Three-dimensional transmission of the fast wave in ion cyclotron resonance plasma heating

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The transmission of the fast Alfvén wave in second harmonic ion cyclotron resonance heating of a deuterium–hydrogen (minority) plasma is analyzed in a general three-dimensional magnetogeometry. The approach is based on the congruent reduction of the Hermitian, anisotropic pressure fluid plasma model. The unreduced, 17th-order, slow-amplitude transport equation, characterizing the problem of interest, is systematically reduced to the fourth-order irreducible transport system describing the coupling of the fast wave to the resonant components of the anisotropic pressure plasma model. This system yields a compact expression for the fast wave transmission coefficient in a general geometry and arbitrary direction of propagation of the incident wave.

I. INTRODUCTION

Plasma heating by waves in the ion cyclotron resonance frequency (ICRF) range attracts the attention of the fusion community more than ever. Both the availability of intense sources, capable of delivering several megawatts of rf power for heating purposes, and recent successful experiments[1,2] are the main reasons for this growing interest in the ICRF heating scheme. In parallel, there exists a significant analytic and numerical effort on the modeling of the ICRF heating process.[3] The problem requires a joint solution of Maxwell and kinetic equations, a difficult task in existing *three-dimensional* plasma magnetogeometries. Typically, the solution encounters additional complicating factors, such as (a) the necessity of solving the near-field problem (antenna region); (b) the existence of resonance, where the conversion to kinetic modes and absorption takes place and the conventional geometric optics approximation is inapplicable; and (c) the presence of the minority ion species (e.g., hydrogen in the deuterium plasma), which increases the order of the system of equations underlying the problem and introduces additional modes and resonances. Because of all these complications, the existing ICRF heating theory is either restricted to one-dimensional analytic modeling or involves computerized, full wave calculations. Although attempts to include realistic three-dimensional features in these calculations, such as the effect of the poloidal magnetic fields, have been reported in the literature,[4] the development of advanced numerical integration techniques is still necessary in order to overcome numerical difficulties encountered in solving high-order partial differential equations in resonant regions. It is therefore important to further develop the analytic tools beyond the currently used one-dimensional models.

The goal of this study is to consider the fast wave transmission problem in the ICRF heating scheme in realistic *three-dimensional* plasmas by using the multidimensional order-reduction[5] and mode conversion[6] theories. A similar approach was recently applied to the electron cyclotron resonance heating (ECRH) problem[7] and led to a compact expression for the transmission coefficient in a general geometry and arbitrary direction of propagation of the incident wave. The transmission in the ECRH case was found in Ref. 7 via the reduction of the Hermitian, anisotropic pressure fluid plasma model. The same plasma model will also be used in the present work (Sec. II) in deriving a three-dimensional expression (Sec. III) for the transmission of the fast Alfvén wave through the resonance in deuterium–hydrogen (minority), second harmonic ($\omega = 2\Omega_D$) ion cyclotron heating.

II. CONGRUENT REDUCTION OF THE ANISOTROPIC PRESSURE MODEL

The starting point of our approach to reduction is the formulation of the underlying, *unreduced* wave problem in a *weakly* varying background as the evolution of an N-component field $\mathbf{Z}(x) = \mathrm{Re}\{\mathbf{A}(x)\exp[i\Psi(x)]\}$ on space-time $[x = (\mathbf{r},t)]$, where the *weakly* varying amplitude $\mathbf{A}$ satisfies the following Nth-order transport equation:[5]

$$\mathbf{D}\cdot\mathbf{A} = i\left[\frac{\partial \mathbf{D}}{\partial k_\mu}\cdot\frac{\partial \mathbf{A}}{\partial x^\mu} + \frac{1}{2}\frac{d}{dx^\mu}\left(\frac{\partial \mathbf{D}}{\partial k_\mu}\right)\cdot\mathbf{A}\right]. \quad (1)$$

Here the wave four-vector k is given by $k_\mu = \partial\Psi/\partial x^\mu$ and $\mathbf{D}(k,x)$ is a *Hermitian* dispersion matrix. We assume that the right-hand side of Eq. (1) is of $O(\delta)$, where $\delta \ll 1$ is a small, dimensionless parameter characterizing the weak variation of the background plasma. Recently an anisotropic, multispecies plasma fluid model was cast into such a Hermitian form.[7] It was shown in that work that in the vicinity of the second harmonic deuterium ion gyroresonance, the system of Maxwell and plasma fluid equations yields the desired transport equation [Eq. (1)] if the field $\mathbf{Z}$ is defined as

$$\mathbf{Z} \equiv \begin{bmatrix} c\mathbf{B}/(4\pi)^{1/2} \\ c\mathbf{E}/(4\pi)^{1/2} \\ \mathbf{V}_e(N_{0e}m_e)^{1/2} \\ \mathbf{V}_D(N_{0D}m_D)^{1/2} \\ \mathbf{V}_H(N_{0H}m_H)^{1/2} \\ P^D_{-+}/(T_D N_{0D})^{1/2} \\ P^H_{z+}/(T_H N_{0H})^{1/2} \end{bmatrix}$$

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$$\equiv \mathrm{Re}\left\{\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{v}_e \\ \mathbf{v}_\mathrm{D} \\ \mathbf{v}_\mathrm{H} \\ p_{-+} \\ p_{z+} \end{bmatrix} \exp(i\Psi)\right\}$$

$$\equiv \mathrm{Re}[\mathbf{A}\exp(i\Psi)]. \qquad (2)$$

Here $\mathbf{B}, \mathbf{E}$ is the electromagnetic field; $\mathbf{v}_e$, $\mathbf{v}_\mathrm{D}$, and $\mathbf{v}_\mathrm{H}$ are the electron, deuterium, and hydrogen (minority) ion fluid velocities, respectively; N_{0e}, $N_{0\mathrm{D}}$, and $N_{0\mathrm{H}}$ are the slowly varying background densities of electrons and various ion species; T_D and T_H are the temperatures of the deuterium and hydrogen ions; and $P^\mathrm{D}_{-+} = \mathbf{e}^*_- \cdot \mathbf{P}^\mathrm{D} \cdot \mathbf{e}_+$ and $P^\mathrm{H}_{z+} = e_z \cdot \mathbf{P}^\mathrm{H} \cdot \mathbf{e}_+$ [$\mathbf{e}_\pm$ being $(\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$] are the resonant components of the anisotropic pressure tensors associated with the deuterium and hydrogen ion species. The slowly varying Hermitian dispersion matrix $\mathbf{D}$ characterizing the evolution of the 17-component vector $\mathbf{A}$, defined in (2), is given by[7]

$$\mathbf{D} = \begin{bmatrix} D_{bb} & D_{ba} & 0 & 0 & 0 & 0 & 0 \\ D^\dagger_{ba} & D_{aa} & D^e_{av} & D^\mathrm{D}_{av} & D^\mathrm{H}_{av} & 0 & 0 \\ 0 & D^{e\dagger}_{av} & D^e_{vv} & 0 & 0 & 0 & 0 \\ 0 & D^{\mathrm{D}\dagger}_{av} & 0 & D^\mathrm{D}_{vv} & 0 & D^\mathrm{D}_{vp} & 0 \\ 0 & D^{\mathrm{H}\dagger}_{av} & 0 & 0 & D^\mathrm{H}_{vv} & 0 & D^\mathrm{H}_{vp} \\ 0 & 0 & 0 & D^{\mathrm{D}\dagger}_{vp} & 0 & \omega/2-\Omega_\mathrm{D} & 0 \\ 0 & 0 & 0 & 0 & D^{\mathrm{H}\dagger}_{vp} & 0 & \omega-\Omega_\mathrm{H} \end{bmatrix}, \qquad (3)$$

where

$$D_{bb} = D_{aa} = \omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$D_{ba} = \begin{bmatrix} -ik_z & 0 & ik_+ \\ 0 & ik_z & -ik_- \\ ik_- & -ik_+ & 0 \end{bmatrix}, \qquad (4)$$

$$D^\alpha_{vv} = \begin{bmatrix} \omega+\Omega_\alpha & 0 & 0 \\ 0 & \omega-\Omega_\alpha & 0 \\ 0 & 0 & \omega \end{bmatrix},$$
$$D^\alpha_{av} = i\omega_\alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (5)$$

and

$$D^\mathrm{D}_{vp} = \begin{bmatrix} 0 \\ \beta_\mathrm{D} k_- \\ 0 \end{bmatrix}, \quad D^\mathrm{H}_{vp} = \begin{bmatrix} 0 \\ \beta_\mathrm{H} k_z \\ \beta_\mathrm{H} k_- \end{bmatrix}. \qquad (6)$$

The $\mathbf{e}_+$, $\mathbf{e}_-$, $\mathbf{e}_z$ basis is used in Eqs. (4)–(6), index $\alpha(\alpha = e,D,H)$ describes the various plasma species, $c\beta_\alpha = (T_\alpha/m_\alpha)^{1/2}$ and the characteristic frequencies are defined as $c\omega = -\partial\Psi/\partial t$, $c\omega_\alpha = \pm(4\pi e^2 N_{0\alpha}/m_\alpha)^{1/2}$, and $c\Omega_\alpha = \pm e|\mathbf{B}_0|/cm_\alpha$, where the sign corresponds to the charge sign of the plasma particle. Note that, for convenience, we divided all the conventional frequency definitions by c, so that, for example, ω and k have the same dimensions. Also, we treat $\beta_\mathrm{D}, \beta_\mathrm{H}$, and the ratio $\omega_\mathrm{H}/\omega_\mathrm{D}$ as small and, formally, of $O(\delta)$. It should be mentioned here that the Hermitian plasma model of Ref. 7 is based on the small gyroradius assumption and thus within the model the thermal effects can only be treated in the vicinity of gyroresonances, excluding such cases as the electron Bernstein mode couplings at the upper hybrid resonance for which a different plasma model should be used. Here and in the following we shall assume the vicinity of the second harmonic deuterium ion gyroresonance, i.e., $\omega \approx 2\Omega_\mathrm{D} \approx \Omega_\mathrm{H}$.

At this stage we proceed to the congruent reduction of order[5] of our transport system [Eq. (1)]. The procedure, basically, is a step-by-step elimination of the components of $\mathbf{A}$, associated with *large* [of $O(\delta^0)$] diagonal elements of the dispersion matrix, characterizing the wave at each reduction step. It was shown in Ref. 5 that if, for example, the element D_{kk} of $\mathbf{D}$ is of $O(\delta^0)$, then the component A_k of the amplitude can be eliminated from the problem, while the remaining components of $\mathbf{A}$, comprising the *reduced* $(N-1)$-component wave amplitude $\mathbf{A} \equiv \{A_i, i \neq k\}$ are described by the transport equation of order $N-1$, but having the same form (1), with $\mathbf{D}$ replaced by a new *reduced* $(N-1)\times(N-1)$ dispersion matrix $\mathbf{D}$ given by

$$\overline{\mathbf{D}}_{ij} = D_{ij} - D_{ik}D_{kj}/D_{kk}, \quad i,j \neq k. \qquad (7)$$

Then we apply the procedure again to the new matrix $\mathbf{D}$ and so on, until the final dispersion matrix $\mathbf{D}^f$ of rank M ($<N$) is obtained with all elements of $O(\delta)$. The final M-component reduced amplitude $\mathbf{A}^f$ is again characterized by the transport equation of form (1) with $\mathbf{D}$ replaced by $\mathbf{D}^f$. This final transport equation is irreducible in the sense that any further elimination of the components of $\mathbf{A}^f$ leads to singular [of $O(1/\delta)$] coefficients in the resulting transport system, which contradicts the slowness of variation assumption for the amplitude. Thus we remain with the final, Mth-order system. In many cases $M=2$, describing the multidimensional pairwise mode conversion problem.[6] We shall see later (Sec. III) that the fast Alfvén wave three-dimensional trans-

mission problem in ICRF plasma heating can be interpreted as an $M=4$ case.

The reduction of the dispersion matrix (3) proceeds as follows. We eliminate the components of the magnetic field $\mathbf{b}$ first by viewing the diagonal elements ω in $\mathbf{D}_{bb}$ [Eq. (4)] as being of $O(\delta^0)$. The resulting reduced dispersion matrix [obtained by a repetitive use of Eq. (7)], which describes the evolution of the reduced amplitude $\mathbf{A}^1 = (\mathbf{a},\mathbf{v}_e,\mathbf{v}_{\rm D},\mathbf{v}_{\rm H},p^{\rm D}_{-+},p^{\rm H}_{z+})$ is

$$\mathbf{D}^1 = \begin{bmatrix} D^1_{aa} & D^e_{av} & D^{\rm D}_{av} & D^{\rm H}_{av} & 0 & 0 \\ D^{e\dagger}_{av} & D^e_{vv} & 0 & 0 & 0 & 0 \\ D^{\rm D\dagger}_{av} & 0 & D^{\rm D}_{vv} & 0 & D^{\rm D}_{vp} & 0 \\ D^{\rm H\dagger}_{av} & 0 & 0 & D^{\rm H}_{vv} & 0 & D^{\rm H}_{vp} \\ 0 & 0 & D^{\rm D\dagger}_{vp} & 0 & \omega/2-\Omega_{\rm D} & 0 \\ 0 & 0 & 0 & D^{\rm H\dagger}_{vp} & 0 & \omega-\Omega_{\rm H} \end{bmatrix}. \tag{8}$$

Here only the matrix $\mathbf{D}^1_{aa}$ is different from other definitions in (4), i.e.,

$$\mathbf{D}^1_{aa} = \begin{bmatrix} \omega - \dfrac{k_z^2+k_+k_-}{\omega} & \dfrac{k^2_+}{\omega} & \dfrac{k_+k_z}{\omega} \\ \dfrac{k^2_-}{\omega} & \omega - \dfrac{k_z^2+k_+k_-}{\omega} & \dfrac{k_-k_z}{\omega} \\ \dfrac{k_-k_z}{\omega} & \dfrac{k_+k_z}{\omega} & \omega - \dfrac{2k_+k_-}{\omega} \end{bmatrix}. \tag{9}$$

Next, we eliminate the components of $\mathbf{v}_e$ and obtain a new reduced matrix,

$$\mathbf{D}^2 = \begin{bmatrix} \mathbf{D}^2_{aa} & \mathbf{D}^{\rm D}_{av} & \mathbf{D}^{\rm H}_{av} & 0 & 0 \\ \mathbf{D}^{\rm D\dagger}_{av} & \mathbf{D}^{\rm D}_{vv} & 0 & \mathbf{D}^{\rm D}_{vp} & 0 \\ \mathbf{D}^{\rm H\dagger}_{av} & 0 & \mathbf{D}^{\rm H}_{vv} & 0 & \mathbf{D}^{\rm H}_{vp} \\ 0 & \mathbf{D}^{\rm D\dagger}_{vp} & 0 & \omega/2-\Omega_{\rm D} & 0 \\ 0 & 0 & \mathbf{D}^{\rm H\dagger}_{vp} & 0 & \omega-\Omega_{\rm H} \end{bmatrix}, \tag{10}$$

describing the amplitude $\mathbf{A}^2 = (\mathbf{a}, \mathbf{v}_{\rm D}, \mathbf{v}_{\rm H}, p^{\rm D}_{-+}, p^{\rm H}_{z+})$. Again, only matrix $\mathbf{D}^2_{aa}$ in (10) differs from the corresponding matrices in $\mathbf{D}^1$, i.e.,

$$\mathbf{D}^2_{aa} = \begin{bmatrix} \omega - \dfrac{k_z^2+k_+k_-}{\omega} - \dfrac{\omega_e^2}{\omega+\Omega_e} & \dfrac{k^2_+}{\omega} & \dfrac{k_+k_z}{\omega} \\ \dfrac{k^2_-}{\omega} & \omega - \dfrac{k_z^2+k_+k_-}{\omega} - \dfrac{\omega_e^2}{\omega-\Omega_e} & \dfrac{k_-k_z}{\omega} \\ \dfrac{k_-k_z}{\omega} & \dfrac{k_+k_z}{\omega} & \omega - \dfrac{2k_+k_- + \omega_e^2}{\omega} \end{bmatrix}. \tag{11}$$

At this stage, we eliminate all the components of $\mathbf{v}_{\rm D}$, which is allowed because we are interested in frequencies ω in the vicinity of $2\Omega_{\rm D}$, so that all the diagonal elements of matrix $\mathbf{D}^{\rm D}_{vv}$ are of $O(\delta^0)$. The resulting reduced matrix, characterizing the remaining amplitude components $\mathbf{A}^3 = (\mathbf{a}, \mathbf{v}_{\rm H}, p^{\rm D}_{-+}, p^{\rm H}_{z+})$, is given by

$$\mathbf{D}^3 = \begin{bmatrix} \mathbf{D}^3_{aa} & \mathbf{D}^{\rm H}_{av} & \mathbf{D}^{\rm D}_{ap} & 0 \\ \mathbf{D}^{\rm H\dagger}_{av} & \mathbf{D}^{\rm H}_{vv} & 0 & \mathbf{D}^{\rm H}_{vp} \\ \mathbf{D}^{\rm D\dagger}_{ap} & 0 & \omega/2-\Omega_{\rm D}+O(\beta^2_{\rm D}) & 0 \\ 0 & \mathbf{D}^{\rm H\dagger}_{vp} & 0 & \omega-\Omega_{\rm H} \end{bmatrix}, \tag{12}$$

where

$$\mathbf{D}_{aa}^{3}=\begin{bmatrix} \omega-\frac{k_z^2+k_+k_-}{\omega}-\frac{\omega_e^2}{\omega+\Omega_e}-\frac{\omega_D^2}{\omega+\Omega_D} & \frac{k_+^2}{\omega} & \frac{k_+k_z}{\omega} \\ \frac{k_-^2}{\omega} & \omega-\frac{k_z^2+k_+k_-}{\omega}-\frac{\omega_e^2}{\omega-\Omega_e}-\frac{\omega_D^2}{\omega-\Omega_D} & \frac{k_-k_z}{\omega} \\ \frac{k_-k_z}{\omega} & \frac{k_+k_z}{\omega} & \omega-\frac{2k_+k_-+\omega_e^2+\omega_D^2}{\omega} \end{bmatrix} \quad (13)$$

and

$$\mathbf{D}_{ap}^{D}=\begin{bmatrix} 0 \\ -i\frac{\omega_D\beta_D k_-}{\omega-\Omega_D} \\ 0 \end{bmatrix}. \quad (14)$$

Now we can eliminate components v_+^{H} and v_z^{H} [the corresponding components of $\mathbf{D}_{vv}^{H}$ are of $O(\delta^0)$], which yields a new reduced amplitude $\mathbf{A}^4=(\mathbf{a}, v_+^{H}, p_{-+}^{D}, p_{z+}^{H})$, characterized, to the lowest order, by the following 6×6 dispersion matrix:

$$\mathbf{D}^4=\begin{bmatrix} \mathbf{D}_{aa}^4 & \mathbf{D}_{av}^{H1} & \mathbf{D}_{ap}^{D} & \mathbf{D}_{ap}^{H} \\ \mathbf{D}_{av}^{H1\dagger} & \omega-\Omega_H & 0 & \beta_H k_z \\ \mathbf{D}_{ap}^{D\dagger} & 0 & \omega/2-\Omega_D & 0 \\ \mathbf{D}_{ap}^{H\dagger} & \beta_H k_z & 0 & \omega-\Omega_H \end{bmatrix}, \quad (15)$$

where

$$\mathbf{D}_{av}^{H1}=\begin{bmatrix} 0 \\ i\omega_H \\ 0 \end{bmatrix},\quad \mathbf{D}_{ap}^{H}=\begin{bmatrix} 0 \\ 0 \\ -i\omega_H\beta_H k_-/\omega \end{bmatrix}, \quad (16)$$

and, on neglecting the terms of $O(\omega_H^2/\omega)$, $\mathbf{D}_{aa}^4=\mathbf{D}_{aa}^3$ [see Eq. (13)].

Now we observe that one diagonal term in matrix $\mathbf{D}_{aa}^4$ is particularly large, i.e.,

$$D^4_{a_za_z}=\omega-\frac{2k_+k_-+\omega_e^2+\omega_D^2}{\omega}\sim O\left(\frac{m_D}{m_e}\frac{\omega_D^2}{\omega}\right). \quad (17)$$

We assume that this term is of $O(1/\delta)$ and, therefore, after the elimination of a_z, the new reduced matrix becomes [see Eq. (7)]

$$\overline{D}_{ij}=D_{ij}-(D_{ia_z}D_{a_zj})/D_{a_za_z}=D_{ij}+O(\delta). \quad (18)$$

Thus, to the lowest order, the component a_z can simply be omitted from the problem with no effect on the rest of the dispersion matrix. As a result, at this stage, our reduced amplitude becomes $\mathbf{A}^5=(a_+, a_-, v_+^{H}, p_{-+}^{D}, p_{z+}^{H})$ and the corresponding dispersion matrix is given by

$$\mathbf{D}^5=\begin{bmatrix} D_+-\frac{k_z^2}{\omega} & \frac{k_+^2}{\omega} & 0 & 0 & 0 \\ \frac{k_-^2}{\omega} & D_--\frac{k_z^2}{\omega} & i\omega_H & -i\frac{\omega_D\beta_D k_-}{\omega-\Omega_D} & 0 \\ 0 & -i\omega_H & \omega-\Omega_H & 0 & \beta_H k_z \\ 0 & i\frac{\omega_D\beta_D k_+}{\omega-\Omega_D} & 0 & \omega/2-\Omega_D & 0 \\ 0 & 0 & \beta_H k_z & 0 & \omega-\Omega_H \end{bmatrix}, \quad (19)$$

where

$$D_\pm=\omega-\frac{k_+k_-}{\omega}-\frac{\omega_e^2}{\omega\pm\Omega_e}-\frac{\omega_D^2}{\omega\pm\Omega_D}$$
$$\approx-\frac{k_\perp^2}{2\omega}+\frac{\omega_D^2\omega}{\Omega_D(\Omega_D\pm\omega)}. \quad (20)$$

Finally, we assume that the element $D_+-k_z^2/\omega$ in $\mathbf{D}^5$ is of $O(\delta^0)$ and reduce a_+. This reduction step yields a four-component amplitude $\mathbf{A}^f=(a_-, v_+^{H}, p_{-+}^{D}, p_{z+}^{H})$, characterized by the dispersion matrix

$$D^f=\begin{bmatrix} D & \alpha & \gamma & 0 \\ \alpha^* & \omega-\Omega_H & 0 & \rho \\ \gamma^* & 0 & \omega/2-\Omega_D & 0 \\ 0 & \rho & 0 & \omega-\Omega_H \end{bmatrix}, \quad (21)$$

where

$$D=D_--k_z^2/\omega-(k_\perp^4/4\omega^2)/(D_+-k_z^2/\omega),$$
$$\alpha=i\omega_H,\quad \rho=\beta_H k_z, \quad (22)$$
$$\gamma=-i[\omega_D\beta_D k_-/(\omega-\Omega_D)].$$

This completes our preliminary reduction steps, which result, at this point, in a fourth-order system suitable for the analysis in the second harmonic ($\omega = 2\Omega_D$) ion cyclotron resonance region.

III. FAST WAVE TRANSMISSION COEFFICIENT

We notice, at this stage, that the diagonal element D in dispersion matrix (22) represents the cold, fast Alfvén wave dispersion. Indeed, after some algebra, we obtain

$$D = \frac{1}{D_+ - k_z^2/\omega}\left[\left(D_+ - \frac{k_z^2}{\omega}\right)\left(D_- - \frac{k_z^2}{\omega}\right) - \frac{k_\perp^4}{4\omega^2}\right]$$
$$= \frac{\omega^2}{c_A^4(D_+ - k_z^2/\omega)}\left[N_z^4 - N_z^2\left(\frac{2}{1-\omega^2/\Omega_D^2} - N_\perp^2\right) - \frac{N_\perp^2 - 1}{1-\omega^2/\Omega_D^2}\right], \qquad (23)$$

where $c_A = \Omega_D/c\omega_D$ is the normalized Alfvén speed and $\mathbf{N} = \mathbf{k}c_A/\omega$. The expression in the square brackets in the last expression is the well known dispersion function for the fast Alfvén wave. Thus when an incident fast wave (given by $D=0$) approaches the cyclotron resonance $[\omega = 2\Omega_D = \Omega_H]$, all the elements of dispersion matrix (21) will become of $O(\delta)$ [recall that objects ω_H, β_D, and β_H are treated here as being of $O(\delta)$]. The matrix $\mathbf{D}^f$ in the resonance region is, therefore, the final *irreducible* dispersion matrix discussed in Sec. II. The transport equation (1) associated with $\mathbf{D}^f$ is

$$\begin{aligned} &Da_- + \alpha v_+^H + \beta p_{-+}^D = i\frac{\partial D}{\partial \mathbf{k}}\cdot\frac{\partial a_-}{\partial \mathbf{r}},\\ &\alpha^* a_- + (\omega - \Omega_H)v_+^H + \rho p_{z+}^H = 0,\\ &\beta^* a_- + (\omega/2 - \Omega_D)p_{-+}^D = 0,\\ &\rho v_+^H + (\omega - \Omega_H)p_{z+}^H = 0. \end{aligned} \qquad (24)$$

Here we neglected the $O(\delta^2)$ contribution associated with all other components of the term $(\partial \mathbf{D}^f/\partial \mathbf{k}_i)\cdot(\partial \mathbf{A}^f/\partial \mathbf{r}_i)$ in Eq. (1). Also, the geometric effect of the convergence of rays in the resonant region [the second term in the square brackets on the left-hand side of Eq. (1)] was neglected in Eqs. (24) because of the narrowness of the resonant region.

At this stage, for simplicity, we shall consider a situation when coupling constant ρ in Eqs. (24) is sufficiently small (small k_Z case) so that during the passage of the fast wave through the resonance we can neglect the excitation of the pressure tensor component p_{z+}^H associated with the minority ions. In this case we set $p_{z+}^H = 0$ in Eqs. (24) and define an effective *hybrid* pressure p via

$$p = (\alpha v_+^H + \beta p_{-+}^D)/\xi,$$

where $\xi^2 = |\alpha|^2 + 2|\beta|^2$. Then, by multiplying the second equation in (24) by α and the third equation by 2β and adding the two equations, we obtain the transport system for a_- and p:

$$\begin{aligned} &Da_- + \xi p = i\frac{\partial D}{\partial \mathbf{k}}\cdot\frac{\partial a_-}{\partial \mathbf{r}},\\ &\xi a_- + (\omega - \Omega_H)p = 0. \end{aligned} \qquad (25)$$

Again this system has form (1) with the following nearly degenerate dispersion matrix:

$$\mathbf{D} = \begin{bmatrix} D & \xi \\ \xi & \omega - \Omega_H \end{bmatrix}. \qquad (26)$$

As mentioned in Sec. II, this $M = 2$ case is characteristic of the multidimensional mode conversion problem.[5] Therefore if initially the mode $D(\mathbf{k},\mathbf{r}) = 0$ (the fast Alfvén wave) was excited, then after passing the crossing point (where both $D = 0$ and $\omega = \Omega_H = 2\Omega_D$) only a part of the initial energy flux remains in the fast wave channel and the transmission coefficient is given by the following expression:[5]

$$T = \exp\left[-\frac{2\pi\xi^2}{\partial D/\partial \mathbf{k}\cdot\partial\Omega_H/\partial\mathbf{r}}\right], \qquad (27)$$

where all the parameters are evaluated at the crossing point.

Now after solving the simplified second-order coupling problem we return to the general, fourth-order system (24), i.e., include the possibility of excitation of the resonant pressure tensor component associated with the minority ions. We shall show that the fast wave transmission coefficient in this general case is independent of the value of the coupling constant ρ and is again given by Eq. (27). Indeed, we can solve system (24) directly by adding a weak collisionality to the problem, i.e., by replacing ω by $\omega + i\nu$ in the kinetic part of Eqs. (24), where the collision frequency ν is assumed to be sufficiently small [ν/ω of $O(\delta)$]. Then, after some algebra, Eqs. (24) yield

$$i\frac{\partial D}{\partial \mathbf{k}}\cdot\frac{\partial a_-}{\partial \mathbf{r}} = \left(D - \frac{|\alpha|^2\Delta}{\Delta^2 - \rho^2} - \frac{2|\gamma|^2}{\Delta} + O\frac{\nu}{\omega}\right) \times a_- + O(\delta^2), \qquad (28)$$

where $\Delta = \omega - \Omega_H + i\nu$. At this stage we define the fast wave phase space trajectory (ray) via

$$\frac{d\mathbf{r}}{d\sigma} = -\frac{\partial D}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{d\sigma} = \frac{\partial D}{\partial \mathbf{r}}, \qquad (29)$$

where σ is the parameter along the trajectory chosen so that it vanishes at the point where the ray passes the resonance surface $\omega = \Omega_H(\mathbf{r})$. Then $D(\mathbf{k},\mathbf{r})$ vanishes along the ray and the integration of Eq. (28) through the resonant region, in which $\omega - \Omega_H \approx \sigma(\partial\Omega_H/\partial\mathbf{r})\cdot(\partial D/\partial\mathbf{k})$ and, to the lowest order, vectors $\partial\Omega_H/\partial\mathbf{r}$ and $\partial D/\partial\mathbf{k}$ can be kept constant and evaluated at the resonance, yields the transmission coefficient

$$\begin{aligned} T &= \left|\frac{a_-(\sigma = +A)}{a_-(\sigma = -A)}\right|^2\\ &= \exp[-(|\alpha|^2 + 2|\gamma|^2)\mathrm{Im}(\mathbf{I}_1)\\ &\quad + \rho^2|\alpha|^2\,\mathrm{Im}(\mathbf{I}_2) + O(\nu/\omega)]. \end{aligned} \qquad (30)$$

Here $|A|$ is sufficiently large and integrals $\mathbf{I}_1$ and $\mathbf{I}_2$ are defined as

$$\mathbf{I}_1 = 2\int_{-A}^{+A}\frac{d\sigma}{\Delta}, \quad \mathbf{I}_2 = 2\int_{-A}^{+A}\frac{d\sigma}{\Delta(\Delta^2 - \rho^2)}.$$

Simple contour integration yields $\mathbf{I}_1 = 2\pi i/|(\partial\Omega_H/\partial\mathbf{r})\cdot(\partial D/\partial\mathbf{k})|$, while $\mathbf{I}_2 = 0$ and, therefore, in the limit $\nu \to 0$, Eq. (30) becomes identical to Eq. (27).

In order to apply Eq. (27) to our problem we first rewrite it in the form

$$T = \exp\left(-\frac{2\pi\xi^2}{|\partial D/\partial\mathbf{k}||\partial\Omega_H/\partial\mathbf{r}|\cos\phi}\right)$$

$$= \exp\left(- \frac{2\pi \xi^2 \sin\theta}{|\partial D/\partial \mathbf{k}_\perp| \, |\partial \Omega_H/\partial \mathbf{r}| \cos\phi} \right) = (T_\perp)^{\sin\theta/\cos\phi}, \tag{31}$$

where ϕ is the angle between the group velocity of the fast wave ($\mathbf{v}_g \propto \partial D/\partial \mathbf{k}$) and the gradient $\partial\Omega_H/\partial\mathbf{r}$, θ is the angle between $\mathbf{v}_g$ and the background magnetic field, and

$$T_\perp = \exp\left(- \frac{2\pi\xi^2}{|\partial D/\partial \mathbf{k}_\perp| \, |\partial\Omega_H/\partial\mathbf{r}|} \right). \tag{32}$$

Equation (31) is the desired expression for the transmission coefficient in a general three-dimensional plasma magnetogeometry for an arbitrary direction of propagation of the incident fast wave. This expression can be further simplified and transformed into a more convenient form. Indeed, the direct differentiation of (23) and use of the dispersion relation $D = 0$ yields

$$\left.\frac{\partial D}{\partial \mathbf{k}_\perp}\right|_{\omega=\Omega_H} = \frac{2\mathbf{k}_\perp}{D_+ - k_z^2/\omega} \left(\frac{k_z^2}{\omega^2} - \frac{\omega_D^2}{\Omega_D^2 - \omega^2} \right) = \frac{2\mathbf{k}_\perp (3N_z^2 + 1)}{3c_A^2 (D_+ - k_z^2/\omega)}, \tag{33}$$

while

$$\xi^2 = \omega_H^2 + 4\Omega_D^2 \beta_D^2 c_A^{-4} N_\perp^2. \tag{34}$$

On the other hand, at the resonance,

$$D_+ = -\frac{k_\perp^2}{2\omega} + \frac{\omega_D^2 \omega}{\Omega_D(\Omega_D + \omega)} = c_A^{-2}\Omega_D \left(\frac{2}{3} - N_\perp^2 \right) \tag{35}$$

and

$$D_+ - k_z^2/\omega = -\Omega_D (3N_z^2 - 1)^2 / 3c_A^2 (3N_z^2 + 1). \tag{36}$$

Thus, if at the resonance we define the scale length $L \equiv \Omega_H / |\partial\Omega_H/\partial\mathbf{r}|$, then Eq. (32) becomes

$$T_\perp = \exp\left(- \frac{\pi L \omega_D [\eta + 2(\beta_D/c_A)^2 N_\perp^2] N_\perp^3}{2(N_z^2 + 1)^2} \right), \tag{37}$$

where $\eta = N_{0H}/N_{0D}$. This expression agrees with the transmission formula of Francis, Bers, and Ram[8] for the case of the perpendicular stratification of the background magnetic field ($\nabla\Omega_H \perp \mathbf{B}_0$) and propagation of the fast wave in the $\nabla\Omega_H$-$\mathbf{B}_0$ plane ($\mathbf{k}_\perp \| \nabla\Omega_H$). This restricted case corresponds to $\theta = \pi/2 - \phi$ in Eq. (31) and therefore $T = T_\perp$.

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[1]PLT Group, in *Proceedings of the 12th European Conference on Controlled Fusion and Plasma Physics* (EPS, Petit-Lancy, Switzerland, 1985), Vol. 9F, Part 2, p. 120.

[2]JET Group, Plasma Phys. Controlled Fusion **28**, 1 (1986).

[3]For a review of the ICRF theory see D. G. Swanson, Phys. Fluids **28**, 2645 (1985).

[4]D. Smith, P. Colestock, T. Kammash, and R. Kashuba, Phys. Rev. Lett. **60**, 801 (1988).

[5]L. Friedland and A. N. Kaufman, Phys. Fluids **30**, 3050 (1987).

[6]L. Friedland, G. Goldner, and A. N. Kaufman, Phys. Rev. Lett. **58**, 1392 (1987).

[7]L. Friedland, Phys. Fluids **31**, 2615 (1988).

[8]G. Francis, A. Bers, and K. Ram, in *Proceedings of the AIP Conference on Applications of Radio-Frequency Power to Plasmas 1987*, Kissimmee, Florida (A.I.P., New York, 1987), p. 230.