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PHASE-SPACE SOLUTION OF THE LINEAR MODE-CONVERSION PROBLEM ☆

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The problem of linear mode conversion in a weakly inhomogeneous medium is posed and solved by phase-space methods. The PDEs for the two coupled modes are transformed to a simple first-order ordinary differential equation by a canonical transformation, wherein the two dispersion functions become essentially a locally conjugate pair of coordinates.

Wave propagation problems have traditionally been solved either in physical four-dimensional space–time (x-space) or in terms of Fourier transforms (k-space). Yet, because the hamiltonian ray equations describe orbits in eight-dimensional phase space, one would expect that more general coordinates would sometimes be of use.

Such is the case in the problem of linear mode conversion, when the components $D_{ij}(k,x)$ of the wave dispersion tensor have spatial variation in different directions [1]. To introduce notation, let us first use reduction techniques [2] to make $\boldsymbol{D}$ a 2×2 matrix field, assumed to be hermitian:

$$\boldsymbol{D}(k,x)=\begin{pmatrix} D_a(k,x) & \eta(k,x) \\ \eta^*(k,x) & D_b(k,x)\end{pmatrix}. \tag{1}$$

Here D_a and D_b are the respective dispersion functions for two modes, **a** and **b**, while η is the (small) coupling. Typically, for a ray of mode **a**, whose orbit is generated by $D_a(k,x)$:

$$\frac{dx^\mu}{d\sigma_a}=-\frac{\partial D_a}{\partial k_\mu},\quad \frac{dk_\mu}{d\sigma_a}=+\frac{\partial D_a}{\partial x^\mu} \tag{2}$$

and lies on the seven-dimensional manifold $D_a(k,x)=0$, the other dispersion function D_b is irrelevant, so long as $D_b\neq 0$.

However, if a ray of mode **a** pierces (at the point (k_c, x_c)) the seven-dimensional manifold $D_b(k,x)=0$, it will transfer wave-action to mode **b**, mediated by the coupling parameter η. Using x-space eikonal methods, we have shown [2] that the transmission ratio (transmitted intensity/incident intensity) is given by

$$T=\exp[-2\pi|\eta|^2/|\{D_a, D_b\}|]\ , \tag{3}$$

in terms of the Poisson bracket

$$\{D_a, D_b\}=\frac{\partial D_a}{\partial x^\mu}\frac{\partial D_b}{\partial k_\mu}-\frac{\partial D_a}{\partial k_\mu}\frac{\partial D_b}{\partial x^\mu}, \tag{4}$$

and the coupling η, both evaluated at the conversion point (k_c, x_c).

The appearance of the Poisson bracket (4) in the result (3) suggests that this problem could benefit from phase-space methods. We shall demonstrate that this is indeed the case, allowing for an extremely simple derivation of the result (2).

We first localize the problem about the conversion point, by expanding D_a, D_b, and η in a Taylor series, and keep *only* the leading terms. Thus D_a and D_b are linear in the first derivatives, evaluated at (k_c, x_c):

$$D_i(k,x)=\frac{\partial D_i}{\partial x^\mu}(x-x_c)^\mu+\frac{\partial D_i}{\partial k_\mu}(k-k_c)_\mu$$

$$(i=a,b)\ , \tag{5}$$

while the leading term in η is its value at (k_c, x_c).

The wave field $\boldsymbol{Z}(x)=(Z_a(x), Z_b(x))$ satisfies [3] the field equation

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$$\boldsymbol{D}(k\to -\mathrm{i}\partial/\partial x, x)\cdot \boldsymbol{Z}(x)=0\,, \tag{6}$$

which is a set of two coupled linear partial differential equations. The operator $\tilde{k}\equiv -\mathrm{i}\partial/\partial x$ satisfies

$$[x^\mu, \tilde{k}_\nu]=\mathrm{i}\delta^\mu_\nu\,, \tag{7a}$$

corresponding to the Poisson bracket

$$\{x^\mu, k_\nu\}=\delta^\mu_\nu\,. \tag{7b}$$

A *general* canonical change of coordinates has the form

$$q_i=q_i(k,x)\,,\quad p_i=p_i(k,x)\quad (i=1,2,3,4)\,, \tag{8a}$$

with

$$\{q_i, p_j\}=\delta_{i,j}\,. \tag{8b}$$

Then the hamiltonian equations are form invariant, and we have, for mode **a**,

$$\frac{\mathrm{d}q_i}{\mathrm{d}\sigma_\mathrm{a}}=-\frac{\partial \bar{D}_\mathrm{a}(q,p)}{\partial p_i}\,,\quad \frac{\mathrm{d}p_i}{\mathrm{d}\sigma_\mathrm{a}}=+\frac{\partial \bar{D}_\mathrm{a}(q,p)}{\partial q_i}\,, \tag{9a}$$

where $\bar{D}_\mathrm{a}(q,p)$ is the original dispersion function in terms of the new variables:

$$\bar{D}_\mathrm{a}(q,p)=D_\mathrm{a}(k,x)\,. \tag{9b}$$

Let us now choose

$$p_1(k,x)\equiv -D_\mathrm{a}(k,x)\,, \tag{10}$$

i.e., $\bar{D}_\mathrm{a}(q,p)=-p_1$. Then (9a) yields

$$\frac{\mathrm{d}q_1}{\mathrm{d}\sigma_\mathrm{a}}=1\,,\quad \frac{\mathrm{d}q_i}{\mathrm{d}\sigma_\mathrm{a}}=0\quad (i\neq 1)\,,\quad \frac{\mathrm{d}p_i}{\mathrm{d}\sigma_\mathrm{a}}=0\,, \tag{11}$$

in other words, q_1 is the orbit parameter of mode **a**. Next we choose

$$q_1(k,x)=\alpha D_\mathrm{b}(k,x)\,, \tag{12}$$

with α a constant determined by (8b): $\{q_1, p_1\}=1$. This yields $\alpha=B^{-1}$, where $B=\{D_\mathrm{a}, D_\mathrm{b}\}$ evaluated at $(x_\mathrm{c}, k_\mathrm{c})$. Then, similarly to (11), we have

$$\frac{\mathrm{d}p_1}{\mathrm{d}\sigma_\mathrm{b}}=B\,,\quad \frac{\mathrm{d}p_i}{\mathrm{d}\sigma_\mathrm{b}}=0\quad (i\neq 1)\,,\quad \frac{\mathrm{d}q_i}{\mathrm{d}\sigma_\mathrm{b}}=0\,, \tag{13}$$

i.e., p_1 is (within the factor B) the orbit parameter of the converted ray of mode **b**. Thus q_1 and p_1 are not only conjugate, but are the natural phase-space coordinates based on the two rays. The dispersion matrix now reads

$$\bar{\boldsymbol{D}}(p,q)=\begin{pmatrix} -p_1 & \eta \\ \eta^* & Bq_1 \end{pmatrix}. \tag{14}$$

The field equation (6), in the q-representation, is

$$\bar{\boldsymbol{D}}(p\to -\mathrm{i}\partial/\partial q, q)\cdot \bar{\boldsymbol{Z}}(q)=0\,, \tag{15}$$

where $\bar{Z}_i(q)$ is the wave field expressed in the q-representation. This is to be contrasted with $Z_i(x)$, which is the wave field in the x-representation. The change of representation implied by this comparison is analogous to, but more general than, the change of representation effected by the Fourier transform, which takes one from the x-representation to the k-representation. Furthermore, just as the Fourier transform is an integral transform, so also is the transformation connecting $Z_i(x)$ and $\bar{Z}_i(q)$. This latter transformation is an example of a *metaplectic transformation*, which are transformations of wave fields analogous to linear canonical transformations in phase space. They are reviewed in ref. [4], in which the explicit forms for the kernels of the integral operators corresponding to the metaplectic transformations are derived and displayed. Since we are mainly interested in transmission coefficients in this paper, and less in the explicit forms of wave fields, we shall not pursue these issues further, except to note that we are effectively using the theory of metaplectic transformations to cast our wave equation in the neighborhood of a mode conversion point into a standard and simple form.

Using (14) in (15), we have

$$\mathrm{i}\partial\bar{Z}_\mathrm{a}(q)/\partial q_1+\eta\bar{Z}_\mathrm{b}(q)=0\,, \tag{16a}$$

$$\eta^*\bar{Z}_\mathrm{a}(q)+Bq_1\bar{Z}_\mathrm{b}(q)=0\,. \tag{16b}$$

Eliminating $\bar{Z}_\mathrm{b}$, we have the first-order ordinary differential equation

$$\frac{\partial\bar{Z}_\mathrm{a}(q)}{\partial q_1}=\frac{\mathrm{i}|\eta|^2}{Bq_1}\bar{Z}_\mathrm{a}(q)\,, \tag{17}$$

whose solution is

$$\bar{Z}_\mathrm{a}(q)=f(q_2,q_3,q_4)\,q_1^{\mathrm{i}|\eta|^2/B}\,, \tag{18}$$

with f arbitrary.

The *amplitude* transmission ratio R is defined to be

$$R \equiv \frac{\bar{Z}_{\mathbf{a}}(+q_1, q_2, q_3, q_4)}{\bar{Z}_{\mathbf{a}}(-q_1, q_2, q_3, q_4)}, \tag{19}$$

i.e., the amplitude of mode **a** is compared at equal $|q_1|$ distances from the conversion point $q_1 = 0$. From (18) we find

$$R = (-1)^{i|\eta|^2/B} = e^{\pm \pi |\eta|^2/B}. \tag{20}$$

Choosing the proper sign for causality, we finally obtain

$$T \equiv |R|^2 = \exp[-2\pi |\eta|^2/|B|]. \tag{21}$$

Comparison of this derivation with our previous x-space approach shows how remarkably simple the phase-space method is.

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