Anisotropy of electron-velocity distributions in weakly ionized gases with large inelastic cross sections

H. Eizenkiet and L. Friedland
Center for Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel
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The problem of anisotropy of the electron-velocity distribution in a weakly ionized gas in an electric field is discussed. In contrast to the conventional *multiple-term* truncated spherical-harmonics expansion method for weakly anisotropic distributions, the present treatment is based on a new *two-term* perturbative solution of the kinetic equation. The expansion parameter δ in the theory is the ratio between the electron-energy gain on a mean free path along the electric field and the electron energy itself. The method allows us evaluation of the angular velocity distribution for an arbitrary degree of anisotropy, provided δ is small. For instance, both the conventional two-term spherical-harmonics expansion result for a weakly anisotropic case and the beamlike distribution characteristic of gases with large inelastic cross sections are obtained as two limiting cases of the theory. The new two-term expansion allows evaluation of coefficients f_n of the conventional spherical-harmonics expansion of the distribution function to arbitrary order via a recurrent relation. The asymptotic form of these coefficients for $n \gg 1$ is obtained from this relation, allowing estimation of the convergence rate of the conventional expansion.

I. INTRODUCTION

The question of the validity of the solution of the Boltzmann equation for the electrons in a weakly ionized plasma in an electric field as a *truncated* spherical-harmonics expansion attracted a periodic attention over the last few decades. For example, one of the earliest works[1] based on the most popular two-term (Lorentz[2]) truncation argued that the applicability of the two-term approximation cannot be test *a priori*. More recent studies[3–5] found that the two-term truncation is insufficient and, in some cases, inclusion of higher harmonics results in appreciable deviations in the distribution function and the corresponding transport coefficients. These deviations were found to be significant in gases characterized by large relative inelastic cross sections. The anisotropy of the velocity distribution in such gases was large and, consequently, one needed more terms in the spherical-harmonics expansion to describe the situation adequately. Because of the absence of theoretical estimates for the convergence rate of the spherical-harmonics expansion, the required number of terms in the series in the abovementioned studies was found numerically.

A recent new approach suggested an alternative direction to overcome the spherical-harmonics expansion convergence problem. The method was based on an unconventional expansion of the integral form of the kinetic equation in powers of dimensionless parameter $\delta = eE/mv\nu$ (E, v, and ν being the electric field, the electron velocity, and the total inelastic collision frequency). If $\delta \ll 1$, i.e., when the electron-energy gain on a mean free path along the electric field is much less than the electron energy $\varepsilon = mv^2/2$, the first two terms of the new expansion yield the electron-energy distribution function $F(\varepsilon)$ [which is proportional to the zeroth-order term $f_0(\varepsilon)$ in the spherical-harmonics expansion] for *any* degree of anisotropy. Indeed, the anisotropy can be conveniently measured by the ratio $k_1 = f_2/f_0$ between the first- and the zeroth-order terms in the spherical-harmonics expansion. Monte Carlo simulations[6,7] showed that in the large inelastic loss energy regions, for $\delta \ll 1$, the anisotropy factor k_1 depends only on the ratio $A = \nu_r/\nu$ (ν_r being the total inelastic collision frequency) rather than on the electric field. Thus the anisotropy may be large in some electron-energy regions, even when the electric field is small, suggesting that the new perturbation expansion scheme in this case is valid uniformly for both low- and high-anisotropy conditions. The present study generalizes the theory of Ref. 6. We shall show that the first *two terms* of the new perturbation scheme not only define the electron-energy distribution function[6] (the symmetric part of the spherical-harmonics expansion) but also allow to evaluate the complete angular dependence of the velocity distribution for an arbitrary degree of anisotropy. The method will yield the coefficients of the spherical-harmonics expansion to arbitrary order, thus allowing us to estimate the convergence rate of the conventional expansion and to explain the results of computer simulations[6,7] such as the dependence of the anisotropy factor k_1 on $A = \nu_r/\nu$.

The presentation will be as follows. In Sec. II we shall derive our basic integral relation between the angular and the energy distribution functions. In Sec. III we shall exploit this relation within our new perturbation expansion in deriving equations describing the coefficients of the conventional spherical-harmonics expansion of the angular distribution function to arbitrary order. Section IV will be devoted to the evaluation of the anisotropy factor k_1 to $O(\delta)$ in our perturbation scheme. Finally, in Sec. V, we shall derive a recurrent relation, allowing us to find all coefficients f_n ($n > 1$) in the spherical-harmonics expansion, provided f_0 and f_1 are known. The recurrent relation will assist us in finding the asymptotic form of f_n

for $n \gg 1$ and estimating the convergence rate of the conventional spherical-harmonics expansion.

II. BASIC EQUATIONS

Define auxiliary angular distribution functions $F^-(\varepsilon,s)$ and $F^+(\varepsilon,s)$ so that $F^+ d\varepsilon ds$ and $F^- d\varepsilon ds$ are the probabilities that the electron energy ε and parameter $s=\cos\theta$ (θ is the angle between the velocity vector and the electric field) are in the intervals $[\varepsilon,\varepsilon+d\varepsilon]$ and $[s,s+ds]$, respectively, *just before* and *just after* a collision with a gas molecule. One can show[7] that if the total collision frequency ν is a constant then $F^-(\varepsilon,s)$ coincides with the conventionally defined angular distribution function $F(\varepsilon,s)$. Otherwise,

$$F(\varepsilon,s)=F^-(\varepsilon,s)\frac{\langle \nu(\varepsilon)\rangle_{\rm av}}{\nu(\varepsilon)} . \tag{1}$$

Now we can write a formal integral relation between F^- and F^+, i.e.,

$$F^-(\varepsilon,s)=\int F^+(\varepsilon_0,s_0)\rho(\varepsilon_0,s_0;\varepsilon,s)d\varepsilon_0 ds_0 , \tag{2}$$

where $\rho(\varepsilon_0,s_0;\varepsilon,s)d\varepsilon ds$ is the probability that an electron, characterized by energy ε_0 and angular parameter s_0 *just after* a collision, has energy ε and angular parameter s *just before* the *next* collision.

At this stage and in the rest of this study, we shall assume that the electron scattering in collisions is isotropic and therefore

$$F^+(\varepsilon_0,s_0)=F^+(\varepsilon_0)\psi(s_0) , \tag{3}$$

where

$$\psi(s_0)=\begin{cases}0, & |s_0|>0\\ \frac{1}{2}, & |s_0|<1\end{cases} \tag{4}$$

and $F^+(\varepsilon)$ is the electron-*energy* distribution function *just after* a collision.[6] Thus we see that, formally, Eqs. (1) and (2) define the angular distribution function $F(\varepsilon,s)$ complete via the distribution F^+, found perturbatively in Ref. 6 for an arbitrary degree of anisotropy. The use of such an approach, however, requires the knowledge of kernel $\rho(\varepsilon_0,s_0;\varepsilon,s)$ in the integral in Eq. (2).

It is convenient, at this point, to expand both $\rho(\varepsilon_0,s_0;\varepsilon,s)$ and $F^-(\varepsilon,s)$ in spherical harmonics:

$$\rho(\varepsilon_0,s_0;\varepsilon,s)=\sum_{n=0}^{\infty} a_n(\varepsilon_0,s_0;\varepsilon)P_n(s) ,$$
$$F^-(\varepsilon,s)=\sum_{n=0}^{\infty} b_n(\varepsilon)P_n(s) . \tag{5}$$

Then Eq. (2) yields

$$b_n(\varepsilon)=\int_0^\infty d\varepsilon_0 F^+(\varepsilon_0)\rho_n(\varepsilon_0,\varepsilon) , \tag{6}$$

where we define

$$\rho_n(\varepsilon_0,\varepsilon)=\int_{-1}^{+1} ds_0\psi(s_0)a_n(\varepsilon_0,s_0;\varepsilon) . \tag{7}$$

Thus the knowledge of $F^+(\varepsilon_0)$ (Ref. 6) and $\rho_n(\varepsilon_0,\varepsilon)$ allows us to evaluate all the coefficients b_n in the conventional spherical-harmonics expansion of the distribution function.

Now we shall complete our basic equations, by deriving an expression for $\rho_n(\varepsilon_0,\varepsilon)$. Similar to the approach of Ref. 6, we consider the case of a constant total collision frequency ν, first. A more general case of slowly varying, energy-dependent collision frequency can then be dealt with within the perturbative solution of the basic equations.[6] For constant ν we can write

$$z\equiv\varepsilon-\varepsilon_0=2us_0\sqrt{\varepsilon_0 B}+Bu^2 \tag{8}$$

and

$$s=[(z+\varepsilon_0 s_0^2)/\varepsilon]^{1/2}\equiv S(u,s_0,\varepsilon_0) , \tag{9}$$

where, as in Ref. 6, we have defined a new dimensionless variable $u=\nu t$ and parameter $B=e^2E^2/2m\nu^2$. Furthermore, using the known probability functions for variables u and s_0 [e^{-u} and $\psi(s_0)$, respectively], we have [see Eqs. (8) and (9)]

$$\rho(\varepsilon_0,s_0;\varepsilon,s)=\int_0^\infty du\, e^{-u}\delta[\varepsilon-(\varepsilon_0+z)]\delta[s-S(u,s_0,\varepsilon_0)] . \tag{10}$$

Then, on using definitions (5) and (7), we obtain

$$\begin{aligned}\rho_n(\varepsilon_0,\varepsilon)&=\frac{2n+1}{2}\int_{-1}^{+1} ds\, P_n(s)\left[\int_{-1}^{+1} ds_0\psi(s_0)\rho(\varepsilon_0,s_0;\varepsilon,s)\right]\\ &=\frac{2n+1}{2}\int_0^\infty du\, e^{-u}\int_{-1}^{+1} ds_0\psi(s_0)\frac{\delta(s_0-\bar{s}_0)}{\partial z/\partial s_0}P_n(S(u,s_0,\varepsilon_0))\\ &=\frac{2n+1}{4\sqrt{\varepsilon_0 B}}\int_0^\infty du\frac{e^{-u}}{u}\psi(\bar{s}_0)P_n(S(u,\bar{s}_0,\varepsilon_0)) ,\end{aligned} \tag{11}$$

where

$$\bar{s}_0=\frac{z-Bu^2}{2u\sqrt{\varepsilon_0 B}} \tag{12}$$

and therefore [see Eq. (9)]

$$S(u,\bar{s}_0,\varepsilon_0)=\frac{z+Bu^2}{2u\sqrt{\varepsilon_0 B}}+O(\delta^2) . \tag{13}$$

Equations (1), (6), and (11) comprise a complete set of equations defining the angular velocity distribution function via the symmetric electron-energy distribution $F^+(\varepsilon)$ just after a collision.

III. SPHERICAL-HARMONICS EXPANSION TO ARBITRARY ORDER

We shall proceed from the expression for the zeroth-order coefficient in the expansion of $F^-(\varepsilon,s)$ in spherical harmonics [see Eq. (6)],

$$b_0(\varepsilon)\equiv\tfrac{1}{2}F^-(\varepsilon)=\int_0^\infty d\varepsilon_0 F^+(\varepsilon_0)\rho_0(\varepsilon_0,\varepsilon)\ , \tag{14}$$

where $F^-(\varepsilon)=\int F^-(\varepsilon,s)ds$ is the *energy* distribution of electrons *just before* a collision. We have already considered Eq. (14) in Ref. 6, where a notation $\frac{1}{2}\rho_{\varepsilon_0}(\varepsilon-\varepsilon_0)$ was used instead of $\rho_0(\varepsilon_0,\varepsilon)$. In that paper, Eq. (14) was viewed as an integral equation for $F^+(\varepsilon)$ which was solved perturbatively in terms of an expansion in powers of dimensionless parameter

$$\delta=eE/mv\nu=(B/\varepsilon)^{1/2}\ll 1\ .$$

The solution was written in the form

$$F^+(\varepsilon)=a(\varepsilon)e^{\psi(\varepsilon)}\ , \tag{15}$$

where both $a(\varepsilon)$ and $\alpha(\varepsilon)=d\psi/d\varepsilon$ were assumed to be slow functions of energy in the sense that

$$\frac{1}{\alpha}\frac{d\ln\xi}{d\varepsilon}\sim O(\delta) \tag{16}$$

(ξ being either a or α), while function ψ itself was assumed to be of $O(1/\delta)$. This ordering allowed us to find a and α to $O(\delta)$ and $O(1)$, respectively, thus defining $F^+(\varepsilon)$ completely.

We observe, at this point, that Eq. (6) relates all the coefficients b_n in the expansion of F^- in spherical harmonics to the objects $a(\varepsilon)$ and $\psi(\varepsilon)$. Indeed, Eq. (6) for an arbitrary value of n has a form similar to that of Eq. (14) with $\rho_0(\varepsilon_0,\varepsilon)$ replaced by $\rho_n(\varepsilon_0,\varepsilon)$. This similarity can be exploited in generalizing the expression for b_0 derived previously [Eq. (35) in Ref. 6]:

$$F^-(\varepsilon)=2b_0(\varepsilon)=e^{\psi}\left[aR+\frac{da}{d\varepsilon}\frac{\partial R}{\partial\alpha}+\frac{a}{2}\frac{d\alpha}{d\varepsilon}\frac{\partial^2 R}{\partial\alpha^2}-\frac{1}{2}\frac{\partial^2 R}{\partial\nu\,\partial\alpha}\frac{d\nu}{d\varepsilon}a+O(\delta^2)\right]\ , \tag{17}$$

where

$$R(\alpha,\varepsilon)=\int_{-\infty}^{\varepsilon}dz\, e^{-\alpha z}\rho_{\varepsilon-z}(z)\ . \tag{18}$$

Similarly, we can write

$$b_n(\varepsilon)=e^{\psi}\left[aR_n+\frac{da}{d\varepsilon}\frac{\partial R_n}{\partial\alpha}+\frac{a}{2}\frac{d\alpha}{d\varepsilon}\frac{\partial^2 R_n}{\partial\alpha^2}-\frac{1}{2}\frac{\partial^2 R_n}{\partial\nu\,\partial\alpha}\frac{d\nu}{d\varepsilon}a+O(\delta^2)\right]\ , \tag{19}$$

with R_n defined as [see also Eq. (11)]

$$\begin{aligned}R_n(\alpha,\varepsilon)&=\int_{-\infty}^{\varepsilon}dz\, e^{-\alpha z}\rho_n(\varepsilon-z,\varepsilon)\\&=\frac{2n+1}{2}\int_{-\infty}^{\varepsilon}dz\, e^{-\alpha z}\int_0^\alpha du\frac{e^{-u}\psi(\bar{s}_0)}{2u\sqrt{\varepsilon_0 B}}P_n(S)\ .\end{aligned} \tag{20}$$

Equations (19) and (20) are the desired equations for evaluating the coefficients of the conventional spherical-harmonics expansion of the angular distribution function to a desired order. It is assumed, of course, that a and ψ are already known.[6] Note that as in Ref. 6, the term proportional to $d\nu/d\varepsilon$ in the large parentheses in (19) accounts for a possible weak energy dependence of ν [i.e., a situation when $(1/\alpha)d\nu/d\varepsilon\sim O(\delta)$].

IV. ANISOTROPY FACTOR k_1

In this section we shall restrict ourselves to the case ν=const and evaluate the anisotropy factor $k_1=f_1/f_0=b_1/b_0$ to the lowest significant order in δ. According to Eq. (19), in this case,

$$k_1=\frac{a\left[R_1+\dfrac{1}{2}\dfrac{d\alpha}{d\varepsilon}\dfrac{\partial^2 R_1}{\partial\alpha^2}\right]+\dfrac{da}{d\varepsilon}\dfrac{\partial R_1}{\partial\alpha}}{a\left[R_0+\dfrac{1}{2}\dfrac{d\alpha}{d\varepsilon}\dfrac{\partial^2 R_0}{\partial\alpha^2}\right]+\dfrac{da}{d\varepsilon}\dfrac{\partial R_1}{\partial\alpha}}+O(\delta^2)\ . \tag{21}$$

In order to simplify this expression we shall assume that $A=\nu_r/\nu$=const. Later we shall remove this restriction and generalize to the case of a weak energy dependence of A.

It was shown in Ref. 6, that in the constant A case, when A is of $O(\delta)$ or larger, we have $da/d\varepsilon=0$ and thus, to $O(\delta)$,

$$k_1=\frac{R_1+\dfrac{1}{2}\dfrac{d\alpha}{d\varepsilon}\dfrac{\partial^2 R_1}{\partial\alpha^2}}{R_0+\dfrac{1}{2}\dfrac{d\alpha}{d\varepsilon}\dfrac{\partial^2 R_0}{\partial\alpha^2}}\ . \tag{22}$$

Also,[6]

$$\alpha=\frac{\alpha'}{2\sqrt{B\varepsilon}}\ , \tag{23}$$

where α' is a constant defined by an algebraic equation

$$\mathcal{S}(\alpha')\equiv\frac{1}{2\alpha'}\ln\left|\frac{1+\alpha'}{1-\alpha'}\right|=\frac{1}{1-A}\ . \tag{24}$$

Furthermore, it was shown in Ref. 6 that

$$\begin{aligned}R_0(\alpha,\varepsilon)&=\tfrac{1}{2}\int_{-\infty}^{\varepsilon}dz\, e^{-\alpha z}\int_0^\infty du\, e^{-u}\frac{\psi(\bar{s}_0)}{2\sqrt{(\varepsilon-z)B}}\\&=\frac{1}{2}\left[\mathcal{S}+\frac{\alpha}{4\varepsilon}\frac{\partial^2\mathcal{S}}{\partial\alpha^2}\right]\ ,\end{aligned} \tag{25}$$

where we view $\mathcal{S}=\mathcal{S}(2\alpha\sqrt{B\varepsilon})$ [see Eq. (23)]. Thus the denominator in (21) is known and we shall proceed to the numerator. Here Eq. (20) yields

$$R_1=\tfrac{3}{2}\int_{-\infty}^{\varepsilon} dz\, e^{-\alpha z}\int_0^{\infty} du\, e^{-u}\frac{\psi(\bar{s}_0)}{2\sqrt{(\varepsilon-z)B}}\,\frac{z+Bu^2}{2u\sqrt{\varepsilon B}} . \tag{26}$$

Now we write $R_1=R_a+R_b$, where

$$R_a=\frac{3}{4\sqrt{\varepsilon B}}\int_{-\infty}^{\varepsilon} dz\, e^{-\alpha z}\int_0^{\infty} du\frac{e^{-u}\psi(\bar{s}_0)z}{2u\sqrt{(\varepsilon-z)B}} \tag{27}$$

and

$$R_b=\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\int_{-\infty}^{\varepsilon} dz\, e^{-\alpha z}\int_0^{\infty} du\, e^{-u}\frac{\psi(\bar{s}_0)u}{2\sqrt{(\varepsilon-z)B}} . \tag{28}$$

Let us show, at this stage, that R_1 can be expressed in terms of R_0 [see Eq. (25)]. We shall define, for this purpose, a new function of parameters α and β,

$$R(\alpha,\beta)=\tfrac{1}{2}\int_{-\infty}^{\varepsilon} dz\, e^{-\alpha z}\int_0^{\infty} du\frac{e^{-\beta u}\psi(\bar{s}_0)}{2\sqrt{(\varepsilon-z)B}} . \tag{29}$$

Note that this function, at $\beta=1$, is just R_0. Then, similar to relation (25) between R_0 and $\mathcal{S}$, one obtains (the derivation is analogous to that given in the Appendix in Ref. 6)

$$R(\alpha,\beta)=\frac{1}{2\beta}\left[\tilde{S}+\frac{\alpha}{4\varepsilon}\frac{\partial^2\tilde{S}}{\partial\alpha^2}\right]+O(\delta^2) , \tag{30}$$

where $\tilde{S}(\alpha,\beta)\equiv\mathcal{S}(\tilde{\alpha})$ and

$$\tilde{\alpha}\equiv\frac{2\alpha\sqrt{\varepsilon B}}{\beta}=\frac{\alpha'}{\beta} . \tag{31}$$

Thus, by definition, we have

$$R_0=R|_{\beta=1} , \tag{32}$$

$$R_a=\frac{3}{4\sqrt{\varepsilon B}}\left[\int d\beta\frac{\partial R}{\partial\alpha}\right]\Bigg|_{\beta=1} , \tag{33}$$

$$R_b=-\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\frac{\partial R}{\partial\beta}\Bigg|_{\beta=1} . \tag{34}$$

Equation (30) then yields [to $O(\delta)$]

$$R_a=\frac{3}{4\sqrt{\varepsilon B}}\left[\int d\beta\frac{1}{\beta^2}\left[2\sqrt{\varepsilon B}\frac{\partial\tilde{S}}{\partial\tilde{\alpha}}+\frac{B}{\beta}\frac{\partial^2\tilde{S}}{\partial\tilde{\alpha}^2}+\frac{2\alpha B\sqrt{\varepsilon B}}{\beta^2}\frac{\partial^3\tilde{S}}{\partial\tilde{\alpha}^3}\right]\right]\Bigg|_{\beta=1} \tag{35}$$

and

$$R_b=\frac{3}{4}\left[R_0-\frac{\partial\tilde{S}}{\partial\beta}\Bigg|_{\beta=1}\right]=\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\left[R_0+\frac{\partial\mathcal{S}}{\partial\alpha'}\right] . \tag{36}$$

At this point, we return to expression (22) for k_1 and evaluate it to $O(\delta^0)$ first. We observe that since α' and $\mathcal{S}$ are viewed as being of $O(\delta^0)$, the objects $\tilde{\alpha}$, $\tilde{S}$, and $\partial^n\tilde{S}/\partial\alpha^n$ are also considered to be of $O(\delta^0)$. Therefore only the first term in the large parentheses in (35) gives the $O(\delta^0)$ contribution in R_a and R_1 [R_b is of $O(\delta)$]:

$$R_1=R_a+O(\delta)=\frac{3}{2}\left[\int\frac{d\beta}{\beta^2}\frac{\partial\tilde{S}}{\partial\tilde{\alpha}}\right]\Bigg|_{\beta=1}+O(\delta) . \tag{37}$$

The integral in the last expression can be evaluated, yielding

$$\begin{aligned}R_1&=\tfrac{3}{2}\int d\beta\frac{1}{\beta^2}\sum_{k=0}^{\infty}2n\frac{(\alpha')^{2n-1}}{\beta^{2n-1}}+O(\delta)\\&=\frac{3[1-\mathcal{S}(\alpha')]}{2\alpha'}+O(\delta) .\end{aligned} \tag{38}$$

Furthermore, since $d\alpha/d\varepsilon\sim O(\delta^{-1})$ and $\partial^2/\partial\alpha^2\sim O(\delta^2)$, we can, to $O(\delta^0)$, drop the contribution of the terms with $(d\alpha/d\varepsilon)(\partial^2R_{0,1}/\partial\alpha^2)$ in expression (22) for k_1 [see expression (25) for R_0]:

$$k_1=\frac{R_1}{R_0}+O(\delta)=\frac{3[1-\mathcal{S}(\alpha')]}{\alpha'\mathcal{S}(\alpha')}+O(\delta) . \tag{39}$$

Finally, on using (24), we get ($\alpha'<0$)

$$k_1=-\frac{3A}{\alpha'}+O(\delta)=\frac{3A}{|\alpha'|}+O(\delta) . \tag{40}$$

If the calculation is performed to $O(\delta)$, one obtains (see the Appendix)

$$\begin{aligned}k_1=\frac{3A}{|\alpha'|}+\frac{3}{2}\left[\frac{B}{\varepsilon}\right]^{1/2}&\left[1+\frac{1}{\mathcal{S}(\alpha')}\sum_{n=0}^{\infty}\frac{(2n-1)\alpha'^{2n}}{(2n+1)(2n+3)}\right]\\&+O(\delta^2) .\end{aligned} \tag{41}$$

It can be also seen that for $|\alpha'|\ll 1$ (or, equivalently, $A\ll 1$) Eq. (24) yields an approximation

$$\alpha'=-\sqrt{3A} \tag{42}$$

and, therefore, expression (41) becomes

$$k_1=\sqrt{3A}+\left[\frac{B}{\varepsilon}\right]^{1/2}+O(\delta^2) , \tag{43}$$

which is the result familiar from the conventional two-term spherical-harmonics expansion.[6] On the other hand, when $A\to 1$, we have $\alpha'\to 1$, so that the first term in (41) dominates. In the limit $A\to 1$, we obtain

$$k_1\xrightarrow[A\to 1]{}3 . \tag{44}$$

The same result for k_1 is also characteristic of a beamlike distribution function $F_{\text{beam}}(\varepsilon,\theta)=f(\varepsilon)\delta(\theta)$. We shall show later that this result is not accidental and that in the limit $A \to 1$, we always have $F(\varepsilon,\theta) \to F_{\text{beam}}$.

We can generalize our results at this stage and include a possible weak energy dependence of collision frequences ν and ν_r. It was shown in Ref. 6, that $da/d\varepsilon \sim O(\delta)$ in this case, while $\alpha' = \alpha'(\varepsilon)$ is a weak function of energy, given again by Eq. (24). Then, to $O(\delta)$, Eq. (40) is still valid and k_1 depends weakly on ε. In order to check this result, we evaluated Eq. (40) for the case

$$A(\varepsilon) = \frac{\nu_r}{\nu} = \begin{cases} 0, & \varepsilon \le \xi \\ A_0\{1-\exp[-(\varepsilon-\xi)/\varepsilon_0]\}, & \varepsilon > \xi \end{cases} \tag{45}$$

where $\xi = 4$ eV, $\varepsilon_0 = 6$ eV, and $A_0 = 0.9$ and compared the results with the predictions of Monte Carlo simulations performed in Ref. 6 for the same conditions with $B = 0.02$ (note that $\delta \le 0.1$ in this case if $\varepsilon > 2$ eV). The results of the comparison are shown in Fig. 1. We can see that above the threshold of the inelastic cross section ($\varepsilon = \varepsilon_0$) the agreement between the analytic and numerical predictions is excellent. Below and near ε_0 the assumption that A is of $O(\delta)$ or more is invalid. In this energy region, the anisotropy is relatively small and thus can be dealt with within the conventional two-term spherical-harmonics expansion.

V. HIGHER-ORDER TERMS IN THE SPHERICAL-HARMONICS EXPANSION

Now we shall evaluate higher-order coefficients $b_n(\varepsilon)$ ($n > 1$) in expansion (5) of distribution $F^-(\varepsilon,s)$ in spherical harmonics. We shall restrict the treatment to the case $\nu = \text{const}$. Furthermore, coefficients b_n will be found to the lowest significant order in δ [i.e., to $O(\delta^0)$]. Note that in the $\nu = \text{const}$ case, $F^-(\varepsilon,s) = F(\varepsilon,s)$ [see Eq. (1)], so that we shall find the *complete expansion* of the angular distribution function of the electrons in the weak electric field limit ($\delta \ll 1$).

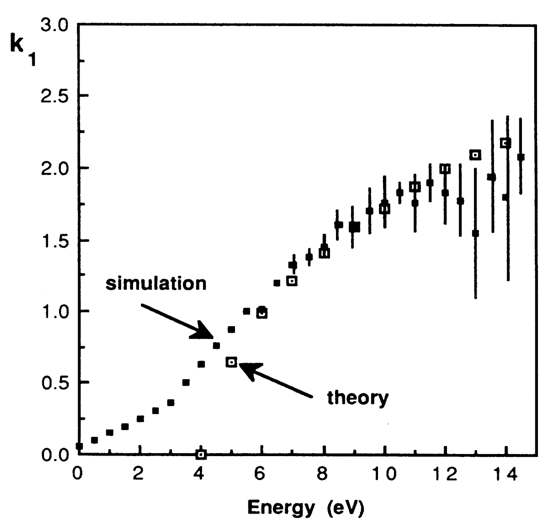

FIG. 1. Anisotropy factor $k_1 = f_1/f_0$ vs energy in the test case. The parameters are $\xi = 4$ eV, $\varepsilon_0 = 6$ eV, $B = 0.02$, $A_0 = 0.9$, and $\nu = \text{const}$. The closed squares with the error bars are the results of the simulation (Ref. 6), while the open squares represent the analytic result [Eq. (41)].

We shall start from Eq. (19), which in the case of interest becomes

$$b_n = ae^{\psi}\left[R_n + \frac{1}{2}\frac{d\alpha}{d\varepsilon}\frac{\partial^2 R_n}{\partial\alpha^2}\right] + O(\delta^2) . \tag{46}$$

Then, to $O(\delta^0)$ [see Eq. (25)],

$$b_n/b_0 = 2R_n^0/\mathcal{S} , \tag{47}$$

where R_n^0 is the $O(\delta^0)$ part of R_n. In order to evaluate R_n, we return to Eq. (20) and replace the Legendre polynomial $P_n(s)$ in the integrand by

$$P_n(s) = \sum_{k=1}^{n} a_k^n s^k . \tag{48}$$

Then R_n can be written as

$$R_n = \frac{2n+1}{2}\sum_{k=1}^{n} a_k^n Q_k , \tag{49}$$

where

$$Q_k = \int_{-\infty}^{\varepsilon} dz\, e^{-\alpha z}\int_0^{\infty} du \frac{e^{-u}\psi(s_0)}{2u\sqrt{\varepsilon_0 B}} s^k \equiv Q(s^k) . \tag{50}$$

Now we can write

$$\begin{aligned} Q(s^k) &= Q\left[\left[\frac{z+Bu^2}{2u\sqrt{\varepsilon_0 B}}\right]^k\right] \\ &= Q\left[\frac{z^k + kBu^2 z^{k-1} + \cdots + (Bu^2)^k}{\left[2u\sqrt{\varepsilon_0 B}\right]^k}\right] . \end{aligned} \tag{51}$$

Note that because of the exponential factor $\exp(-\alpha z)$, the integration over z in Eq. (50) extends effectively over distance $1/\alpha \sim O(\delta)$. Therefore factor z^k in the integrand in (50) contributes as $O(\delta^k)$ to the integral, so that to $O(\delta^0)$:

$$Q_k = Q\left[\left[\frac{z}{2u\sqrt{\varepsilon_0 B}}\right]^k\right] . \tag{52}$$

Then, on using definition (29), we can write

$$Q_k = \frac{2}{(2\sqrt{\varepsilon B})^k}\left[\int d^k\beta \frac{\partial^k}{\partial\alpha^k} R(\alpha,\beta)\right]_{\beta=1} + O(\delta) , \tag{53}$$

which, after substituting Eq. (30), yields

$$\begin{aligned} Q_k &= \left[\int d^k\beta \frac{1}{\beta^{k+1}}\frac{\partial^k}{\partial\tilde{\alpha}^k} S(\tilde{\alpha})\right]\Bigg|_{\beta=1} + O(\delta) \\ &= \sum_{n=n_0}^{\infty} (-1)^k \frac{\alpha'^{2n-k}}{2n+1} + O(\delta) , \end{aligned} \tag{54}$$

where n_0 is either m, or $m+1$, depending on whether $k = 2m$, or $2m+1$, respectively.

At this point, we wish to derive a recurrent relation for R_n. We use the known relation for Legendre polynomials,[8]

$$P_{n+1}(s)=\frac{1}{n+1}[(2n+1)sP_n(s)-nP_{n-1}(s)] , \quad (55)$$

and substitute it into expression (49) for R_n. The result is

$$R_{n+1}=\frac{2n+1}{2(n+1)}[(2n+1)Q(sP_n(s))-nQ(P_{n-1}(s))] . \quad (56)$$

But [see Eq. (20)]

$$\frac{2n-1}{2}Q(P_{n-1}(s))=R_{n-1} , \quad (57)$$

while [see Eq. (50)]

$$Q(sP_n(s))=Q\left[s\sum_{k=0}^{n}a_k^n s^k\right]=\sum_{k=0}^{n}a_k^n Q(s^{k+1}) = \sum_{k=0}^{n}a_k^n Q_{k+1} . \quad (58)$$

On the other hand, to $O(\delta)$ [see Eq. (54)],

$$Q_{k+1}=\begin{cases}-Q_k/\alpha', & k=2m+1\\ -\left[Q_k-\dfrac{1}{k+1}\right]\Big/\alpha', & k=2m .\end{cases} \quad (59)$$

The substitution of this result into (58) yields

$$Q(sP_n(s))=-\frac{1}{\alpha'}\sum_{k=0}^{n}a_k^n Q_k+\frac{1}{\alpha'}\sum_{\text{even } k}a_k^n/(k+1) . \quad (60)$$

It can be easily shown, by using the properties of Legendre polynomials, that the last sum in (60) vanishes, and therefore [see Eq. (49)]

$$Q(sP_n(s))=-\frac{2R_n}{(2n+1)\alpha'} . \quad (61)$$

Thus, finally, if one defines coefficients $k_n=b_n/b_0$, then, to $O(\delta^0)$, Eq. (56) yields

$$k_{n+1}=-\frac{2n+3}{n+1}\left[\frac{k_n}{\alpha'}+\frac{nk_{n-1}}{2n-1}\right], \quad n=1,2,3,\ldots . \quad (62)$$

Since $k_0=1$ and to $O(\delta^0)$, $k_1=3A/|\alpha'|$ (see Sec. III), recurrent relation (62) allows us to evaluate higher-order coefficient k_n easily. An interesting special case corresponds to the limit $A\to 1$ $(\alpha'\to 1)$. The recurrent relation then yields $k_n\to 2n+1$. It is precisely this result that one also gets for the beamlike distribution function $F(\varepsilon,s)=F(\varepsilon)\delta(s)$, showing that indeed, one has a beamlike distribution in energy regions dominated by inelastic collisions. Note that even this highly anisotropic case is described very well by our perturbation scheme provided δ is small. The $O(\delta^0)$ result for k_n is sufficient, of course, to describe the anisotropy in this case. The case $A<1$ requires numerical evaluation of Eq. (62). The results of such calculations are presented in Fig. 2, where coefficients k_n are shown versus n for different values of A. We observe that for higher values of A, one needs to include more terms in the spherical-harmonics expansion to adequately describe the distribution function. For example, when $A=0.3$ (the case characteristic of N_2, or CO at the maximum of the vibrational excitation cross sections) it is only when $n>6$, that one can neglect the higher-order terms in the spherical-harmonics expansion ($f_n<0.1f_0$ for $n>6$).

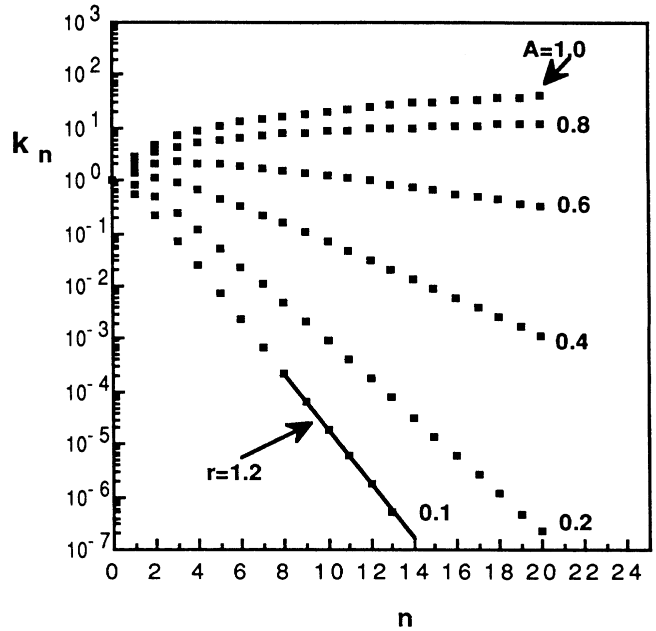

FIG. 2. Coefficients k_n vs n for different values of parameter $A=\nu_r/\nu$. The asymptotic slope of the solid line for $A=0.1$ is $r=1.2$ [Eq. (65) predicts $r=1.29$ in this case].

We also see in Fig. 2 that asymptotically, for large n, k_n decreases exponentially with n. This fact has a simple explanation. Indeed, for large n, Eq. (62) can be approximated as

$$\frac{k_{n+1}+k_{n-1}}{2}=-\frac{k_n}{\alpha'} . \quad (63)$$

This difference equation has a solution of form $k_n=A\exp(-rn)$ where r is given by

$$\cosh r=\frac{1}{|\alpha'|} . \quad (64)$$

For an easy comparison, one can use the simple approximations

$$\cosh r\approx\begin{cases}1+r^2/2, & r<1.4\\ \frac{1}{2}\exp r, & r\geq 1.4\end{cases} \quad (65)$$

which after the substitution into (64) yield

$$r\approx\begin{cases}\sqrt{2(1-|\alpha'|)/|\alpha'|}, & 1\geq|\alpha'|>0.5\\ \ln(2/|\alpha'|), & |\alpha'|<0.5 .\end{cases} \quad (66)$$

It can be checked in Fig. 2 that these slopes are in a good agreement with the results predicted by recurrent relation (62) for large n.

APPENDIX: FIRST-ORDER CORRECTION FOR k_1

Consider the denominator of Eq. (22) first. The $O(\delta)$ contribution in the denominator is [see Eq. (25)]

$$\left[\frac{\alpha}{4\varepsilon}+\frac{1}{2}\frac{d\alpha}{d\varepsilon}\right]\frac{\partial^2\mathcal{S}}{\partial\alpha^2}, \tag{A1}$$

which vanishes in view of (33). In the numerator of Eq. (22), in contrast, the $O(\delta)$ contribution survives and comes from R_b [Eq. (36)], from the second and the third terms in Eq. (35) for R_a and from

$$\tfrac{1}{2}(d\alpha/d\varepsilon)(\partial^2 R_1/\partial\alpha^2),$$

where, as in the denominator, to $O(\delta)$, we can use the zeroth-order result R_1^0 for R_1. Thus, to $O(\delta)$, the numerator of (22) is

$$\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\left[S-\frac{\partial\tilde{S}}{\partial\beta}+\int\frac{d\beta}{\beta^3}\frac{\partial^2\tilde{S}}{\partial\tilde{\alpha}^2}+\alpha'\int\frac{d\beta}{\beta^4}\frac{\partial^3\tilde{S}}{\partial\tilde{\alpha}^3}-\alpha'\frac{\partial^2 R_a^0}{\partial\alpha'^2}\right]\Bigg|_{\beta=1}. \tag{A2}$$

Finally, we expand $\mathcal{S}$ in powers of α and perform the differentiation and integration in (A2). The result is

$$\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\left[\mathcal{S}+\sum_{n=1}^{\infty}\left[\frac{2n}{2n+1}\alpha'^{2n}-\frac{2n-1}{2n+1}\alpha'^{2n-2}\right]\right]=\frac{3}{4}\left[\frac{B}{\varepsilon}\right]^{1/2}\left[\mathcal{S}+\sum_{n=0}^{\infty}\frac{(2n-1)\alpha'^{2n}}{(2n+1)(2n+3)}\right] \tag{A3}$$

and, therefore, the $O(\delta)$ contribution in k_1 is

$$k_1^1=\frac{3}{2}\left[\frac{B}{\varepsilon}\right]^{1/2}\left[1+\frac{1}{\mathcal{S}(\alpha')}\sum_{n=0}^{\infty}\frac{(2n-1)\alpha'^{2n}}{(2n+1)(2n+3)}\right]. \tag{A4}$$

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