Strong autoresonance excitation of Rydberg atoms: The Rydberg accelerator

B. Meerson and L. Friedland

Center for Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem, 91904 Israel

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A classical mechanism is proposed for the many-photon ionization of Rydberg atoms in a quasi-monochromatic oscillating electric field with an amplitude smaller than the stochasticity threshold. The mechanism is based on the dynamic autoresonance between the slowly varying frequency of the oscillating field and the Keplerian frequency (or one of its harmonics) of atomic electrons. It is shown that a large fraction of an initial ensemble of atoms can be efficiently excited by this coherent mechanism and then ionized via the stochastic instability.

A significant experimental and theoretical effort has been recently devoted to the investigation of different regimes of many-photon ionization of highly excited (Rydberg) atoms (see, e.g., Refs. 1–10). One of the most interesting is the chaotic ionization regime, i.e., the ionization via the (classical) stochastic instability of the atomic electron.[4] This phenomenon can take place even in the simplest case of a Rydberg hydrogen atom in a monochromatic oscillating electric field. The necessary condition for such a "chaotic ionization" is provided by the Chirikov's criterion of the overlapping of resonances,[4] which yields the threshold oscillating field amplitude F_0 for a given principal quantum number n and driving frequency ω. Below the threshold, most of the perturbed classical trajectories remain regular corresponding to the Kolmogorov-Arnold-Moser (KAM) surfaces in an appropriate phase space.[4,11] This means that the ionization from such states is practically absent. Therefore the question arises of whether it is possible to achieve the multiphoton ionization of Rydberg atoms by an oscillating electric field with an amplitude much smaller than the above-mentioned threshold value. It has been shown recently[12] that the answer to this question is positive in the case of a broadband-noise radiation field. In the present work we propose a novel mechanism for the excitation and ionization of Rydberg atoms by an essentially subthreshold *quasimonochromatic* oscillating electric field, whose frequency slowly decreases with time. We find conditions under which a large fraction of the initial ensemble of atoms can be ionized, while the precise form of the time dependence of the frequency of the oscillating field is not essential. The term "autoresonance excitation" will be used in describing the initial stage of the process because it is based upon a self-sustained (classical) resonance between the slowly varying external driving frequency and the Keplerian frequency (or one of its harmonics) of the atomic electrons. This mechanism is surprisingly similar to that of a number of charged-particle acceleration schemes (see, e.g., Ref. 13) and therefore it can be called "the Rydberg accelerator."

For the sake of simplicity we consider here a one-dimensional (1D) hydrogen atom. The 1D model is adequate at least when the initial state is an extremal Stark level. It is convenient to express the classical Hamiltonian of the atom in terms of the canonical action-angle variables I and λ, determined by the unperturbed Keplerian motion of the electron. Since the quantization of the unperturbed motion converts the action variable I to the principal quantum number of the atom n, we shall write n instead of I. In the 1D case, n is the only quantum number of the atom. The classical Hamiltonian in atomic units becomes (cf. Ref. 4)

$$H=-\frac{1}{2n^2}+n^2F(t)\left[\tfrac{3}{2}-2\sum_{k=1}^{\infty}x_k\cos(k\lambda)\right], \tag{1}$$

where $F(t)=F_0\cos\Phi(t)$ is the oscillating electric field, F_0 is the (constant) amplitude of the field, $\Phi(t)$ is the phase, and $x_k=J_k'(k)/k$ and J_k' are derivatives of the Bessel functions with respect to their arguments. The time derivative of the phase $\Phi(t)$ is equal, by definition, to the frequency $\omega(t)$, which is assumed to vary slowly with time.

Consider an atom whose electron has an initial value of $n=n_0$ such that the initial value of the external frequency $\omega(t=0)=\omega_0$ is in a kth resonance with corresponding Keplerian frequency of the electron, i.e., $\omega_0=kn_0^{-3}$, where $k=1,2,\ldots$. Let parameter $F_0n_0^4$ be much less than the corresponding stochasticity threshold of Hamiltonian (1) with $\omega=\omega_0=\text{const}$ (for example, this threshold is close to 0.02 for k of the order of 1; see Refs. 4, 8, and 9). In this case we can start with the isolated resonance approximation similarly to the case of $\omega=\text{const}$.[4,11] Then, introducing a new phase $\xi=\Phi-k\lambda-\pi$, normalized action $N=n/n_0$, time $\tilde{t}=tn_0^{-3}$, frequency $\tilde{\omega}=\omega n_0^3$, and the interaction parameter $\epsilon=F_0n_0^4kx_k=F_0n_0^4J_k'(k)\ll 1$, we obtain the following equations of motion:

$$\dot{\xi}=\omega(t)-kN^{-3}-2\epsilon N\cos\xi\ , \tag{2}$$

$$\dot{N}=-\epsilon N^2\sin\xi \tag{3}$$

(here and in the following dots mean time derivatives and we omit all tildes for simplicity). Now we consider an ensemble of Rydberg atoms and assume that initially each atom has an initial value of n close to the above-mentioned n_0 (the allowable spread of n will be settled later). It is clear that the initial value of the normalized frequency is equal to k: $\omega(t=0)=k$. The initial values of the classical phase λ (and, consequently, of the phase ξ) can be arbitrary, i.e., $-\pi\le\xi_0\le\pi$, where $\xi_0=\xi(t=0)$.

In the case ω=const, the solutions of Eqs. (2) and (3) describe relatively slow and small periodic modulations of N with a "nonlinear" frequency ν (see Refs. 4 and 11). The phase ξ, in contrast, behaves differently depending on whether the corresponding electron is "trapped" or "untrapped" in the phase plane ξ,N (see Refs. 4 and 11). The phase ξ of a trapped particle oscillates with the same nonlinear frequency ν around the equilibrium point $\xi=0$, while the phases of untrapped particles vary monotonically. Now let us return to the case of slowly varying ω. We shall consider only initially trapped particles and find conditions under which these particles remain trapped for a long time, despite possible large (but slow) changes in the driving frequency ω. We shall show that almost all initially trapped electrons can be coherently "accelerated" (excited) by a subthreshold oscillating field until they enter the stochasticity domain (i.e., the resonance overlapping domain) and rapidly escape from the atoms.

Let us start with numerical solutions of Eqs. (2) and (3) (see Fig. 1). In these calculations we set $N(t=0)=1$ and chose different initial phases (two of them are shown in Fig. 1). We considered the main resonance $k=1$ and slowly decreasing frequency $\omega(t)=1/(1+\alpha t^2)$ [similar results are also obtained for other slow dependences $\omega(t)$]. We observe in Fig. 1 that, in contrast to ω=const, the average value of N for each of the two cases grows in time considerably, which means strong excitation. In addition, the average value of ξ remains small and negative, the amplitude of oscillations of N grows in time, while both the amplitude of oscillations of ξ and the nonlinear frequency decrease.[14] Now we shall find the conditions under which this is the characteristic behavior of the system (2) and (3), and develop an analytic theory of the Rydberg accelerator.

Let $\omega(t)$ be a slowly varying function on a time scale of the nonlinear period $2\pi/\nu$, i.e., $|\dot{\omega}|/\nu^2\ll 1$. Then we can solve Eqs. (2) and (3) perturbatively by separating the fast and slow time dependences of N and ξ. In other words, we seek solutions of Eqs. (2) and (3) in the form $N(t)=\bar{N}(t)+\delta N(t)$ and $\xi(t)=\bar{\xi}(t)+\delta\xi(t)$, where $\langle\delta N(t)\rangle=\langle\delta\xi(t)\rangle=0$, $\delta N(t)\ll\bar{N}(t)$, and $\langle\ \rangle$ means the time averaging over the nonlinear period, which now is considered as a "fast" time scale. Note that for the trapped particles considered here $\bar{\xi}(t=0)$ always vanishes. Seeking a regime in which the phase drift remains small, i.e., $\bar{\xi}\ll 1$ (the corresponding criterion will be found *a posteriori*), and separating the fast and slow variables in Eq. (2) and (3), we obtain the following two sets of equations:

$$\delta\dot{\xi}=3k\bar{N}^{-4}\delta N\ , \tag{4}$$

$$\delta\dot{N}=-\epsilon\bar{N}^{2}\sin\delta\xi \tag{5}$$

for the fast variables, and

$$\dot{\bar{\xi}}=\omega(t)-k\bar{N}^{-3}\ , \tag{6}$$

$$\dot{\bar{N}}=-\epsilon\bar{N}^{2}\bar{\xi}\langle\cos\delta\xi\rangle \tag{7}$$

for the slow ones. Note that we have neglected the small term, proportional to ϵ, in Eq. (2).

To the lowest order we assume now that $\bar{N}$ in Eqs. (4) and (5) is time independent, and obtain the well-known nonlinear pendulum equation for the fast phase:

$$\delta\ddot{\xi}+\nu_0^2\sin\delta\xi=0\ , \tag{8}$$

where $\nu_0^2=3\epsilon k\bar{N}^{-2}$. Then the dependences of $\delta\xi$ and δN on the fast time are given by[11]

$$\cos\delta\xi(t)=1-2m^2\mathrm{cn}^2(m,\nu t)\ , \tag{9}$$

$$\delta N(t)=(2m\nu_0/3k)\bar{N}^{4}\mathrm{sn}(m,\nu t)\ , \tag{10}$$

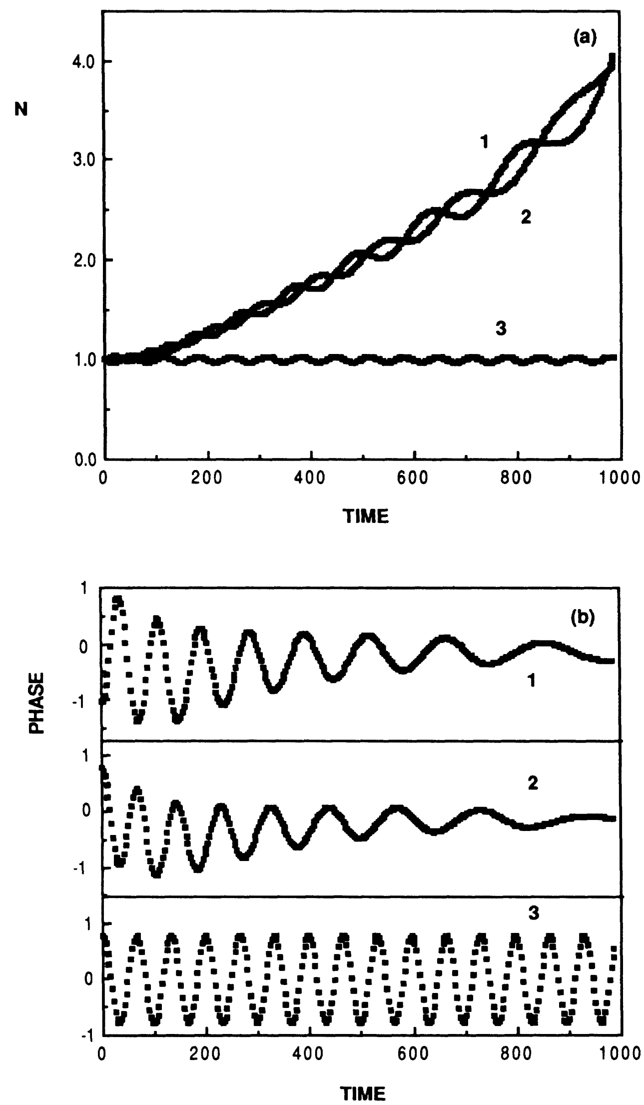

FIG. 1. Numerical examples of the strong autoresonance excitation, as described by Eqs. (2) and (3): (a) N and (b) ξ as functions of the scaled time for a slowly decreasing driving frequency [$\alpha=2.5\times10^{-5}$, graphs 1 (initial phase $\xi_0=-\pi/3$) and 2 (initial phase $\pi/4$)]. Also shown is the case of a constant frequency $\alpha=0$ for the initial phase $\pi/4$ (graphs 3). In all cases $\epsilon=0.003\,25$, corresponding to $F_0n_0^4=0.01$.

where $m=\sin(\xi_0/2)$, while sn and cn are the Jacobian elliptic functions of parameter m. For definiteness, we have chosen the following initial conditions for $\delta\xi$ and δN: $\delta\xi(t=0)=\xi_0$ and $\delta N(t=0)=0$. The nonlinear frequency ν of a particle depends on ξ_0 and is given by $\nu=\pi\nu_0/2\mathbf{K}(m)$ where $\mathbf{K}$ (and later $\mathbf{E}$) are the complete elliptic integrals. Now we can take into account the slow time dependences in Eqs. (9) and (10). Indeed there exists an adiabatic invariant J of the fast oscillations[11,15] that is preserved under slow (on the time scale of ν^{-1}) changes of $\omega(t)$:

$$J=\frac{8N^3\epsilon^{1/2}}{\pi(3k)^{1/2}}[\mathbf{E}(m)-(1-m^2)\mathbf{K}(m)]=\text{const} . \quad (11)$$

Once the slow time dependence $\bar{N}(t)$ is known (see below), Eq. (11) yields the slow time dependence of $\xi_0=2\sin^{-1}m$. The quantity $\xi_0(t)$ here has the meaning of the slowly varying amplitude of the fast oscillations of the phase of the trapped particles. Similarly, the slowly varying amplitude ΔN of the fast oscillations of δN is given by

$$\Delta N=2\bar{N}^3(t)(\epsilon/3k)^{1/2}|\sin(\xi_0(t)/2)| . \quad (12)$$

Before proceeding to the slow variables, we calculate $\langle\cos\delta\xi\rangle$ in Eq. (7). This can be done directly from Eq. (9) yielding

$$\langle\cos\delta\xi\rangle=1-\frac{4\pi^2 q}{\mathbf{K}^2(1+q)^2} , \quad (13)$$

where $q=\exp(-\pi\mathbf{K}'/\mathbf{K})$ and $\mathbf{K}'=\mathbf{K}[\cos(\xi_0(t)/2)]$. Now we differentiate Eq. (6) with respect to time and use Eq. (7) to obtain

$$\ddot{\bar{\xi}}+\nu_0^2\langle\cos\delta\xi\rangle\bar{\xi}=\dot{\omega}(t) , \quad (14)$$

where $\langle\cos\delta\xi\rangle$ is given by Eq. (13). Equation (14) describes a linear oscillator with a slowly varying natural frequency $\nu_0(t)\langle\cos\delta\xi\rangle^{1/2}$, which is perturbed by an external force proportional to $\dot{\omega}(t)$. We observe that Eq. (14) describes a *slow* motion if and only if the second derivative term is negligible, so that $\bar{\xi}(t)$ can be simply written as

$$\bar{\xi}=\dot{\omega}(t)/(\nu_0^2\langle\cos\delta\xi\rangle) . \quad (15)$$

The substitution of this expression into Eq. (6) and integration yields the autoresonance condition $\omega(t)=k\bar{N}^{-3}(t)$. Thus the average value of N *for almost all trapped particles* (see below) varies similarly, i.e., as $[\omega(t)/k]^{-1/3}$, provided ω is a slowly varying function of time. We observe that by slowly (but otherwise arbitrarily) lowering the driving frequency, we can continuously increase $\bar{N}$ until the electron reaches the stochasticity domain and escapes from the atom. It is also clear that one can strongly excite an ensemble of atoms having initially slightly different values of n. The allowable initial spread of n must only be smaller than the corresponding initial width of the resonance region, which is proportional to $\epsilon^{1/2}$.

Returning to Eqs. (11) and (12) and replacing $\bar{N}$ by $[\omega(t)/k]^{-1/3}$, we see that when $\omega(t)$ decreases, the amplitude of the fast oscillations of the phase $\xi_0(t)$ of the trapped particles and the nonlinear frequency decrease, while the amplitude ΔN of the fast oscillations of δN grows in agreement with numerical results presented in Fig. 1. The decrease of $\xi_0(t)$ means that the particles become "increasingly trapped" with time. In the limit of such "deeply trapped" particles we simply have $\xi_0(t)=\xi_0(0)[\omega(t)]^{1/2}$ and $\Delta N(t)=\Delta N(0)[\omega(t)]^{-1/2}$. Quantitative comparisons of Eqs. (11), (12), and (15) with numerical results show very good agreement.

It follows from (15) that the phase drift $\bar{\xi}$ remains small if $|\dot{\omega}|/(3\epsilon k^{1/3}\omega^{2/3}\langle\cos\delta\xi\rangle)\ll 1$. Similarly to the abovementioned inequality $|\dot{\omega}|/\nu^2\ll 1$, this criterion imposes a

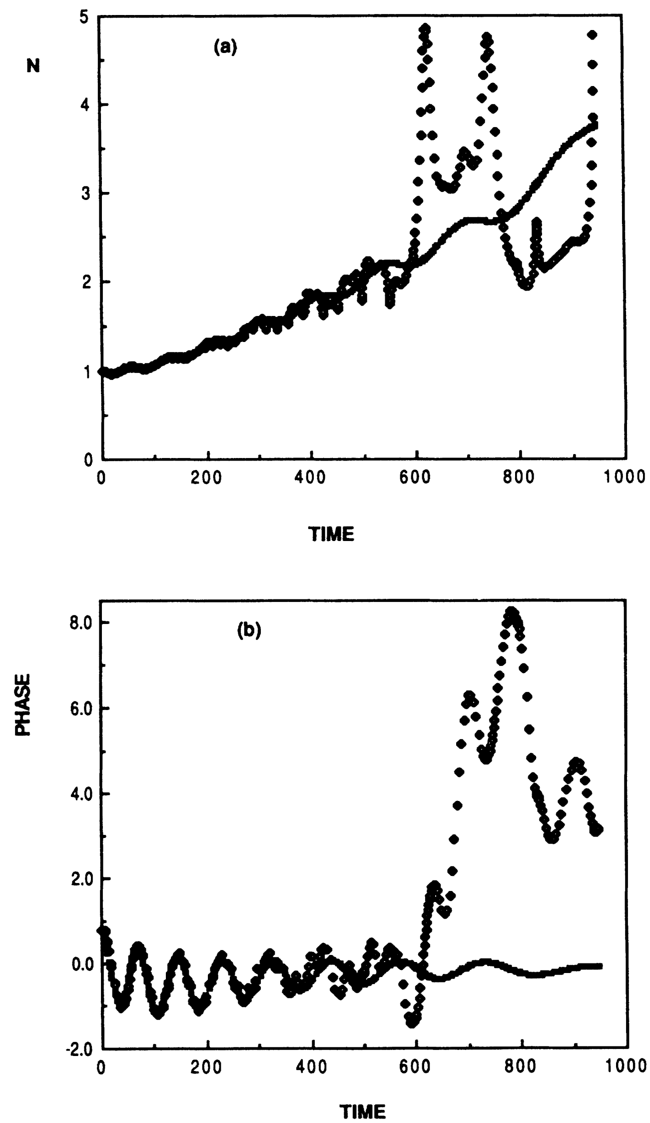

FIG. 2. A numerical example of the time-dependent transition to chaos [(a) N and (b) ξ vs the scaled time], as described by the Hamiltonian (1) for $\alpha=2.5\times10^{-5}$, $F_0 n_0^4=0.01$, and $\xi_0=\pi/4$ (open squares). Also shown are single-resonance solutions for the same parameters and initial phase (solid line).

constraint on the "chirping rate" of the frequency of the oscillating electric field. The stronger the phase trapping, the easier each of these criteria is satisfied. Note that since ν decreases with the increase of m, and $\nu \rightarrow 0$ when $m \rightarrow 1$, the condition $|\dot{\omega}|/\nu^2 \ll 1$ cannot be satisfied in a narrow region near the separatrix, dividing the trapped and untrapped phase trajectories, and the corresponding (small) fraction of atoms cannot be excited by the proposed mechanism, regardless of how slow $\omega(t)$ is.

The isolated autoresonance approximation applied above fails when the particle approaches the stochasticity domain. In this limit the strong coherent autoresonance excitation is replaced by the chaotic ionization of the atom. An approximate estimate of the *time moment* t_* of the onset of the stochasticity can be obtained using the constant-frequency results of Refs. 4, 8, and 9, i.e., $F_0 n_0^4 N^4(t_*) \approx 0.02$ (for k of the order of 1) or $F_0 n_0^4 \omega^{-4/3}(t_*) \approx 0.02$. Figure 2 presents an example of numerical integration of the equations of motion corresponding to the exact Hamiltonian (1). We have chosen the same subthreshold value of the amplitude of the oscillating field and the same "chirping" form of the driving frequency as in the numerical examples in Fig. 1. Also, in these calculations we followed Casati, Guarneri, and Shepelyansky[9] to introduce the eccentric anomaly ζ and a new time η, for which the unperturbed motion of ζ is uniform. It is seen from Fig. 2 that the initial stage of the ionization process (strong autoresonance excitation) is very well described by the isolated resonance approximation. Later the particle enters the stochasticity domain and executes a chaotic motion, leading in practice to the ionization. The observed time of the onset of the stochasticity agrees well with the estimate given above. Calculations for different initial phases of the trapped particles and other slow dependences of ω on time gave similar results.

The classical theory of the Rydberg accelerator provides a basis for the experimental and further theoretical study of the phenomena. Quantum mechanics may impose a constraint on the validity time scale of the theory,[11,12,16,17] [apart from the obvious constraints $n \gg 1$ and $\omega \ll (2n^2)^{-1}$, ω is *not* scaled anymore]. The precise form of this constraint will be known only after the corresponding quantum theory is developed. The development of such a theory is urgent because the particular model considered here exemplifies more general phenomena of the strong excitation and transition to chaos in Hamiltonian systems with slowly varying parameters.

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