$$m\left(\frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}(v_0v_1) + \frac{3v_0}{\gamma_0}\frac{d\gamma_0}{dx}v_1\right) + \frac{e}{\gamma_0^3}E_1 = 0. \quad (50)$$

Finally, we define a new variable

$$v_1' = \gamma_0^3 v_1 \tag{51}$$

for which (50) becomes

$$m\left(\frac{\partial v_1'}{\partial t} + \frac{\partial}{\partial x}(v_0 v_1')\right) + eE = 0. \tag{52}$$

This equation must be combined with linearized Maxwell and continuity equations [Eqs. (10) and (11)]:

$$\epsilon_0 \frac{\partial E_1}{\partial t} - e\Gamma_1 = 0,$$

$$\frac{\partial n}{\partial t} + \frac{\partial \Gamma_1}{\partial x} = 0, \tag{53}$$

where now [compare to (13)]

$$n_1 = (1/v_0)(\Gamma_1 - n_0 v_1'/\gamma_0^3). \tag{54}$$

By introducing, similarly to the nonrelativistic case, new dependent variables

$$E = \epsilon_0^{1/2} E_1,$$

$$U = (mn_0)^{1/2} v_1',$$

$$\Gamma = (m/n_0)^{1/2}\Gamma_1, \tag{55}$$

we transform (52) and (53) into

$$\frac{\partial E}{\partial t} - \omega_p \Gamma = 0,$$

$$\frac{\partial \Gamma}{\partial t} + v_0 \frac{\partial \Gamma}{\partial x} - \left(\frac{\Gamma}{2}\right)\frac{dv_0}{dx} - \left(\frac{1}{\gamma_0^3}\right)\frac{\partial U}{\partial x} = 0,$$

$$\frac{\partial U}{\partial t} + v_0 \frac{\partial U}{\partial x} + \left(\frac{3U}{2}\right)\frac{dv_0}{dx} + \omega_p E = 0, \tag{56}$$

which is the desired Hermitian form for the relativistic case [compare to Eqs. (20)–(22)] characterized by the dispersion matrix [compare to (26)]:

$$\mathbf{D} = \begin{bmatrix} \omega & 0 & -i\omega_p \\ 0 & -\omega/\gamma_0^3 & \omega - kv_0 - iv_0' \\ i\omega_p & \omega - kv_0 + iv_0' & 0 \end{bmatrix}. \tag{57}$$

In conclusion, we see that formally the transition to the one-dimensional relativistic theory is equivalent to the replacements

$$v_1 \Rightarrow \gamma_0^3 v_1,$$

$$n_0 \Rightarrow n_0/\gamma_0^3, \tag{58}$$

in the nonrelativistic description. For example, in the reduced relativistic case $\widetilde{\Gamma}$ is again described by the scalar transport equation of form (36), where [see Eq. (35)]

$$D^2 = -[\omega_p^2/\gamma_0^3 - (\omega - kv_0)^2]/\omega \tag{59}$$

so that

$$k(x) = (\omega \pm \omega_p/\gamma_0^{3/2})/v_0. \tag{60}$$

Then Eq. (39) yields

$$\frac{\widetilde{\Gamma}(x)}{\widetilde{\Gamma}(0)} = \left(\frac{v_0(x)\omega_p(x)\gamma_0^{3/2}(0)}{v_0(0)\omega_p(0)\gamma_0^{3/2}(x)}\right)^{-1/2} = \left(\frac{\gamma_0^3(x)n_0(x)}{\gamma_0^3(0)n_0(0)}\right)^{1/4}. \tag{61}$$

Also, Eqs. (58) allow us to use directly results (28) and (30) for calculating the relativistic energy density and flux ($Q = A^{\dagger}\cdot\partial\mathbf{D}/\partial\omega\cdot\mathbf{A}$ and $J = -A^{\dagger}\cdot\partial\mathbf{D}/\partial k\cdot A$). The results are

$$Q = 4\langle \epsilon_0 E_1^2/2 + \gamma_0^3 n_0 m v_1^2/2 + mv_0\gamma_0^3 n_1 v_1\rangle_{\text{av}} \tag{62}$$

and

$$J = 4mv_0\gamma_0^3\langle v_1\Gamma_1\rangle_{\text{av}}. \tag{63}$$

This checks (again within a factor of 4) with what one would obtain from the small-amplitude conservation of energy for relativistic e beams.[7]

IV. COUPLED BEAM PROBLEM

Consider two coupled thin-sheet beams propagating parallel to each other in x direction with a free space separation between the beams. The coupling between the beams is assumed to be the result of the extension of the electric field in free space. We shall also restrict ourselves, for simplicity, to the nonrelativistic case.

The linearized momentum equations describing the beams can be written as

$$m\left(\frac{\partial v_{1i}}{\partial t} + \frac{\partial(v_{0i}v_{1i})}{\partial x}\right) + e\mathscr{E}_{1i} = 0, \tag{64}$$

where $\mathscr{E}_{1i}$ is the electric field perturbation acting on the ith beam. We write $\mathscr{E}_{1i}$ as a linear combination

$$\mathscr{E}_{1i} = R_i^2 E_{1i}' + C^2 E_{1j}', \tag{65}$$

where either $i=1, j=2$, or $i=2, j=1$, and E_{1i}' are the electric fields one would have for fully decoupled and infinite cross-section beams, i.e., by definition,

$$\epsilon_0 \frac{\partial E_{1i}'}{\partial t} - e\Gamma_{1i} = 0. \tag{66}$$

The "heuristic" parameters R_i^2 and C^2 in our model are slowly varying self-field reduction factors resulting for the thin-sheet beam i and the coupling coefficient, respectively. The rigorous derivation of the expressions for these parameters is out of the scope of the present study. Finally, we write the linearized continuity equations for the beams

$$\frac{\partial n_{1i}}{\partial t} + \frac{\partial \Gamma_{1i}}{\partial x} = 0 \tag{67}$$

or, on using $n_{1i} = (\Gamma_{1i} - n_{0i}v_{1i})/v_{0i}$,

$$\frac{\partial \Gamma_{1i}}{\partial t} + v_{0i}\frac{\partial \Gamma_{1i}}{\partial x} - n_{0i}\frac{\partial v_{1i}}{\partial t} = 0. \tag{68}$$

In order to symmetrize the momentum and Ampère–Maxwell equations (64) and (66) we now introduce new fields E_{1i} defined by the linear transformation

$$E_{1i} = \alpha_i E_{1i}' + \beta E_{1j}', \tag{69}$$

where α_i and β are parameters to be related to R_i^2 and C^2. The new fields are described by the equations

$$\epsilon_0 \frac{\partial E_{1i}}{\partial t} - e(\alpha_i\Gamma_{1i} + \beta\Gamma_{1j}) = 0. \tag{70}$$

At this point we make the choice of α_i and β so that Eq. (64) expressed in terms of E_{1i} becomes

$$m\left(\frac{\partial v_{1i}}{\partial t}+\frac{\partial(v_{0i}v_{1i})}{\partial x}\right)+\alpha_i eE_{1i}+\beta eE_{1j}=0. \quad (71)$$

The proper choice is

$$\alpha_i^2+\beta^2=R_i^2,$$
$$\beta(\alpha_1+\alpha_2)=C^2. \quad (72)$$

Note now that the complete system of equations (70), (71), and (68) in the new variables yields the conservation law in the general form (15), where W and S are

$$W=\sum_{i=1,2}\left(\frac{\epsilon_0 E_{1i}^2}{2}+n_{0i}\frac{mv_{1i}^2}{2}+mv_{0i}v_{1i}n_{1i}\right),$$

$$S=\sum_{i=1,2} mv_{0i}v_{1i}\Gamma_{1i}. \quad (73)$$

At this stage, similarly to the single beam case, we define new dependent variables

$$E_i=\epsilon_0^{1/2}E_{1i},$$
$$U_i=(mn_{0i})^{1/2}v_{1i},$$
$$\Gamma_i=(m/n_{0i})^{1/2}\Gamma_{1i}, \quad (74)$$

and use the eikonal representation for the unknown six-component vector

$$\mathbf{Z}=\begin{bmatrix}E_1\\U_1\\\Gamma_1\\E_2\\U_2\\\Gamma_2\end{bmatrix}=\operatorname{Re}\left[\begin{bmatrix}\tilde E_1\\\tilde U_1\\\tilde\Gamma_1\\\tilde E_2\\\tilde U_2\\\tilde\Gamma_2\end{bmatrix}\exp[-i\omega t+i\psi(x)]\right]$$
$$=\operatorname{Re}\{\mathbf{A}(x)\exp[-i\omega+i\psi(x)]\}. \quad (75)$$

The slow amplitude $\mathbf{A}$ in (75) is again described by the Hermitian transport equation of form (25), where in the present case [compare to Eq. (26)]

$$\mathbf{D}=\begin{bmatrix}\omega & 0 & -i\alpha_1\omega_{p1} & 0 & 0 & -i\beta\omega_{p_2}\\ 0 & -\omega & \omega-kv_{01}-iv_{01}' & 0 & 0 & 0\\ i\alpha_1\omega_{p1} & \omega-kv_{01}-iv_{01}' & 0 & i\beta\omega_{p1} & 0 & 0\\ 0 & 0 & -i\beta\omega_{p1} & \omega & 0 & -i\alpha_2\omega_{p2}\\ 0 & 0 & 0 & 0 & -\omega & \omega-kv_{02}-iv_{02}'\\ i\beta\omega_{p2} & 0 & 0 & i\alpha_2\omega_{p2} & \omega-kv_{02}+v_{02}' & 0\end{bmatrix}. \quad (76)$$

Now, in searching for embedded pairwise mode coupling situations, we proceed to the reduction of order, assuming slow variation of the equilibrium beam parameters [$n_{0i}(x)$ and $v_{0i}(x)$]. We eliminate components $\tilde E_1$ and $\tilde E_2$ of $\mathbf{A}$ first. The reduced amplitude $\mathbf{A}^{(1)}=(\tilde U_1,\tilde\Gamma_1,\tilde U_2,\tilde\Gamma_2)^T$ after this reduction step is described by the reduced dispersion matrix [the rule (32) was used twice to get this result]:

$$\mathbf{D}^{(1)}=\begin{bmatrix}-\omega & \omega-kv_{01}-iv_{01}' & 0 & 0\\ \omega-kv_{01}+iv_{01}' & -(\alpha_1^2+\beta^2)\omega_{p1}^2/\omega & 0 & -(\alpha_1+\alpha_2)\beta\omega_{p1}\omega_{p2}/\omega\\ 0 & 0 & -\omega & \omega-kv_{02}-iv_{02}'\\ 0 & -(\alpha_1+\alpha_2)\beta\omega_{p1}\omega_{p2}/\omega & \omega-kv_{02}+iv_{02}' & -(\alpha_2^2+\beta^2)\omega_{p2}^2/\omega\end{bmatrix}. \quad (77)$$

Next, we reduce $\tilde U_1$ and $\tilde U_2$ [the corresponding diagonal elements $(-\omega)$ are assumed being of $O(1)$]. The two-component reduced amplitude $A^2=(\tilde\Gamma_1,\tilde\Gamma_2)^T$ is then described by the dispersion matrix

$$\mathbf{D}^{(2)}=\begin{bmatrix}D_1 & \eta\\ \eta & D_2\end{bmatrix}, \quad (78)$$

where to $O(\delta)$

$$D_{i=1,2}=[(\omega-kv_{0i})^2-(\alpha_i^2+\beta^2)\omega_{pi}^2]/\omega, \quad (79)$$
$$\eta=-(\alpha_1+\alpha_2)\beta\omega_{p1}\omega_{p2}/\omega. \quad (80)$$

Hermitian dispersion matrix $\mathbf{D}^{(2)}$ has the standard form allowing local pairwise linear mode conversion situations. The mode conversion happens when in some region in space, $\eta\sim O(\delta)$ (normal degeneracy[6]) and both $D_1[k(x),x]$ and $D_2[k(x),x]$ become simultaneously small [also of $O(\delta)$]. In our case this happens when, for instance, the coupling constant β is of $O(\delta)$ (the beams are sufficiently far away from each other) and, simultaneously,

$$D_{i=1,2}\approx[(\omega-kv_{0i})^2-\alpha_i^2\omega_{pi}^2]/\omega\sim O(\delta). \quad (81)$$

These conditions are always satisfied near the crossing point x_0 such that $D_1[k(x_0),x_0]=D_2[k(x_0),x_0]$. An interesting situation occurs when the crossing is between the negative energy mode of beam 1 and the positive energy mode of beam 2, i.e., when [see the definitions in Eqs. (37) and (41)]

$$(\omega-kv_{01}+\alpha_1\omega_{p1})_{x=x_0}=(\omega-kv_{02}-\alpha_2\omega_{p2})_{x=x_0}=0. \quad (82)$$

Near the crossing

$$D_1\approx-(2\alpha_1\omega_{p1}/\omega)(\omega-kv_{01}+\alpha_1\omega_{p1}),$$
$$D_2\approx+(2\alpha_2\omega_{p2}/\omega)(\omega-kv_{02}-\alpha_2\omega_{p2}). \quad (83)$$

Therefore, if one initially excites mode 1, the transmission coefficient for this mode after passing the crossing point is [see Eq. (15) in Ref. 8 with the substitution $(-1) = \exp(\pm i\pi)$]

$$T = (|\widetilde{\Gamma}_1|^2_{\text{out}}/|\widetilde{\Gamma}_1|^2_{\text{in}}) = \exp(\pm 2\pi\eta^2 |B|_{x=x_0}), \quad (84)$$

where B is the Poisson bracket

$$B = \frac{\partial D_1}{\partial x}\frac{\partial D_2}{\partial k} - \frac{\partial D_1}{\partial k}\frac{\partial D_1}{\partial x}.$$

In the final result of Ref. 8 it was stated that, based upon causality, the negative sign in the expression for T should be used. However, this is in general not correct. In addition to causality, one must take into account the conservation of energy flux, especially in systems where the small-amplitude energies may be both positive and negative. Indeed, for the two coupled waves of Eq. (81):

$$J = -\mathbf{A}^{\dagger}\cdot\frac{\partial \mathbf{D}}{\partial k}\cdot\mathbf{A}$$

$$\approx -\mathbf{A}^{(2)\dagger}\cdot\frac{\partial \mathbf{D}^{(2)}}{\partial k}\cdot\mathbf{A}^{(2)}$$

$$= (2/\omega)(\omega_{p2}\alpha_2 v_{02}|\widetilde{\Gamma}_2|^2 - \omega_{p1}\alpha_1 v_{01}|\widetilde{\Gamma}_1|^2)$$

$$= \text{const}(x). \quad (85)$$

If $v_{01}, v_{02} > 0$ and before the crossing $\widetilde{\Gamma}_1 = \Gamma_{10}$ and $\widetilde{\Gamma}_2 = 0$, then, after the crossing and excitation of $\widetilde{\Gamma}_2$ ($|\widetilde{\Gamma}_2|^2 > 0$), $|\widetilde{\Gamma}_1|^2$ must also grow to satisfy Eq. (85). Thus, the sign in (84) is positive and we have an "amplification" of mode 1 resulting from the coupling. The reason for such an "unstable" coupling are the *opposite* signs in the contributions of modes in Eq. (85), which, in turn, is the result of the *opposite* energy density sign of the two coupled modes. The calculation of the transmission (amplification) coefficient (84) can now be completed by using approximation (83) for local dispersion functions of the coupled modes. After some algebra one obtains

$$T = \exp\left(\frac{\pi\omega_{p1}\omega_{p2}(\alpha_1+\alpha_2)^2\beta^2/2\alpha_1\alpha_2}{|v_{01}\omega'_{p2}[(2kv_{02}/\omega_{p2}) - \alpha_2] - v_{02}\omega'_{p1}[(2kv_{01}/\omega_{p1}) + \alpha_1]|}\right). \quad (86)$$

A much simpler expression for T is obtained when, for instance, $v_{02} = 0$. Then

$$T = \exp\left(\frac{\pi\omega_{p1}\omega_{p2}(\alpha_1+\alpha_2)^2\beta^2}{2\alpha_1\alpha_2^2 v_{01}\omega'_{p2}}\right). \quad (87)$$

In the Appendix we give an independent derivation of this result by solving the differential equation for the small-amplitude beam current density in a thin-sheet beam coupled to a thin-sheet plasma.

Finally, let us consider the conditions for the mode crossing in our system. First, in Fig. 1, we show the typical

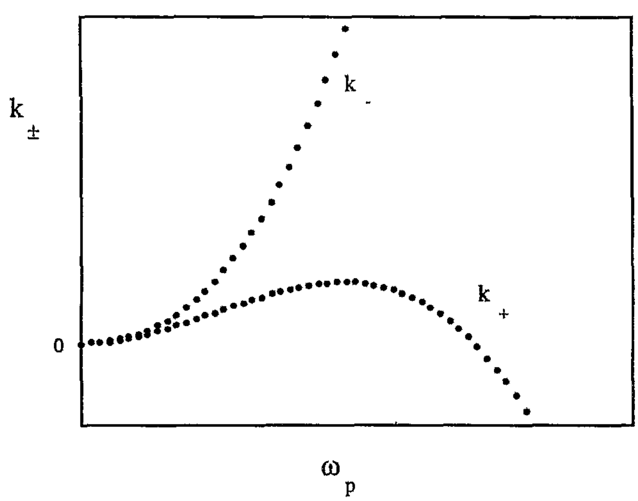

FIG. 1. The typical variation with $\omega_p \sim [n_0(x)]^{1/2}$ of the positive and negative energy mode wave vectors $k_\pm(x) \sim \omega_p^2(\omega \pm \omega_p)$ in a single beam case and for a fixed frequency ω.

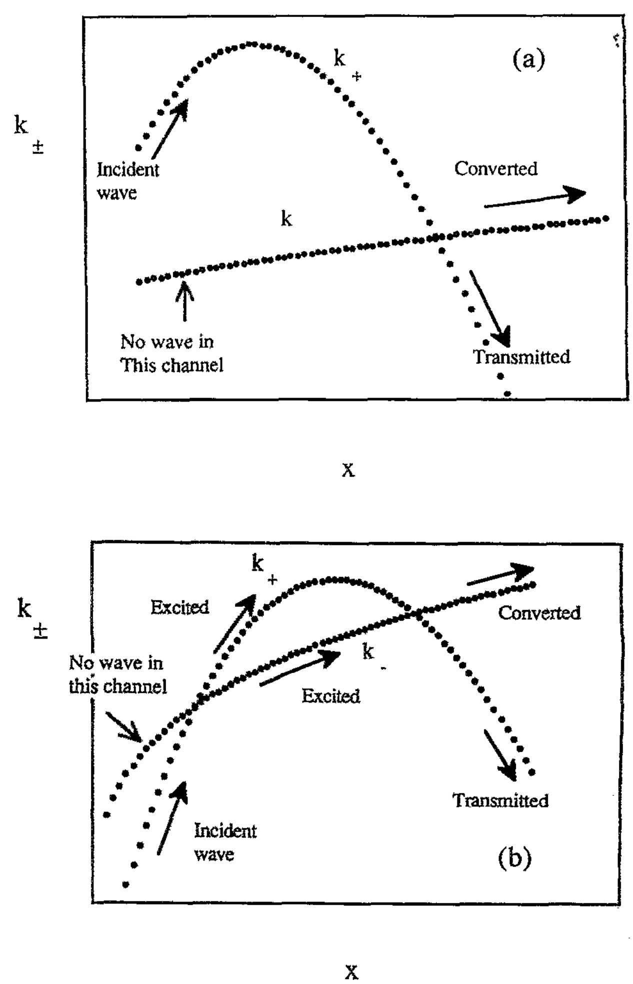

FIG. 2. Possible crossing events in the decelerating two-beam system.

dependence of $k_{\pm}(x)$ (positive and negative energy mode wave vectors) on ω_p. Obviously, the two modes do not cross (unless, $\omega_p \to 0$). In the two-beam case, in contrast, the crossing is possible at a point x_0 at which

$$\frac{\omega + \alpha_1\omega_{p1}}{v_{01}} = \frac{\omega - \alpha_2\omega_{p2}}{v_{02}} \Rightarrow \omega = \frac{v_{01}\alpha_2\omega_{p2} + v_{02}\alpha_1\omega_{p1}}{v_{01} - v_{02}}. \quad (88)$$

The only necessary condition for (88) is $v_{01} > v_{02}$. Furthermore, it follows from the characteristic dependences of $k_{\pm}$ on ω_p (see Fig. 1), that the crossings for the two-beam system when the beams are *decelerated* in the x direction are as shown in Fig. 2. We observe that, in some cases, two coupling events [Fig. 2(b)] are possible here. The analysis above was related to a single crossing and initial excitation of only one mode at the entrance to the crossing. Only the crossing in Fig. 2(a) corresponds to this case. The crossing of the two excited modes [Fig. 2(b)] and further flux redistribution, possibly accompanied by an additional energy gain due to the coupling, comprises a more difficult problem lying outside the scope of the present work.

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APPENDIX: INHOMOGENEOUS PLASMA-BEAM COUPLING

In this appendix we formulate and solve the differential equation for the beam current density in a weakly coupled system of a thin-sheet electron beam and a thin-sheet plasma. The two thin-sheet systems can be imagined separated by free space. They are then coupled by the fields from one thin sheet extending to and acting on the electrons of the other thin sheet.

We use a cold-plasma model for both sheets and assume one-dimensional dynamics in (x,t), as would be the case in a strongly confining, external magnetic field. Also, for simplicity only, we ignore here relativistic effects. For comparing the results with Eq. (87) we consider the unperturbed beam density to be a constant, n_{b0}, and the unperturbed beam velocity to be also a constant, v_0, and in the x direction; however, the unperturbed plasma density $n_{p0}(x)$ is allowed to vary in the x direction of beam fluid.

For the small-amplitude, perturbed variables it is convenient to use the beam kinetic voltage $U_b = (mv_0/-e)v_b$, where v_b is the perturbed beam velocity, the beam current density $J_b = -e(n_{b0}vb + n_b v_0)$ where n_b is the perturbed beam density, and plasma current density $J_p = -en_{p0}(x)v_p$, where v_p is perturbed plasma velocity. Assuming a time dependence $\exp(-i\omega t)$ for all perturbed variables, including the electrostatic fields, the small-amplitude continuity and force equations (68) and (64) with (65) for the thin-sheet beam weakly coupled to the thin-sheet plasma are

$$L_x J_b = -i\omega\epsilon_0 k_b^2 U_b, \quad (A1)$$

$$L_x U_b = R_b^2 E_b + C^2 E_p. \quad (A2)$$

Here $L_x = [(d/dx) - i(\omega/v_0)]$ with $\omega_b = (e^2 n_{b0}/m\epsilon_0)^{1/2}$ the beam plasma frequency, R_b^2 is the field reduction factor for the thin-sheet beam, which we allow to be a function of x (more generally it is an operator in x), and C^2 is the weak-coupling coefficient between the beam and plasma thin sheets, which we can also allow to vary in x. The force equation for the thin-sheet plasma weakly coupled to the thin sheet beam is

$$-i\omega\epsilon_0 J_p = \omega_p^2(x)(R_p^2 E_p + C^2 E_b), \quad (A3)$$

where $\omega_p = [e^2 n_{p0}(x)/m\epsilon_0]^{1/2}$, R_p^2 is the fluid reduction factor for the thin-sheet plasma (also allowed to be a function of x), and C^2 is the same as in Eq. (A2) to assure an overall conservative coupled system. Finally we have the Maxwell–Ampère equations (66):

$$i\omega\epsilon_0 E_b = J_b, \quad (A4)$$

$$i\omega\epsilon_0 E_p = J_p. \quad (A5)$$

Equations (A1)–(A5) describe the small-amplitude dynamics of the coupled system.

The differential equation for J_b is obtained as follows. Take L_x (A1) and in the resultant equation use (A2) with (A4) and (A5) to obtain

$$L_x(L_x J_b) = -k_{bR}^2 [J_b + (C^2/R_b^2) J_p], \quad (A6)$$

where $k_{bR} = \omega_{bR}/v_0$ with $\omega_{bR} = R_b\omega_b$ the reduced plasma frequency of the beam. Next, use (A4) and (A5) in (A3) to find

$$\epsilon_{pR}(x) J_p = (C^2/R_p^2)[\omega_{pR}^2(x)/\omega^2] J_b, \quad (A7)$$

where $\omega_{pR} = R_p\omega p$, and

$$\epsilon_{pR}(x) = 1 - \omega_{pR}^2(x)/\omega^2 \quad (A8)$$

is the permittivity function for the thin-sheet, inhomogeneous plasma. Combining (A6) and (A7), and letting

$$J_b = \hat{J}_b(x)\exp(i\omega x/v_0), \quad (A9)$$

we obtain the sought after differential equation

$$\epsilon_{pR}(x)\left(\frac{d^2\hat{J}_b}{dx^2} + k_{bR}^2\hat{J}_b\right) = \chi^2 \frac{\omega_{bR}^2(x)}{v_0^2\omega^2}\hat{J}_b, \quad (A10)$$

where

$$\chi = C^2/R_b R_p \quad (A11)$$

and $\chi^2 \ll 1$ by the initial assumption of weak coupling. Note that (A10) represents the coupling between the plasma oscillations modes $\epsilon_{pR} = 0$ and the two beam modes, the negative and positive energy waves $[(\omega/v_0) \pm k_{bR}]$ in the beam.

We can now proceed to solve (A10) for a simple example to compare with Eq. (87). Assume χ^2 is independent of x, and let

$$\omega_{pR}^2(x) = \omega^2 + ax, \quad (A12)$$

where a is a constant. The plasma sheet permittivity (A8) becomes

$$\epsilon_{pR} = -ax/\omega^2 \quad (A13)$$

and note that $a = 2\omega_{pR}(d\omega_{pR}/dx)$. Introducing (A13) into (A10); and changing to a normalized coordinate

$$\xi = (1 - x^2)^{1/2} k_{bR} x \approx k_{bR} x, \quad (A14)$$

we find

$$\frac{d^2 \hat{J}_b}{d\xi^2} + \left(1 - \frac{\eta}{\xi}\right) \hat{J}_b = 0, \quad (A15)$$

where

$$\eta = \chi^2 \omega^2 k_{bR} / a (1 - \chi^2)^{1/2} \approx \chi^2 \omega^2 k_{bR} / a. \quad (A16)$$

Thus we obtained the Budden equation with asymptotic solutions giving

$$\frac{|J_b(\xi = +\infty)|^2}{|J_b(\xi = -\infty)|^2} = e^{\pm \pi\eta} = \exp\left[\pm \frac{\pi}{2} \left(\frac{C^2}{R_b R_p} \right)^2 R_b \frac{\omega_b}{v_0} \frac{\omega_{p0}}{\omega'_{p0}} \right], \quad (A17)$$

where we have used (A11) and (A12), written $[d\omega_p(x=0)/dx] = \omega'_{p0}$. The sign in the exponent in Eq. (A17) depends on whether one excites the positive or negative energy beam modes at $\xi = -\infty$ with the positive (amplification) sign corresponding to the *unstable* coupling between the negative energy beam mode and the positive energy plasma oscillations. Using the relations in Eq. (72), we find that (A17) with the positive sign is exactly the same result obtained in the text, Eq. (87), for the same problem.

In conclusion, it should be noted that the cold beam–plasma interaction is rather singular. When both the beam and the plasma are homogeneous, small-amplitude perturbations evolve in an absolute instability manner and the assumption of a steady-state response is meaningless. When the electrons in the plasma are given a finite drift velocity in the direction of the beam (or a finite thermal spread in velocities)[9] the instability becomes convective and solving for a response at a given frequency is meaningful. For the inhomogeneous plasma and uniform beam treated in this appendix we have found the response of the weakly coupled system at a real frequency. For the plasma with an arbitrarily small velocity in the direction of the beam and the negative energy beam mode excited at $x = -\infty$ the result of Eq. (87) is then appropriate.

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