Electron-beam confinement by rotational stabilization in a linear wiggler free electron laser

L. Friedland and R. E. Shefer
Science Research Laboratory, Inc., Somerville, Massachusetts 02143

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Finite radius electron-beam transport in a linear wiggler free-electron laser with a guide magnetic field is investigated. The addition of a guide magnetic field improves the beam confinement, but also leads to a detrimental drift in the direction transverse to the wiggler magnetic field. The introduction of a rotational transformation of the wiggler magnetic field is proposed to further improve the beam confinement. It is shown that the transformation results in a stable, uniform, solid body rotation of the beam provided (a) the guide field is larger than the amplitude of the wiggler field, (b) the electron drift velocity is much smaller than the rotation speed of the wiggler field as seen by the beam, and (c) the wiggler field rotates in the direction opposite to the direction of the electron gyromotion in the guide field. Theoretical predictions of the improvement of the radial beam transport with the introduction of the rotation of the wiggler field are confirmed in numerical simulations.

I. INTRODUCTION

Linear wigglers based on permanent rare-earth cobalt magnets are now widely used in free-electron lasers (FELs). Among the numerous advantages of using the permanent magnet technology in constructing wigglers for FELs are engineering simplicity, large field amplitude on the axis, adjustability, and ease of access to the beam line. There exists, however, a serious disadvantage of linear wigglers for FEL applications, i.e., the poor beam stability in the direction transverse to the wiggler magnetic field. The problem of beam transport becomes especially important in collective, low-gamma FELs.

Conventionally, quadrupole magnetic fields are added to the linear wiggler for stabilization, complicating the configuration significantly. Alternatively, the use of curved or canted poles and the addition of a guide magnetic field have been investigated. Canted magnets introduce focusing in the transverse direction, but, like quadrupoles, lead to slow betatron oscillations in the transverse plane. Focusing without the introduction of betatron oscillations may be obtained using curved pole faces. However, this technique is very difficult to implement in a pure permanent magnet design. A sufficiently strong guide field may also improve the transport, but leads at the same time to transverse drifts of off-axis electrons.

In the present work a new stabilization scheme is proposed, i.e., the rotation of the direction of polarization of the wiggler magnetic field in the presence of a guide magnetic field. Under certain conditions this technique yields a solid body rotation of the electron beam, thus preserving the circular beam cross section along the length of the wiggler. In Sec. II, the finite radius beam dynamics in the combined linear wiggler and guide magnetic fields is considered. The transverse electron drift is described by using an averaging technique. Slow time-scale equations of motion derived in Sec. II are generalized in Sec. III to describe beam transport with the addition of a slow rotation of the direction of polarization of the wiggler field. Parameter regimes corresponding to stable beam propagation are investigated in Sec. III using these slow time-scale equations. Finally, the results of the calculations are summarized in Sec. IV. Numerical simulations are employed throughout the paper in order to both check the predictions of the analytic theory and to add such important effects as a finite emittance and self-space-charge fields.

II. BEAM DYNAMICS IN CROSSED LINEAR WIGGLER AND GUIDE MAGNETIC FIELDS

Consider a finite radius relativistic electron beam propagating along the axis of combined linear wiggler and guide magnetic fields given by

$$\mathbf{B} = F(z)B_1 \cos(k_0\varphi)\cosh(k_0y) \hat{\varphi} + [B_0 - F(z)B_1 \sin(k_0\varphi)\sinh(k_0y)] \hat{z},$$

where $k_0$ is the wiggler wave number, the magnitude of the guide field $B_0$ is a constant, and the transition function $F(z)$ describes the adiabatic tapering of the wiggler field amplitude, $B_1$, at the entrance. In the following, we shall use

$$F(z) = [1 + \exp(-z/L)]^{-1},$$

where $L$ is the adiabatic transition scale length $L \gg 2\pi/k_0$.

Neglecting, at this stage, the self-fields of the electron beam, we write the momentum equations describing the individual electrons orbits as
\[ u_x = -\frac{\Omega_{\parallel}}{\gamma} - \frac{\Omega_{\parallel}}{\gamma} F \sinh(k_0y)\sin(k_0z) \]
\[ + u_x \frac{\Omega_{\parallel}}{\gamma} \cosh(k_0y)\sin(k_0z) , \]
\[ \dot{u}_x = u_x \left( -\frac{\Omega_{\parallel}}{\gamma} - \frac{\Omega_{\parallel}}{\gamma} \sinh(k_0y)\sin(k_0z) \right) , \]
\[ \dot{u}_z = -u_x \frac{\Omega_{\parallel}}{\gamma} \cosh(k_0y)\cos(k_0z) , \]

where \( u = \nu/c \) is the electron velocity normalized to the speed of light, \( \cdot \cdot \cdot = d(\cdot \cdot \cdot)/dt \), \( \tau = ct \), and \( \Omega_{\perp} = eB_{\perp}/mc^2 \). Note that this definition of the cyclotron frequency \( \Omega_{\perp} \) differs from the conventional one by a factor of \( 1/c \), and thus \( \Omega_{\perp} \) has dimensions of \( \text{cm}^{-1} \). We now assume the existence of two different spatial scales, i.e., the short spatial scale \( L_s \) defined by the wiggler period: \( L_s = \lambda_w = 2\pi/k_0 \) and the long spatial scale
\[ L_l = \min \left( L, \frac{2\pi\gamma}{\Omega_{\perp}}, \frac{2\pi\gamma}{\Omega_{\parallel}} \right) , \]

such that
\[ L_s/L_l \approx \delta < 1 . \]

Since the transition scale length \( L_s \) is much longer than \( L_l \), by definition, Eq. (5) implies that the beam must be sufficiently relativistic such that
\[ k_0\gamma \gg \max(\Omega_{\perp}, \Omega_{\parallel}) . \]

Equation (3) may be expanded in the small parameter \( \delta \) [see Eq. (5)] to obtain solutions of the form
\[ u_x = u_{xs} + w_x \cosh(k_0y)\sin(k_0z) + O(\delta^2) , \]
\[ u_y = u_{ys} + w_y \cosh(k_0y)\cos(k_0z) + O(\delta^3) , \]
\[ u_z = u_{zs} + O(\delta^3) , \]

where \( u_{xs}, u_{ys}, u_{zs}, w_x, w_y, \) and \( y \) are slowly varying quantities in the sense that if \( A \) is one of these quantities then
\[ L_s \left| \frac{1}{A} \frac{dA}{dt} \right| \sim O(\delta) . \]

We also assume that
\[ w_x \sim O(\delta) , \quad w_y \sim O(\delta^2) , \quad u_{xs} \sim O(\delta) , \]
\[ u_{ys} \sim O(\delta^2) , \quad u_{zs} \sim O(1) . \]

This ordering assumption will be tested \textit{a posteriori} for consistency. By substituting Eqs. (7) into Eqs. (3), keeping the lowest significant order, and averaging over the short spatial scale, we get
\[ \dot{u}_{xs} = -u_{ys} \frac{\Omega_{\parallel}}{\gamma} , \]
\[ \dot{u}_{ys} = u_{xs} \frac{\Omega_{\parallel}}{\gamma} - u_x \frac{\Omega_{\parallel}}{4\gamma} \sinh(2k_0y) , \]
\[ \dot{u}_{zs} = 0 . \]

Thus \( u_{zs} = \text{const} \). Evaluating the rapidly varying part of (3) yields
\[ \frac{\Omega_{\parallel} F}{\gamma k_0} \sim O(\delta) , \]
\[ \frac{w_x}{\gamma k_0 m_{\perp}} = \frac{\Omega_{\parallel} U_x}{\gamma^2 k_0^2 m_{\perp}} \sim O(\delta^3) . \]

The ordering assumptions (9) are indeed satisfied, as can be seen from Eqs. (10) and (11). The slow-scale equations (10) can be written in the vector form, i.e.,
\[ \dot{u}_x = \frac{\Omega_{\parallel}}{\gamma} \hat{e}_z \times u_x + F_y \hat{e}_y , \]

where the effective slowly varying "force" \( F_y \) is given by
\[ F_y = -\frac{\Omega_{\parallel}^2 \nu^2}{4k_0^2 \gamma} \sinh(2k_0y) . \]

Equation (13) represents a \textit{ponderomotive-type} force, i.e., has its origin in the vector product between the rapidly oscillating \textit{off-axis} \( z \) component of the wiggler field \( B_{\text{osc}} = -B_{\parallel} F \sinh(k_0y)\sin(k_0z) \) and \( x \) component of the electron velocity \( u_x \nu \sin(k_0y)\sin(k_0z) \) averaged over the fast-time scale. One recognizes, of course, that Eq. (13) describes the usual focusing force in linear wigglers without the guide field. The general solution of (12) is
\[ u_{ys} = u_0 \cos \left( \frac{\Omega_{\parallel}}{\gamma} \tau + \phi_0 \right) , \]
\[ u_{zs} = u_0 \sin \left( \frac{\Omega_{\parallel}}{\gamma} \tau + \phi_0 \right) + u_d , \]

where the drift velocity in the \( x \) direction is given by
\[ u_d = \frac{\Omega_{\parallel}^2}{4k_0^2 \gamma} \sinh(2k_0y) . \]

and \( \phi_0 \) is the initial phase. The first equation in (14) predicts slow betatron oscillations in the \( y \) direction, the amplitude of which can be negligible if the transverse velocity \( u_0 \) is sufficiently small. This transverse velocity depends on the beam emittance and entrance conditions into the wiggler. It will be shown below that reasonable values of \( u_0 \) are sufficiently small to justify our assumption of the approximate constancy of \( y \) in (7) and in expression (13) for the ponderomotive force. Figure 1 shows the numerical solution of Eqs. (3) in the case \( \gamma = 9.6, \lambda_w = 2 \text{ cm}, B_{\parallel} = 5 \text{ kG}, B_{\perp} = 2 \text{ kG}, \) and \( L = 10 \text{ cm}, \) subject to the following initial conditions (at \( \tau = 0 \)): \( x = 0, y = 0.2 \text{ cm}, z = -15 \text{ cm}, u_x = u_y = 0, \) and \( u_z = (1 - 1/\gamma^2)^{1/2} . \) Fast oscillations with the wiggler period, slow betatron oscillations due to the guide field, and the drift in the \( x \) direction are seen in the figure. The drift motion of the \( \text{off-axis} \) electrons in a linear wiggler FEL with a guide field was first observed in calculations by Pasos et al. In Ref. 6, the drift velocity was obtained from a fit to the numerical data and was found to have the same functional form as Eq. (15). The analytic theory developed above describes the beam transport in
combined linear wiggler and guide magnetic fields. We shall show in the next section how this theory can be modified to describe more complex magnetic field configurations which result in a complete radial confinement and stabilization of the beam.

III. ROTATIONAL TRANSFORMATION

In this section the possibility of a slow spatial variation of the ponderomotive force [see Eq. (13)] is considered. The previous analysis remains valid in this case with Eq. (12) replaced by

\[ \mathbf{u} = -\frac{\Omega}{\gamma} \mathbf{e}_x \times \mathbf{u}_0 + \mathbf{F}_p(\tau), \]  

where \( \mathbf{F}_p \) may vary slowly in amplitude, or direction, or both. Since a constant \( \mathbf{F}_p \) in the \( y \) direction leads to the drift in the \( x \) direction, one can try to suppress the drift by introducing an oscillating ponderomotive force. This may be accomplished by adiabatically rotating the direction of polarization of the wiggler field along the wiggler axis, so that at a given \( z \) the direction of the wiggler field is along the vector

\[ \mathbf{e}_y = \mathbf{e}_x \sin(k_x z) + \mathbf{e}_y \cos(k_x z), \]  

where \( k_x = 2\pi/\lambda_x \) and \( \lambda_x \) \( \lambda_y \) \( \lambda_w \) is the rotation pitch. The wiggler field in the laboratory frame will then be [see Eq. (1)]

\[ B_x = B_0 \cos(k_0 z) \cos(k_0 y) \sin(k_x z), \]  

\[ B_y = B_0 \cos(k_0 z) \cos(k_0 y) \cos(k_x z), \]  

where

\[ \gamma_x = \gamma_x \sin(k_x z) + \gamma_y \cos(k_x z), \]  

is the projection of the radius vector \( r = (x, y, z) \) in the direction of vector \( \mathbf{e}_y \). The slow-scale equation (16) for this case becomes

\[ \mathbf{u} = -\frac{\Omega}{\gamma} \mathbf{e}_y \times \mathbf{u}_0 + f \sinh(2k_0 \mathbf{u}_y) \mathbf{e}_y, \]  

where

\[ f = \frac{1}{\kappa_0^2} \left( \frac{\Omega}{\gamma} \right)^2. \]  

In order to analyze Eq. (20), we first write this equation in component form:

\[ \mathbf{u}_x = -\frac{\Omega}{\gamma} \mathbf{e}_y \times \mathbf{u}_0 - f \sinh(2k_0 \mathbf{u}_y) \sin(k_x z), \]  

\[ \mathbf{u}_y = -\frac{\Omega}{\gamma} \mathbf{e}_y \times \mathbf{u}_0 - f \sinh(2k_0 \mathbf{u}_y) \cos(k_x z). \]  

Here we can use [see Eq. (19)]

\[ z = u_0 \tau \]

and

\[ y = R \sin(k_x z + \beta) = R \sin(\omega \tau + \beta), \]  

where

\[ R = (x^2 + y^2)^{1/2}, \quad \tan \beta = y/x, \quad \omega = k_x u_{z0} \approx k_x, \]

and \( \tau = 0 \) is associated with the axial position \( z = 0 \).

The ponderomotive force terms in (22) are now products of two correlated oscillating functions and thus, in principle, the drift cannot be averaged to zero. Nevertheless, we shall show below that the drift can be reduced by the rotation of the wiggler polarization. Under certain conditions the drift becomes purely azimuthal, thus resulting in the radial stability of the beam. Assuming the existence of a slow azimuthal drift, characterized by a linear velocity \( u_{0y} \) such that

\[ \omega_\beta = \frac{|u_{0y}|}{R} \approx \omega \approx k_x, \]  

Eqs. (22) can be averaged over the “fast” rotation scale to yield

\[ \langle \mathbf{u}_x \rangle = -\langle u_y \rangle \frac{\Omega}{\gamma} - k_0 f \mathbf{R} \cos \beta, \]  

\[ \langle u_y \rangle = \langle u_{0y} \rangle + k_0 f \mathbf{R} \sin \beta. \]  

For simplicity, the analysis will be limited to small values of \( k_0 y \), so that \( \sinh(2k_0 y) \approx 2k_0 y \), in Eqs. (22). By writing

\[ \beta = \omega_\beta \tau + \beta_0, \]  

where \( \beta_0 \) is the initial phase, and by defining

\[ \langle u \rangle = \langle u_{0y} \rangle + i \langle u_{0y} \rangle, \]  

Eqs. (25) can be rewritten as

\[ \langle \mathbf{u} \rangle = \frac{\Omega}{\gamma} (u) - k_0 R f e^{i(\omega_\beta \tau + \beta_0)} \]  

\[ \langle \mathbf{u} \rangle = \frac{\Omega}{\gamma} (u) - k_0 R f e^{i(\omega_\beta \tau + \beta_0)}. \]
This equation yields a rotating solution with constant radius \( R \)
\[
\langle u \rangle = \langle u_0 \rangle e^{i (\omega_D + f_0)}.
\]  

(29)

where
\[
\langle u_0 \rangle = \frac{i k_0 f R}{\omega_D - \Omega/\gamma} = \text{const}.
\]  

(30)

We then observe that
\[
\omega_D = \frac{|\langle u_0 \rangle|}{R} \frac{\text{sign} (\omega_D - \Omega/\gamma)}{\omega_D - \Omega/\gamma} = \frac{k_0 f}{\omega_D - \Omega/\gamma}.
\]  

(31)

The last equation yields two values for \( \omega_D \):
\[
\omega_D = \frac{\Omega}{\gamma} \pm \sqrt{\left( \frac{\Omega}{\gamma} \right)^2 + \frac{\Omega^2}{\gamma^2}}.
\]  

(32)

One can see from Eq. (32) that the slowness condition, Eq. (24), is only satisfied when
\[
(\Omega/\Omega) < 1,
\]  

(33)

and the negative sign in Eq. (32) is used. In this case
\[
\omega_\phi \approx - \Omega^2/4\gamma \Omega_\phi.
\]  

(34)

The linear, azimuthal drift velocity associated with \( \omega_\phi \) in Eq. (34) is given by
\[
\omega_\phi = |\omega_\phi| = R = \frac{\Omega^2 R}{4\gamma \Omega_\phi},
\]  

(35)

which is equal to one half of the drift velocity in the non-rotating wiggler case [see Eq. (15)].

Figure 2(a) shows the results of a numerical solution of the exact momentum equations with the same parameters used in Fig. 1, but with the addition of a rotation with \( k_r = 0.15 \text{ cm}^{-1} \) and an initial transverse velocity \( u_t = 0.006 \phi_z \), representing finite-beam emittance. The figure clearly demonstrates the predicted stabilizing role of the rotational transform. Furthermore, we observe that the azimuthal drift does not alter the form of the cross section of the beam if this cross section is initially circular. In this case the self-fields of the beam can be easily modeled for a uniform density \( N_\phi \).
\[
|E| = 2\pi e N_\phi r, \quad |B| = 2\pi e N_\phi u_{2s},
\]  

(36)

where \( r < r_{beam} \) is the radial position in the beam and \( r_{beam} \) is the beam radius. The self-fields (36) have been included in calculations shown in Fig. 2(a) with the total beam current \( I = 600 \text{ A} \) and \( r_{beam} = 0.2 \text{ cm} \). It may be noted that the case shown in the figure corresponds to \( r < 0.2 \), so that \( 2k_r r = 1.25 \) and therefore the small parameter expansion \( \sinh(2k_r r) \approx 2k_r r \phi_z \), used in deriving Eq. (34), is violated. Nevertheless, the azimuthal drift and the resulting radial stabilization are still apparent. Finally, the averaging technique is tested in Fig. 2(b) which shows the solution of the slow-slope equations (22) for the same parameters as Fig. 2(a). Comparing the two calculations we see that the slow-scale equations correctly slow the evolution of the trajectory and thus can be used in studying the beam transport. This allows us to avoid the numerical difficulties associated with resolving the fast scale in the exact momentum equations when studying the beam behavior for long times.

In order to compare the results of the numerical calculations with the predictions of Eq. (35), slow-scale calculations were performed with the same parameters as Fig. 2(b), but with \( B_t = 10 \text{ kG} \) and the initial values \( x = 0.1 \text{ cm} \) and \( y = 0 \). A much longer integration time of \( \tau = 500 \text{ cm} \) corresponding to a longer wiggler length was used. The results of these calculations are shown in Fig. 3. The dots in the figure represent the mapping of the transverse electron position along the wiggler with the step size \( \Delta \tau = 2 \)
cm. A high degree of radial beam stability can be seen in the figure. Also, in the case considered, \( \omega_D \approx 0.00625 \text{ cm}^{-1} \) so that the azimuthal drift angle during the integration time (\( \tau = 500 \text{ cm} \)) should be \( \Delta \beta = \omega_D \tau \approx \pi \). The 30\% larger angle \( \Delta \beta \approx 1.3 \pi \) obtained in the calculations can be explained by the azimuthal drift with the velocity \( E_{\text{wigg}}/\gamma^2 B_y \) due to the self-fields of the beam.

We now discuss the conditions necessary for stable beam operation in the rotating wiggler configuration. The radial stabilization is destroyed if any of the assumptions which led to the azimuthal drift theory are violated. For example, Fig. 4 shows a case with the same parameters as Fig. 2(a) (exact momentum equations), but with \( I = 0 \) and \( k_y = 0.018 \text{ cm}^{-1} \) so that the rotation is too slow, i.e., \( \omega_D \sim k_y \) in violation of (24). These conditions result in a gradual but steady increase in the radial position of the test electron. Another example of poor stability is shown in Fig. 5, in which the same parameters as Fig. 2(a) are again used except \( I = 0 \) and \( B_y = B_l (= 2 \text{ kG}) \) in violation of condition (33). In this case the drift is too fast and the circular motion is only a very rough approximation of the actual trajectory shown in Fig. 5. Finally, the condition that determines which of the two solutions \( \omega_D \) and \( \omega_D \) is valid was investigated. We found that the solution is determined by the direction of the rotation of the wiggler polarization relative to the cyclotron motion in the guide field. The situation of rotation in the wrong direction is shown in Fig. 6. The parameters are again as in Fig. 2(a), but \( I = 0 \) and \( k_y = -0.15 \text{ cm} \), so that the wiggler polarization vector rotates in the direction of the electron gyration in the guide field. The figure shows that a very rapid azimuthal electron motion results. The motion is characterized by the fast frequency, \( \omega_D \approx \Omega_y/\gamma_c \), is radically unstable, and thus inappropriate for FEL applications.

Finally, we shall address the important issue of the FEL radiation coupling to the rotating electron beam. The wiggle velocity in the rotating beam is always perpendicular to the direction of the polarization of the wiggler field. Therefore, the polarization of the wiggle velocity rotates slowly with distance along the axis. This rotation may, in principle, destroy the FEL gain for a conventional, linearly...
IV. CONCLUSIONS

(i) Off-axis electrons in combined linear wiggler and guide magnetic fields drift transversely to the direction of the wiggler field, thus destroying the radial localization of the beam.

(ii) The origin of the transverse motion lies in the $F \times B$ drift in the guide field due to a ponderomotive-type force which is the result of the product of the rapidly oscillating transverse electron velocity and the off-axis axial component of the wiggler field averaged over the wiggler period. An expression for the drift velocity is obtained by averaging equations of motion over the wiggler wavelength scale.

(iii) Rotation of the wiggler polarization direction around the axis results in a slow azimuthal drift with a magnitude equal to one half the drift velocity without the rotation. Radial confinement of the beam is obtained for rotation wave numbers which are much smaller than the wiggler wave number provided certain conditions are satisfied.

(iv) The theory shows that radial confinement and stabilization are achieved when (a) $(B_1/B_2)^2 < 1$, (b) the wiggler rotation speed is much faster than the drift velocity, and (c) the wiggler polarization vector rotates in the direction opposite to the gyromotion of the electrons in the guide field.

(v) The theory was tested in numerical simulations which included the effects of beam self-fields and finite beam emittance. These calculations showed that radial stabilization can be achieved in the presence of these real beam effects.

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