Congruent reduction in parametrically unstable space- and time-varying plasmas

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A recently developed congruent reduction technique is applied to the parametric amplification problem in space- and time-varying plasmas. The systematic reduction procedure yields a second-order system of partial differential equations describing mode coupling in the presence of a strong pump wave. This system is similar in form to that characterizing the general multidimensional mode conversion problem so that the solution in both cases proceeds similarly. The method allows a consistent derivation of the expression for the parametric amplification coefficient in general geometry. An unmagnetized plasma case is considered as an example.

I. INTRODUCTION

In a recent paper[1] Friedland and Kaufman suggested a new method of extracting lower-order equations embedded in higher-order linear systems describing a large class of dissipationless multicomponent waves in space- and time-varying plasmas. The method is based on congruent transformations and comprises a step-by-step procedure of elimination of wave components, lowering the order of the system by one at each step and preserving, at the same time, the form and the first differential order of the system describing the remaining wave components. The approach is particularly suitable for dealing with the pairwise linear mode conversion problem in general geometry, where the final reduced system comprises a characteristic set of two coupled first-order partial differential equations (the coupled mode equations). This final system is characterized by the *local* dispersion relation of the form

$$D_a(k,x)D_b(k,x) - \eta^2 = 0\,, \qquad (1)$$

where $x = (\mathbf{r},t)$, $k = (\mathbf{k}, -\omega)$, and η^2 is a small *positive* constant. Equation (1) describes two linear, weakly coupled modes described by local dispersions, $D_a(k,x) = 0$ and $D_b(k,x) = 0$, respectively, while η plays the role of a small coupling coefficient. The coupling is only important in the vicinity of the crossing point (k_0,x_0) in the phase space, at which $D_a(k_0,x_0) = D_b(k_0,x_0) = 0$. In the neighborhood of the crossing point the wave energy is redistributed between the two modes and if, for example, the mode associated with $D_a = 0$ is excited at the entrance into the crossing region, only a fraction of the energy flux in this mode is transmitted through the region, while the remainder is mode converted into the mode $D_b = 0$. The transmission coefficient T in such a four-dimensional mode conversion process is fully described[2] in terms of D_a, D_b, and η^2:

$$T = \exp(-2\pi\eta^2/|B|), \qquad (2)$$

where B is the Poisson bracket,

$$B = \frac{\partial D_a}{\partial k_\mu}\frac{\partial D_b}{\partial x^\mu} - \frac{\partial D_a}{\partial x^\mu}\frac{\partial D_b}{\partial k_\mu}, \qquad (3)$$

evaluated at the crossing point (k_0,x_0).

It was recently noticed by Williams[3] that Eq. (1) is reminiscent of the dispersion relation for two modes coupled parametrically by a third, strong pump wave. In this case a typical local dispersion relation is

$$D_a(k,x)D_b(k_p - k,x) + \gamma^2 = 0, \qquad (4)$$

where $k_p = k_p(x)$ is a given wave four-vector of the pump wave and $\gamma^2 > 0$ characterizes the parametric growth rate in a uniform medium. On the basis of the similarity between (1) and (4) Williams argued[3] that the transmission coefficient in the parametric excitation case in a four-dimensional medium, similar to Eq. (2), is given by

$$T_p = \exp(2\pi\gamma^2/|B_p|)\,, \qquad (5)$$

where

$$B_p = \frac{\partial D_a}{\partial k_\mu}\left(\frac{\partial D_b}{\partial x^\mu} + \frac{\partial D_b}{\partial k_\nu}\frac{\partial k_{p\nu}}{\partial x^\mu}\right) + \frac{\partial D_a}{\partial x^\mu}\frac{\partial D_b}{\partial k_\mu} \qquad (6)$$

is now evaluated at the crossing point (k_0,x_0), satisfying $D_a(k_0,x_0) = D_b(k_p - k_0,x_0) = 0$. It was also observed[3] that Eq. (6) is a four-dimensional generalization of the well-known Rosenbluth result[4] in one dimension.

In spite of the simplicity of the generalization (5), one basic problem still should be resolved prior to application. Indeed, the similarity between dispersions (1) and (4) is only a *necessary* condition for the validity of Eq. (5). Only the *similarity between* the underlying *systems* of coupled mode equations describing the parametric excitation and linear mode conversion cases in four-dimensional media will assure the validity of (5). Since, at present, there exists no general procedure for extracting such lower-order systems for the parametric case in plasmas of general magnetogeometry,[5] Eq. (5) still remains a guess, though verified in some applications (the one-dimensional case[4]). The goal of the present work is to close the remaining gap by developing a general procedure for finding lower-order parametric couplings embedded in high-order systems, characterizing magnetized, space- and time-varying plasmas. As in the linear mode conversion case we shall exploit the congruent reduction idea (Sec. II), which allows the systematic lowering of the order of underlying equations, keeping, at the same time, the information on the energy conservation in the unreduced system. As an example of reduction, we shall consider a cold unmagnetized plasma case in Sec. III.

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II. REDUCTION PROCEDURE

Consider an N-component real perturbation $\mathbf{Z}(x)$ on space-time $[x=(\mathbf{r},t)]$ in a weakly varying background. Suppose that $\mathbf{Z}$ is governed by the integrodifferential equation of the form

$$\int d^4x'\, D_{lm}(x,x')Z_m(x') + Z_m(x)S^l_{mn}Z_n(x) = 0 . \tag{7}$$

Here the *symmetric* dispersion kernel $[D_{lm}(x',x) = D_{ml}(x,x')]$ describes a linear but nonlocal part of the response of the background and, by assumption, $\mathbf{D}(x,x')$ varies with $x-x'$ on a scale that is much faster than the variation with respect to $(x+x')$. Let a dimensionless parameter $\delta \ll 1$ describe the ratio between these two characteristic variation scales. The nonlinear coupling operator S^l_{mn} (7) is defined as

$$S^l_{mn} = c^l_{mn}(x) + d^l_{mn\mu}(z)\frac{\partial}{\partial x^\mu}, \tag{8}$$

where $d^l_{mn}(x)$ and $c^l_{mn\mu}$ are small [of $O(\delta)$] *slowly* varying coefficients so that possible nonlocal effects associated with the inhomogeneity of the background in the nonlinear terms are of $O(\delta^2)$ and can be neglected. The choice (8) for the nonlinear coupling operator is characteristic of fluid models. Nevertheless, the following reduction method proceeds similarly for any other bilinear coupling of tensorial form.

We seek a three-wave solution of Eq. (7), i.e.,

$$\mathbf{Z} = \mathbf{Z}^p + \mathbf{Z}^a + \mathbf{Z}^b , \tag{9}$$

where

$$\mathbf{Z}^q = \mathrm{Re}\{\mathbf{A}^q(x)\exp[i\psi^q(x)]\} \tag{10}$$

(q being p, a, or b) with *slowly* varying amplitudes $\mathbf{A}^q(x)$ and wave four-vectors $k^q_\mu = \partial\psi^q/\partial x^\mu$ and *rapidly* varying *real* phases $\psi^q(x)$. Now we substitute solution (9) into Eq. (7), perform the conventional geometric optics expansion in the linear integral part of this equation,[6] and observe that the required solution is obtained if the phases of the three waves in (9) are related as

$$\psi^a - \psi^b = \psi^p \tag{11}$$

(another possible case $\psi^a + \psi^b = \psi^p$ is treated similarly by substituting $\psi^b \to -\psi^b$), while the slowly varying amplitudes are described by the system of equations

$$i(\mathbf{D}^p\cdot\mathbf{A}^p + \mathbf{A}^a\cdot\mathbf{S}^{pp}\cdot\mathbf{A}^{b*}) = \mathbf{L}^p(\mathbf{A}^p) , \tag{12}$$

$$i(\mathbf{D}^a\cdot\mathbf{A}^a + \mathbf{A}^p\cdot\mathbf{S}^{ab}\cdot\mathbf{A}^b) = \mathbf{L}^a(\mathbf{A}^a) , \tag{13}$$

$$i(\mathbf{D}^b\cdot\mathbf{A}^b + \mathbf{A}^{p*}\cdot\mathbf{S}^{ba}\cdot\mathbf{A}^a) = \mathbf{L}^b(\mathbf{A}^b) , \tag{14}$$

where the dispersion matrix $\mathbf{D}^q$ is defined as

$$\mathbf{D}^q(k^q,x) = \int d^4s\, D\left(x+\frac{1}{2}s, x-\frac{1}{2}s\right)\exp(-ik^q s) \tag{15}$$

and

$$D^q_{mn}(k,x) = D^{q*}_{mn}(k,x).$$

The linear differential operator $\mathbf{L}^q$ is given by

$$[\mathbf{L}^q(\mathbf{A}^q)]_l = -\left[\frac{\partial D^q_{lm}}{\partial k^q_\mu}\partial_\mu A^q_m + \frac{1}{2}d_\mu\left(\frac{\partial D^q_{lm}}{\partial k_\mu}\right)A^q_m\right], \tag{16}$$

and

$$(\mathbf{S}^{pp})^l_{mn} = \tfrac{1}{2}[(c^l_{mn} + c^l_{mn}) + i(d^l_{mn\mu}k^a_\mu - d^l_{nm\mu}k^b_\mu)] , \tag{17}$$

$$(\mathbf{S}^{ab})^l_{mn} = \tfrac{1}{2}[(c^l_{mn} + c^l_{nm}) + i(d^l_{mn\mu}k^b_\mu + d^l_{nm\mu}k^p_\mu)] , \tag{18}$$

$$(\mathbf{S}^{ba})^l_{mn} = \tfrac{1}{2}[(c^l_{mn} + c^l_{nm}) + i(d^l_{mn\mu}k^a_\mu - d^l_{nm\mu}k^p_\mu)] . \tag{19}$$

At this point, following the conventional, linear parametric amplification picture, we assume that the amplitude $\mathbf{A}^p$ is large (the pump wave), while the $\mathbf{A}^a$ and $\mathbf{A}^b$ describe small perturbations. Then, to the lowest order, we can keep only the linear terms in Eq. (12) for $\mathbf{A}^p$:

$$i\mathbf{D}^p\cdot\mathbf{A}^p = \mathbf{L}^p(\mathbf{A}^p) . \tag{20}$$

This is the usual transport equation for a linear wave in a weakly varying background.[6] Note that due to the hermiticity of $\mathbf{D}^p$, the pump wave conserves the action four-flux J^μ:

$$\left[\frac{\partial}{\partial x^\mu} + \left(\frac{\partial k_\nu}{\partial x^\mu}\right)\frac{\partial}{\partial k_\nu}\right]J^\mu \equiv d_\mu J^\mu = 0 . \tag{21}$$

The transport equation (20) can be solved by the usual methods, i.e., by integrating along the conventional geometric optics rays generated by the dispersion relation $\mathrm{Det}\,\mathbf{D}^p = 0$ of the pump wave. The phase of the pump wave is also readily obtained by integration along the rays. Thus at this point we can assume that the pump wave $\mathbf{Z}^p$ is completely known throughout the medium. Then Eqs. (13) and (14) describe the propagation of two *linear*, weakly coupled waves $\mathbf{Z}^a$ and $\mathbf{Z}^b$ in a slowly varying background. Therefore, formally, the problem is reduced to that of the linear wave conversion problem in a four-dimensional geometry. Indeed, define a $2N$-component vector $\mathbf{A} = (\mathbf{A}^a,\mathbf{A}^b)$. To $O(\delta)$, this vector is described by the transport equation

$$i\mathbf{D}\cdot\mathbf{A} = \mathbf{L}\cdot\mathbf{A}, \tag{22}$$

where

$$\mathbf{D} = \begin{bmatrix}\mathbf{D}^a & \gamma^{ab}\\ \gamma^{ba} & \mathbf{D}^b\end{bmatrix}, \tag{23}$$

$$\gamma^{ab} = \mathbf{A}^p\cdot\mathbf{S}^{ab}, \quad \gamma^{ba} = \mathbf{A}^{p*}\cdot\mathbf{S}^{ba}, \tag{24}$$

and

$$\mathbf{L} = \begin{bmatrix}\mathbf{L}^a & 0\\ 0 & \mathbf{L}^b\end{bmatrix}. \tag{25}$$

At this stage we shall reduce the order of system (22) by using the congruent reduction technique. We define two transformation matrices $\mathbf{Q}^a(x,k^a)$ and $\mathbf{Q}^b(x,k^b)$ such that if one defines $\bar{\mathbf{A}}^a$ and $\bar{\mathbf{A}}^b$ via

$$\mathbf{A}^a = \mathbf{Q}^a\cdot\bar{\mathbf{A}}^a, \quad \mathbf{A}^b = \mathbf{Q}^b\cdot\bar{\mathbf{A}}^b, \tag{26}$$

all the components of $\bar{\mathbf{A}}^a$ and $\bar{\mathbf{A}}^b$ except $\bar{A}^a_\alpha \equiv A_\alpha$ and $\bar{A}^b_\beta \equiv A_\beta$, respectively, are annihilated $[A^a_m(m\neq\alpha) = A^b_m(m\neq\beta)\approx O(\delta)]$. The details of the step-by-step evaluation of $\mathbf{Q}^a$ and $\mathbf{Q}^b$ are given in Ref. 1, so that here we shall assume that these transformation matrices are known. Then (see Ref. 1), the remaining (nonvanishing) components A_α, A_β are described by the system of equations of form (22) with the $(2N\times 2N)$ matrix $\mathbf{D}$ replaced by the *reduced* (2×2) dispersion matrix $\mathbf{D}$ given by

$$\bar{\mathbf{D}} = \begin{bmatrix} D_\alpha & \gamma_{\alpha\beta} \\ \gamma_{\beta\alpha} & D_\beta \end{bmatrix}, \tag{27}$$

and

$$\bar{\mathbf{L}} = \begin{bmatrix} L_\alpha & 0 \\ 0 & L_\beta \end{bmatrix}, \tag{28}$$

where

$$D_\alpha = (\mathbf{Q}^{a\dagger}\cdot\mathbf{D}^a\cdot\mathbf{Q}^a)_{\alpha\alpha}, \tag{29}$$

$$D_\beta = (\mathbf{Q}^{b\dagger}\cdot\mathbf{D}^b\cdot\mathbf{Q}^b)_{\beta\beta}, \tag{30}$$

$$\gamma_{\alpha\beta} = (\mathbf{Q}^{a\dagger}\cdot\gamma^{ab}\cdot\mathbf{Q}^b)_{\alpha\beta}, \tag{31}$$

$$\gamma_{\beta\alpha} = (\mathbf{Q}^{b\dagger}\cdot\gamma^{ba}\cdot\mathbf{Q}^a)_{\beta\alpha}, \tag{32}$$

and $L_{\alpha,\beta}$ in (28) are defined as in (16) with $\mathbf{D}$ replaced by D_α and D_β, respectively. Explicitly, the reduced transport system is

$$i(D_\alpha A_\alpha + \gamma_{\alpha\beta}A_\beta) = -\left[\frac{\partial D_\alpha}{\partial k^a_\mu}\partial_\mu A_\alpha + \frac{1}{2}d_\mu\left(\frac{\partial D_\alpha}{\partial k^a_\mu}\right)A_\alpha\right], \tag{33}$$

$$i(\gamma_{\beta\alpha}A_\alpha + D_\beta A_\beta) = -\left[\frac{\partial D_\beta}{\partial k^b_\mu}\partial_\mu A_\beta + \frac{1}{2}d_\mu\left(\frac{\partial D_\beta}{\partial k^b}\right)A_\beta\right]. \tag{34}$$

At this point, we notice that D_α is fully defined via k^a, while only k^b enters the definition of D_β. The coupling coefficients, in contrast, can be evaluated only if both four-k-vectors k^a and k^b are known. Let k^a satisfy

$$D_\alpha(k^a(x),x)\equiv 0. \tag{35}$$

Then, if, as in the conventional linear mode conversion case, A_α is excited initially and A_β is small, Eqs. (33) and (34) predict an effective excitation of A_β along the rays generated by (35) in the vicinity of the crossing point x_0 given by

$$D_\alpha(k^a(x_0),x_0) = D_\beta(k^b(x_0),x_0), \tag{36}$$

where $k_0^{a,b}$ and x_0 are evaluated along the ray. Since for a given k^a, k^b is known, i.e.,

$$\mathbf{k}^b = \mathbf{k}^a - \mathbf{k}^p, \tag{37}$$

Eq. (36) can alternatively be written as

$$D_\alpha(k^a(x_0);x_0) = D_\beta(k^p(x_0) - k^a(x_0);x_0). \tag{38}$$

Also, following the steps of Ref. 2, one shows that the transmission ratio for A_α through the coupling region is

$$T = |A_\alpha(+\infty)|^2/|A_\alpha(-\infty)|^2 = \exp\{-[2\pi\,\mathrm{Re}(\gamma_{\alpha\beta}\gamma_{\beta\alpha})/|B_p|]\}, \tag{39}$$

where B_p is the Poisson bracket

$$B_p = \{D_\alpha(k^a,x), D_\beta(k^a - k^p,x)\} \tag{40}$$

leading to expression (6). This completes the proof of Williams' conjecture.

III. EXAMPLE OF REDUCTION

Consider a cold, unmagnetized plasma case, described by the system of equations

$$\nabla\times\mathbf{B}' = -4\pi eN\mathbf{V}' + \frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t}, \tag{41}$$

$$\nabla\times\mathbf{E}' = -\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t}, \tag{42}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}'\cdot\nabla\right)\mathbf{V}' = -\frac{e}{m}\mathbf{E}' - \frac{e}{mc}\mathbf{V}'\times(\mathbf{B}_0 + \mathbf{B}'), \tag{43}$$

$$\frac{\partial N}{\partial t} + \nabla\cdot(N\mathbf{V}') = 0, \tag{44}$$

where $N = N_0 + N'$, while B_0 and N_0 are the slowly varying, unperturbed background magnetic field and plasma density, respectively (we also assume that $\mathbf{V}_0 = 0$).

Define

$$(e/mc^2)(\mathbf{E}',\mathbf{B}')\equiv(\mathbf{E},\mathbf{B}), \tag{45}$$

and

$$\mathbf{V} = \omega_e(\mathbf{V}'/c), \tag{46}$$

where $\omega_e = (4\pi e^2N_0/mc^2)^{1/2}$. Then, if Δ is defined by

$$\sqrt{4\pi e^2N'/mc^2} = \omega_e + \Delta \tag{47}$$

and we seek a solution of system (41)–(43) in form (9), Eq. (44) yields

$$\Delta^q = [(\mathbf{k}^q\cdot\mathbf{V}^q)/2\omega^q] + O(\delta). \tag{48}$$

Since Δ^q enters only the nonlinear terms in Eqs. (41) and (44), one can approximate it by the zeroth order of (48). If one then defines $\mathbf{A} = (\mathbf{A}^a;\mathbf{A}^b)\equiv(\mathbf{B}^a,\mathbf{E}^a,\mathbf{V}^a;\mathbf{B}^b,\mathbf{E}^b,\mathbf{V}^b)$, our unreduced system [Eqs. (41)–(43)] can be rewritten in the standard form (22), i.e.,

$$i\mathbf{D}\cdot\mathbf{A} = i\begin{bmatrix}\mathbf{D}^a\cdot\mathbf{A}^a & \gamma^{ab}\cdot\mathbf{A}^b \\ \gamma^{ba}\cdot\mathbf{A}^a & \mathbf{D}^b\cdot\mathbf{A}^b\end{bmatrix} = \mathbf{L}(\mathbf{A}), \tag{49}$$

where

$$\mathbf{D}^a\cdot\mathbf{A}^a = \begin{bmatrix}\omega^a\,\mathbf{B}^a - \mathbf{k}^a\times\mathbf{E}^a \\ \mathbf{k}^a\times\mathbf{B}^a + \omega^a\mathbf{E}^a + i\omega_e\mathbf{V}^a \\ i\omega_e\mathbf{E}^a + \omega^a\mathbf{V}^a\end{bmatrix}, \tag{50}$$

$$\mathbf{D}^b\cdot\mathbf{A}^b = \mathbf{D}^a\cdot\mathbf{A}^a(a\to b), \tag{51}$$

$$\gamma^{ab}\cdot\mathbf{A}^b = \begin{bmatrix}0 \\ \xi_1 \\ \xi_2\end{bmatrix}, \tag{52}$$

$$\gamma^{ba}\cdot\mathbf{A}^a = \gamma^{ab}\cdot\mathbf{A}^b(b\to a;\ \mathbf{k}^p,\omega^p\to -\mathbf{k}^p, -\omega^p;\ \mathbf{A}^p\to\mathbf{A}^{p*}). \tag{53}$$

The nonlinear couplings in Eq. (52) are given by

$$\xi_1 = -(i/2)(\Delta^b\mathbf{V}^p + \Delta^p\mathbf{V}^b), \tag{54}$$

$$\xi_2 = \frac{i}{2}(\Delta^b\mathbf{E}^p + \Delta^p\mathbf{E}^b) - \frac{1}{2\omega_e}[(\mathbf{k}^b\cdot\mathbf{V}^p)\mathbf{V}^b$$

$$+ (\mathbf{k}^p\cdot\mathbf{V}^b)\mathbf{V}^p] + \frac{i}{2}(\mathbf{V}^p\times\mathbf{B}^b + \mathbf{V}^b\times\mathbf{B}^p) . \quad (55)$$

Now we proceed to the congruent reduction,[1] i.e., eliminating components of **A**, associated with large diagonal elements of **D** in Eq. (49). If, for example, D_{kk} is of $O(1)$, then the component A_k can be eliminated and the remaining components of **A**, comprising the reduced wave amplitude $\mathbf{A}[A_j = A_j\,(j\neq k)]$, are described by the transport equation of form (49) with matrix **D** replaced by the reduced matrix **D** given by[1]

$$D_{ij} = D_{ij} - (D_{ik}D_{kj}/D_{kk}), \quad i,j\neq k . \quad (56)$$

In applying this reduction procedure, we view ω^a and ω^b in **D** [see Eq. (50)] as objects of $O(1)$ and thus eliminate $\mathbf{B}^a, \mathbf{B}^b, \mathbf{V}^a$, and $\mathbf{V}^b$ in vector **A**. Note again that the nonlinear coupling parts of **D** are small [of $O(\delta)$] and, therefore, according to (56), do not affect $\bar{\mathbf{D}}^a$ and $\bar{\mathbf{D}}^b$. The reduced dispersion matrix after these reduction steps is given by

$$\bar{\mathbf{D}}^a\cdot\bar{\mathbf{A}}^a = \{[(\omega^a)^2 - (k^a)^2 - (\omega_e)^2]/\omega^a\}\mathbf{E}^a + (\mathbf{k}^a/\omega^a)(\mathbf{k}^a\cdot\mathbf{E}^a) \quad (57)$$

and

$$\bar{\gamma}^{ab}\cdot\bar{\mathbf{A}}^b = \frac{i\omega_e^2}{2\omega^a\omega^b\omega^p}\left(\mathbf{k}^a(\mathbf{E}^b\cdot\mathbf{E}^p) + \frac{\omega^p}{\omega^b}(\mathbf{k}^b\cdot\mathbf{E}^b)\mathbf{E}^p + \frac{\omega^b}{\omega^p}(\mathbf{k}^p\cdot\mathbf{E}^p)\mathbf{E}^b\right) . \quad (58)$$

We used the relation $\mathbf{k}^a = \mathbf{k}^b + \mathbf{k}^p$ in writing Eq. (58).

Now we choose the convenient representation, i.e.,

$$\mathbf{k}^p = k^p\mathbf{e}_x, \quad k_z^a = k_z^b = 0 , \quad (59)$$

so that

$$k_y^a = k_y^b \equiv k_y . \quad (60)$$

Suppose also that the *pump is an electromagnetic wave*, i.e.,

$$E_x^p = 0. \quad (61)$$

In this case matrices $\mathbf{D}^a$, $\mathbf{D}^b$ and γ^{ab}, γ^{ab} transform into

$$\bar{\mathbf{D}}^a = \frac{1}{\omega^a}\begin{bmatrix} (\omega^a)^2 - (k_y)^2 - (\omega_e)^2 & (k_x^a)^2k_y & 0 \\ (k_x^a)^2k_y & (\omega^a)^2 - (k_x^a)^2 - (\omega_e)^2 & 0 \\ 0 & 0 & (\omega^a)^2 - (k^a)^2 - (\omega_e)^2 \end{bmatrix}, \quad (62)$$

$$\bar{\mathbf{D}}^b = \bar{\mathbf{D}}^a(a\rightarrow b) , \quad (63)$$

and

$$\bar{\gamma}^{ab} = (i\omega_e^2/2\omega^a\omega^b\omega^p)\mathbf{\Gamma}_{ab} , \quad (64)$$

where

$$\mathbf{\Gamma}_{ab} = \begin{bmatrix} 0 & k_x^aE_y^p & k_x^aE_z^p \\ \frac{\omega^p}{\omega^b}k_x^bE_y^p & \frac{\omega^a}{\omega^b}k_yE_y^p & k_yE_z^p \\ \frac{\omega^p}{\omega^b}k_x^bE_z^p & \frac{\omega^p}{\omega^b}k_yE_z^p & 0 \end{bmatrix} \quad (65)$$

and, as before,

$$\gamma_{ba} = \gamma_{ab}(a,b\rightarrow b,a;\, \mathbf{k}^p,\omega^p\rightarrow -\mathbf{k}^p, -\omega^p;\, \mathbf{E}^p\rightarrow\mathbf{E}^{p*}). \quad (66)$$

It can easily be seen that the interaction exists only when one of the coupled waves is a Langmuir wave while the other is an electromagnetic wave. Let $\mathbf{A}^a$ be a Langmuir wave. Then after the elimination of E_z^a, E_x^a, and E_x^b [again, by using Eq. (56) at each step], we obtain a new reduced dispersion matrix

$$\begin{bmatrix} D_a & \alpha E_y^p & \alpha E_z^p \\ \beta E_y^{p*} & gD_b & 0 \\ \beta E_z^{p*} & 0 & D_b \end{bmatrix}, \quad (67)$$

where

$$D_a = \{[(\omega^a)^2 - (\omega_e)^2]/\omega^a\}\,[(k^a)^2/k_y^2] , \quad (68)$$

$$D_b = [(\omega^b)^2 - (\omega_e)^2 - (k^b)^2]/\omega^p , \quad (69)$$

$$g = (k^b/k_x^b)^2 , \quad (70)$$

$$\alpha = (i\omega_e^2/2\omega^a\omega^b\omega^p)\,[(k^a)^2/k_y] , \quad (71)$$

$$\beta = [i\omega_e^2/2(\omega^a)^2\omega^b]\,[(k^a)^2/k_y] . \quad (72)$$

In the cases $k^b\approx k^p\approx\omega^p\approx -\omega^b$ and $\omega^a = \omega_e \ll \omega^b,\omega^p$, dispersion matrix (67) yields the dispersion relation obtained previously[7] for parametric decay in a homogeneous plasma. The transport system at this stage is

$$\begin{aligned} &iD_aE_y^a + i\alpha E_y^pE_y^b + i\alpha E_z^pE_z^b = L_a(E_y^a), \\ &i\beta E_y^{p*}E_y^a + igD_bE_y^b = L_b(E_y^b) , \\ &i\beta E_z^{p*}E_y^a + iD_bE_z^b = L_b(E_z^b) , \end{aligned} \quad (73)$$

where L_a and L_b are defined as in (16) with **D** replaced by D_a and D_b, respectively. Now, define the "hybrid field"

$$E = (E_y^pE_y^b + E_z^pE_z^b)/\mathscr{E}, \quad (74)$$

where

$$\mathscr{E}^2 = (E_y^{p*}E_y^p/g) + E_z^{p*}E_z^p . \quad (75)$$

By multiplying the second equation in (72) by $E_y^p/g\mathscr{E}$, the third equation by $E_z^p/\mathscr{E}$, and adding the results, we obtain

$$i\beta\mathscr{E}E_y^a + iD_bE = \frac{E_y^p}{g\mathscr{E}}L_b(E_y^b) + \frac{E_z^p}{\mathscr{E}}L_b(E_z^b) = L_b(E) + \left[E_y^b\frac{\partial}{\partial x^\mu}\left(\frac{E_y^p}{g\mathscr{E}}\right)\right.$$

$$+E_z^b \frac{\partial}{\partial x^\mu}\left(\frac{E_z^p}{\mathscr{E}}\right)\Bigg]\frac{\partial D_b}{\partial k_\mu}. \quad (76)$$

Because of the assumption that the pump wave is an intense wave, the variation of $E_y^p/g\mathscr{E}$ and $E_z^p/\mathscr{E}$ near the crossing point can be assumed to be small compared to that of E^a and E^b. Therefore we can neglect the last term in the right-hand side of Eq. (65). We then obtain the final second-order transport system:

$$\begin{aligned} &iD_a E_y^a + i\alpha\mathscr{E}E = L_a(E_y^a), \\ &i\beta\mathscr{E}E_y^a + iD_b E = L_b(E), \end{aligned} \quad (77)$$

characterized by the final dispersion matrix

$$\begin{bmatrix} D_a & \alpha\mathscr{E} \\ \beta\mathscr{E} & D_b \end{bmatrix}. \quad (78)$$

This system yields the amplification coefficient

$$T = \exp[-2\pi\,\mathrm{Re}(\alpha\beta)\,\mathscr{E}^2/|B_p|], \quad (79)$$

where B_p is given by Eq. (16). In the stationary case,

$$B_p = (4\omega_e/\omega^a\omega^b)[(k^a)^2/k_y^2](\mathbf{k}^b\cdot\nabla\omega_e), \quad (80)$$

and we finally have

$$T = \exp\left(\frac{\pi\omega_e^3(k^a/\omega^a)^2}{8(\omega^a-\omega^p)\omega^p|\mathbf{k}^b\cdot\nabla\omega_e|}\,\mathscr{E}^2\right), \quad (81)$$

which describes the amplification in a general three-dimensional geometry for an arbitrary direction of propagation of the incident wave.

IV. CONCLUSIONS

(i) The congruent reduction technique was applied to the general parametric amplification problem in space- and time-varying plasmas in the presence of an intense pump wave.

(ii) The method allows a systematic derivation of the reduced coupled mode equations which, in turn, yield a compact, four-dimensional expression for the amplification coefficient for a large class of nonlinear couplings.

(iii) The reduction approach is illustrated in the example of a nonmagnetized plasma case, demonstrating the systematic approach to the problem. The same reduction technique can also be applied to much more complicated situations (e.g., magnetized, multispecies plasmas) by automating the reduction algorithm on a computer.

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[1] L. Friedland and A. Kaufman, Phys. Fluids **30**, 3050 (1987).
[2] L. Friedland, G. Goldner, and A. Kaufman, Phys. Rev. Lett. **58**, 1392 (1987).
[3] E. Williams, Phys. Rev. Lett. **59**, 2709 (1987).
[4] M. Rosenbluth, Phys. Rev. Lett. **29**, 565 (1972).
[5] For a review of the current status of the theory for magnetized plasmas, see C. S. Liu and V. K. Tripathi, Phys. Rep. **130**, 143 (1986).
[6] I. Bernstein and L. Friedland, in *Handbook of Plasma Physics*, edited by M. Rosenbluth and R. Sagdeev (North-Holland, Amsterdam, 1983), Vol. 1, Chap. 25.
[7] Y. Drake, P. Kaw, Y. Lee, G. Schmidt, C. Liu, and M. N. Rosenbluth, Phys. Fluids **17**, 778 (1974).