Four-Dimensional Eikonal Theory of Linear Mode Conversion

Lazar Friedland,[(1),(2)] Galya Goldner,[(1)] and Allan N. Kaufman[(2)]
[(1)]Center for Plasma Physics, Racah Institute of Physics, Hebrew University, Jerusalem, Israel
[(2)]Lawrence Berkeley Laboratory and Physics Department, University of California, Berkeley, California 94720
(Received 2 February 1987)

The problem of linear mode conversion in nonuniform media is solved for general geometry, in that the dispersion functions $D_a(k,x)$, $D_b(k,x)$ of the two, orthogonally polarized, coupled modes a,b have spatial gradients $\partial D_a/\partial x^\mu, \partial D_b/\partial x^\mu$ which need not be parallel. The transmission ratio is found to be $\exp(-2\pi|\eta|^2/|B|)$, where η is the coupling coefficient, and B is the Poisson bracket $\{D_a,D_b\}=(\partial D_a/\partial x^\mu)(\partial D_b/\partial k_\mu)-(\partial D_a/\partial k_\mu)(\partial D_b/\partial x^\mu)$. The further generalization to weak dissipation and ray divergence is included.

PACS numbers: 03.40.Kf, 11.80.Fv, 42.20.−y, 52.35.−g

The propagation of nondegenerate waves in an anisotropic weakly nonuniform medium has been successfully treated by eikonal theory.[1] The medium supports a set of N modes, each mode having its own dispersion function $D_a(k,x)$, which is one of the N local eigenvalues of the dispersion matrix $D_{ij}(k,x)$. It is assumed that the variation of $\mathbf{D}$ and of D_a with respect to $x^\mu=(\mathbf{x},t)$ is slow, and with respect to $k_\mu=(\mathbf{k},-\omega)$ is smooth. The rays of each mode are generated independently in the eight-dimensional phase space (k,x), by the respective ray Hamiltonians $D_a(k,x)$: $dx^\mu/d\sigma_a=-\partial D_a/\partial k_\mu$, $dk_\mu/d\sigma_a=\partial D_a/\partial x^\mu$; the ray-orbit parameter σ_a is related to time by $dt/d\sigma_a=\partial D_a/\partial\omega$. For a coherent wave of mode a, its rays form a four-dimensional surface, imbedded in the seven-dimensional dispersion surface $D_a(k,x)=0$.

This conventional eikonal analysis breaks down when two of the eigenvalues (say, D_+ and D_-) are near degenerate near zero ($D_+\approx D_-\approx 0$) in some (k,x) region, causing these eigenvalues and their eigenvectors to become rapidly varying; wave action can then be effectively transferred between the two corresponding modes. The geometrical picture is explained in Fig. 1.

Almost all studies of this linear mode conversion process have been limited to the one-dimensional case; i.e., it has been assumed that the components of the dispersion matrix vary in only one direction locally.[2-4] In general, however, the two interacting modes, with polarizations $\mathbf{e}_a$ and $\mathbf{e}_b$, have nonparallel dispersion gradients, $\partial D_a/\partial x^\mu$ and $\partial D_b/\partial x^\mu$, in the conversion region. In that case, we conclude that an effectively two-dimensional treatment is required, which is presented in this Letter. Our approach generalizes previous work[5] by one of us (L.F.) on the one-dimensional case; that work was inspired by the unifying local analyses of Cairns and Lashmore-Davies[3] and of Fuchs, Ko, and Bers.[4]

Before presenting our analysis, let us summarize our results, for the case that the dispersion matrix is Hermitian. In the mode-conversion region, centered at (k_0, x_0), its two relevant eigenvalues $D_\pm(k,x)$ can be expressed as

$$D_\pm(k,x)=\tfrac{1}{2}(D_a+D_b)\pm\tfrac{1}{2}[(D_a-D_b)^2+4|\eta|^2]^{1/2}, \quad (1)$$

where D_a and D_b are locally *linear* functions on phase space,

$$D_a(k,x)=(k-k_0)_\mu V_a^\mu+(x-x_0)^\mu R_\mu^a, \quad (2)$$

and η is a small constant, representing one-half the minimum eigenvalue separation $|D_+-D_-|$. The $\pm$ signs in (1) are to be chosen so that (see Fig. 1) $D_+\to D_a$ on the incident ray, $D_+\to D_b$ on the converted ray, and $D_-\to D_a$ on the transmitted ray. Thus D_a and D_b are eigenvalues of $\mathbf{D}$ in the asymptotic region $|D_+-D_-|\gg|\eta|$. The vector coefficients in (2) are the group four-velocity $V_a^\mu=\partial D_a/\partial k_\mu$ and spatial gradient (or four-refraction) $R_\mu^a=\partial D_a/\partial x^\mu$, evaluated at

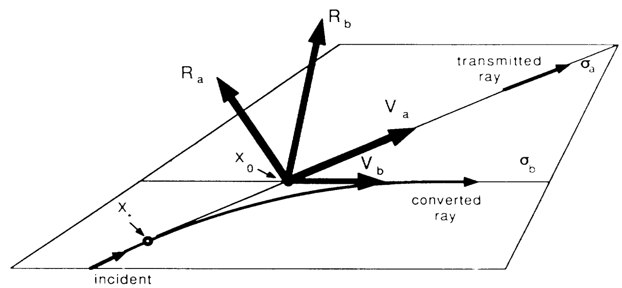

FIG. 1. Geometry of the linear mode conversion process in four-dimensional space-time. The incident and transmitted modes, with polarization $\mathbf{e}_a$, have group four-velocity V_a^μ and dispersion gradient $R_\mu^a=\partial D_a(k,x)/\partial x^\mu$. The converted mode, with polarization $\mathbf{e}_b$ (orthogonal to $\mathbf{e}_a$), has group velocity V_b^μ and dispersion gradient $R_\mu^b=\partial D_b(k,x)/\partial x^\mu$. Note that the four-vectors are not coplanar, in general. The curved ray is generated by the eigenvalue $d_+(k,x)$ of the dispersion matrix $\mathbf{D}(k,x)$. The two dispersion functions D_a,D_b are the asymptotic limits of D_+.

(k_0,x_0); similarly for mode b.

The transmission ratio T is defined as the absolute square of the ratio of the amplitudes of the transmitted and incident rays, at equal distances from x_0, in the asymptotic region. We derive the result

$$T=\exp(-2\pi|\eta|^2/|B|), \quad (3)$$

where B is the Poisson bracket,

$$B=\{D_a,D_b\}=R^a\cdot V_b-R^b\cdot V_a, \quad (4)$$

of the dispersion functions of the two modes. In the one-dimensional case, (3) reduces to the results of Refs. 2 and 4.

Our starting point is the linear integral equation

$$\int d^4x_2\hat{D}_{ij}(x_1,x_2)Z_j(x_2)=0 \quad (5)$$

for the N-component field $\mathbf{Z}(x)$ on space-time. The two-point $N\times N$ kernel matrix $\hat{D}_{ij}(x_1,x_2)$ is considered given; its local Fourier transform

$$\mathbf{D}(k,x)=\int d^4s\,\hat{\mathbf{D}}(x+\tfrac{1}{2}s,x-\tfrac{1}{2}s)\exp(-ik\cdot s) \quad (6)$$

was referred to above. We allow $\mathbf{D}=\mathbf{D}^{\mathrm{H}}+\mathbf{D}^{\mathrm{A}}$ to have a small anti-Hermitian part $\mathbf{D}^{\mathrm{A}}$, representing dissipation. The eikonal form $\mathbf{Z}(x)=\mathbf{A}(x)\exp[i\Psi(x)]$, with *slowly varying* amplitude $\mathbf{A}(x)$ and wave vector $k_\mu(x)=\partial\Psi/\partial x^\mu$, leads, to *first* order, to

$$\mathbf{D}^{\mathrm{H}}\cdot\mathbf{A}=i\left[\frac{\partial\mathbf{D}^{\mathrm{H}}}{\partial k_\mu}\cdot\frac{\partial\mathbf{A}}{\partial x^\mu}+\frac{1}{2}\frac{d(\partial\mathbf{D}^{\mathrm{H}}/\partial k_\mu)}{dx^\mu}\cdot\mathbf{A}\right]-\mathbf{D}^{\mathrm{A}}\cdot\mathbf{A}. \quad (7)$$

Expanding $\mathbf{A}(x)=\mathbf{A}^{(0)}(x)+\mathbf{A}^{(1)}(x)+\ldots$, we have, to zero order,

$$\mathbf{D}^{\mathrm{H}}(k,x)\cdot\mathbf{A}^{(0)}(x)=0. \quad (8)$$

In the nondegenerate region, we would conclude that the N vector $\mathbf{A}^{(0)}(x)$ is a scalar multiple of a local eigenvector $\mathbf{e}_a(k(x),x)$ of $\mathbf{D}^{\mathrm{H}}(k,x)$ with zero eigenvalue $D_a(k,x)$. However, in the mode-conversion region, the eigenvectors vary rapidly, and so we use the method of Ref. 5. At a point (x_*,k_*) where an incident ray *enters* the conversion region, we construct the *constant* basis pair $\mathbf{m}_a=\mathbf{e}_a(x_*)$, $\mathbf{m}_b=\mathbf{e}_b(x_*)$. Setting $\mathbf{A}(x)=A_a(x)\mathbf{m}_a+A_b(x)\mathbf{m}_b$, we obtain

$$\begin{aligned}D_aA_a+\eta A_b&=i[(\partial D_a/\partial k)\cdot(\partial A_a/\partial x)+\tfrac{1}{2}(d/dx)\cdot(\partial D_a/\partial k)A_a]-\mathbf{m}_a^*\cdot\mathbf{D}^{\mathrm{A}}\cdot\mathbf{A},\\ \eta^*A_a+D_bA_b&=i[(\partial D_a/\partial k)\cdot(\partial A_a/\partial x)+\tfrac{1}{2}(d/dx)\cdot(\partial D_b/\partial k)A_b]-\mathbf{m}_b^*\cdot\mathbf{D}^{\mathrm{A}}\cdot\mathbf{A},\end{aligned} \quad (9)$$

where $D_a(k,x)=\mathbf{m}_a^*\cdot\mathbf{D}^{\mathrm{H}}(k,x)\cdot\mathbf{m}_a$, similarly for b and $\eta(k,x)=\mathbf{m}_a^*\cdot\mathbf{D}^{\mathrm{H}}(k,x)\cdot\mathbf{m}_b$. We note that at (x_*,k_*), $D_a=0=\eta$ and $D_b\neq0$. We now continue on the ray a generated by $D_a(k,x)$ (see Fig. 1), to locate the point (x_0,k_0) where $D_b=0$. [The distance along ray a is found to be $\Delta\sigma_a=-D_b(k_*,x_*)/B$.] We set $\eta=\eta(k_0,x_0)$ (to lowest order). The left-hand side of (9) now yields Eq. (1). Further, we use the underdeterminacy of our system to set $k=k_0$ throughout the conversion region.[5] After some algebra, we obtain

$$V_a\cdot\partial A_a/\partial x-(ix'\cdot R^a+C_a)A_a=i\eta_{ab}A_b,\quad V_b\cdot\partial A_b/\partial x-(ix'\cdot R^b+C_b)A_b=i\eta_{ba}A_a, \quad (10)$$

where $x'=x-x_0$, $C_a=i\mathbf{m}_a^*\cdot\mathbf{D}^{\mathrm{A}}\cdot\mathbf{m}_a-\tfrac{1}{2}dV_a^\mu/dx^\mu$, and $\eta_{ab}=\mathbf{m}_a^*\cdot\mathbf{D}(k_0,x_0)\cdot\mathbf{m}_b$. The (real) coefficients C incorporate dissipation and divergence due to nonuniform group velocity, and are evaluated at (k_0,x_0). These equations yield the conservation law for total wave action:

$$(\partial/\partial x^\mu)[V_a^\mu(x)|A_a|^2(x)+V_b^\mu(x)|A_b|^2(x)]=i\mathbf{A}^*\cdot\mathbf{D}^{\mathrm{A}}\cdot\mathbf{A};$$

for each mode, $V_n^\mu(x)|A_n|^2(x)$ is the action flux four-vector.

With the coefficients V, R, C, and η in (10) all constants, we proceed to the solution. The change of dependent variable,

$$Y_n(x)=A_n(x)\exp\{-[i(x'\cdot R^n)^2/2+C_nx'\cdot R^n]/(V_n\cdot R^n]\},$$

and of coordinates $x\to\sigma$ in the conversion plane, $x'^\mu=V_a^\mu\sigma_a+V_b^\mu\sigma_b$, yields

$$\partial Y_a/\partial\sigma_a=i\eta_{ab}Y_b\exp[i\phi(\sigma_a,\sigma_b)],\quad \partial Y_b/\partial\sigma_b=i\eta_{ba}Y_a\exp[+i\phi(\sigma_a,\sigma_b)], \quad (11)$$

with

$$\phi=c_a\sigma_a^2/2-c_b\sigma_b^2/2-B\sigma_a\sigma_b+id_b\sigma_a-id_a\sigma_b,\quad c_a=(V_a\cdot R^a)-(V_a\cdot R^b)^2/(V_b\cdot R^b),\quad d_a=C_b-C_a(V_b\cdot R^a)/(V_a\cdot R^a).$$

Eliminating Y_b from (11) and substituting

$$g(\sigma_a,\sigma_b)=Y_a(\sigma_a,\sigma_b)\exp(-ic_b\sigma_b^2/2)$$

yields

$$\partial^2 g/\partial\sigma_a \partial\sigma_b = -\eta_{ab}\eta_{ba} g + (iB\sigma_a - d_a)\partial g/\partial\sigma_a. \tag{12}$$

By separation of variables, the solution of (12) is

$$g(\sigma_a,\sigma_b) = \int d\beta f(\beta)\exp(i\beta\sigma_b)[\sigma_a - (\beta - id_a)/B]^{-i\rho}, \tag{13}$$

with $\rho = \eta_{ab}\eta_{ba}/B$, and $f(\beta)$ arbitrary. To determine $f(\beta)$, we return to (10), and consider the initial state ($\sigma_a \to \infty$), with no excitation of mode b; we obtain

$$g \to \alpha\sigma_a^{-i\rho}\exp(-iV_b \cdot R^b \sigma_b^2/2), \tag{14}$$

with α a constant. Obtaining $f(\beta)$ as the Fourier transform of (14) with respect to σ_b, $f(\beta) = \alpha' \exp[i\beta^2/(2V_b R^b)]$, we pass to the transmission ratio:

$$T = \lim_{\sigma_a \to \infty} |A_a(+\sigma_a,\sigma_b)/A_a(-\sigma_a,\sigma_b)|^2 = |(-1)^{-i\rho}|^2 \exp(-4C_a\Delta\sigma_a). \tag{15}$$

With $(-1) = \exp(\pm\pi i)$, we choose the causal sign, and obtain

$$T = \exp[(-2\pi|\mathrm{Re}(\eta_{ab}\eta_{ba})/B| - 4C_a\Delta\sigma_a)]. \tag{16}$$

In the case of zero dissipation and no divergence ($\eta_{ab} = \eta_{ba}^*$, $C_a = 0$), this is the result (3).

In summary, we have derived an expression for the transmission ratio in linear mode conversion, for general geometry, on the basis of a local analysis. We plan to report on applications of this result in future publications.

This research was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, and by a grant from the U.S.-Israel Binational Science Foundation.

[1] I. Bernstein and L. Friedland, in *Handbook of Plasma Physics,* edited by M. Rosenbluth and R. Sagdeev (North-Holland, Amsterdam, 1983), Vol. 1, Chap. 2.5.

[2] T. Stix and D. Swanson, in Ref. 1, Chap. 2.4.

[3] R. Cairns and C. Lashmore-Davies, Phys. Fluids **26**, 1268 (1983).

[4] V. Fuchs, K. Ko, and A. Bers, Phys. Fluids **24**, 1251 (1981).

[5] L. Friedland, Phys. Fluids **28**, 3260 (1985).