Linear mode conversion and singularities of geometric optics approximation in plasmas

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The general, unreduced, one-dimensional, linear wave propagation problem in weakly inhomogeneous plasmas is considered. Conventional geometric optics perturbation expansion reduces this high-order, multimode problem to the solution of a first-order ordinary differential equation for the wave amplitude along the rays given by $\epsilon_i = 0$, where ϵ_i is one of the eigenvalues of the generalized tensor E characterizing the problem. In some plasma regions, however, this perturbation expansion, based on following only one mode at a time, fails, predicting fast variation of the zero-order amplitude and local wave vector **k** of the wave, or leading to large first-order corrections to the amplitude. Typically, in this region either (a) an additional eigenvalue of E becomes small and couples a new mode into the problem, or (b) all the eigenvalues except ϵ_i remain large; however, $\partial\epsilon_i/\partial k_x \to 0$, x being the direction of the inhomogeneity. Following an earlier work on case (a), a renormalization technique is employed in formulating a consistent, nonsingular, eikonal type perturbation expansion in case (b). The method yields a single, second-order, ordinary differential equation for amplitude of the wave, in contrast to case (a), more naturally described by a system of two first-order equations for the amplitudes of the coupled modes. Both theories thus comprise a complete general description of pairwise, linear mode conversion in the aforementioned singular regions. The approach is illustrated by the example of wave interaction in a cold magnetized plasma with plane parallel stratification. The importance of studying the unreduced problem is demonstrated by showing that in case (a) the information on the type of singularity can be lost by formulating the problem in terms of the conventional, reduced dielectric tensor.

I. INTRODUCTION

The possibility of linear mode conversion is one of the most fascinating, nontrivial consequences of plasma inhomogeneity. Even in the case of weakly inhomogeneous plasmas in which, intuitively, the waves can be described by their local wave vector $\mathbf{k}(\mathbf{r})$ and frequency ω, two or more waves can couple linearly to each other in regions, where these waves have almost identical dispersion characteristics [nearly equal values of $\mathbf{k}(\mathbf{r})$ and ω]. The coupling may result in a significant redistribution of energy between the modes after passing the mode conversion region, the effect of which has no analog in homogeneous plasmas.

At present the linear mode conversion phenomenon is being extensively studied in many branches of physics, and in particular in plasma physics, the branch richest in wave phenomena.[1] Nevertheless, until recently most studies were confined to special cases of wave propagation, each case being treated separately by a wide range of methods and techniques. The most successful approach in trying to unify and generalize the theory of linear mode conversion was certainly the approach based on the idea of association of a differential equation for slowly varying amplitude of the wave with the local plasma dispersion relation.[2] The method leads to a high-order (usually 4th or 6th) ordinary differential equation for the amplitude, asymptotic solutions of which outside the mode conversion region describe possible transformations of modes in the plasma. Recently, the technique was further improved by suggesting that, despite the complexity and multitude of possible modes in plasmas, in most cases only two modes are coupled at a time. This process should be described by a second-order differential equation for the amplitude. A method of finding such pairwise coupling events "embedded" in general plasma dispersion relations was developed by Fuchs *et al.*[3,4] The resulting, second-order amplitude equation in this theory should be solved subject to an additional external constraint postulating the conservation of energy. Although successfully used in many examples, the method faced difficulties in a class of mode conversion situations described by local dispersion relations of the form $[\omega - \omega_1(k,x)]\,[\omega - \omega_2(k,x)] = \eta$ with η small and positive and $\omega_{1,2}$ smooth, differentiable functions of k and x. The problem was resolved by Cairns and Lashmore-Davies,[5] who assumed that this type of coupling events is more naturally associated with a *pair* of coupled, first-order energy conserving differential equations for the amplitude. The method again was successfully applied to a variety of situations and gave results similar to those found by other, more refined and complex theories.

Despite the success of the two methods, the theories still left an uncertainty in the choosing of the correct number, or order, of the embedded amplitude equations. The enforcement of the energy conservation as an additional constraint,[3,4] or by choosing a certain form of the amplitude equation,[5] seems a successful guess rather than the result of a consistent theoretical prescription. An attempt to provide such a prescription was reported recently.[6] The new approach, basically, was a consistent renormalized geometric optics expansion procedure used in solving a complete system of Maxwell-kinetic equations. Conventionally,[7] the geo-

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metric optics perturbation scheme leads, in the zero order with respect to a small inhomogeneity parameter δ, to the local dispersion relation. It was this dispersion relation that was employed in the aforementioned mode conversion theories. The dispersion relation defines the rays of geometric optics but, nevertheless, does not contain such information as polarizations of the modes and energy conservation. This lack of information was exactly the source of the above-mentioned uncertainty regarding mode conversion theories based only on the dispersion relation. The difficulty can be resolved by going to higher orders in the geometric optics perturbation scheme. For instance, in a single, nondegenerate mode propagation problem, when only one eigenvalue of the dielectric tensor vanishes along the ray, the perturbation scheme to first order in δ yields an energy conserving first-order, ordinary differential equation for the amplitude along the ray.[7] If, in contrast, *two* eigenvalues of the local dielectric tensor become simultaneously small (of order δ), then as was shown in Ref. 6 the first-order corrections to the amplitude become large, and the conventional perturbation procedure fails to adequately describe the situation. This singular behavior in the mode conversion region (each of the vanishing eigenvalues defines a mode in the plasma) was removed in Ref. 6 by suitably renormalizing the original vector amplitude equation prior to application of the perturbation analysis. After the renormalization one obtains in first order a pair of coupled, energy conserving, first-order differential equations identical in form to those suggested by Cairns and Lashmore-Davies.[5] In contrast to Ref. 5, however, the renormalization procedure was not based on the study of a certain class of dispersion relations, but rather started from the basic energy conserving Maxwell-kinetic equations, leading to a description via the general properties of the local plasma dielectric tensor (its eigenvalues and eigenvectors). The theory provided a complete physical description of the coupled modes and their amplitudes and polarizations—the information missing in the studies based only on the dispersion relations.

An additional subject discussed in Ref. 6 was a generalization of the renormalization method to the case of the *unreduced* multicomponent wave propagation problem. In this most general situation, the problem is initially defined by a set of n, first-order, linear differential equations, describing the evolution of plasma parameters such as the average velocities of various plasma particles, electromagnetic fields, etc. This unreduced description yields a $(n\times n)$ local matrix equivalent of the dielectric tensor (carrying a maximum of information) and is thus more suitable for studying the mode conversion phenomenon than the reduced dielectric tensor (a 3×3 matrix). If any two of the n eigenvalues of the unreduced problem vanish simultaneously, then, similar to the reduced case, an appropriate renormalization procedure again yields a system of two coupled first-order amplitude equations. Thus we conclude that the method of Ref. 5 describes a situation when two *different* eigenvalues of, in general, an unreduced, multidimensional problem become simultaneously of order δ. Note that in this type of coupling, the study of the *unreduced* problem is important. Indeed, if after the reduction the two modes originally belonging to different eigenvalues of the unreduced problem become associated with a single eigenvalue of the reduced dielectric tensor, the information on the type of singularity is lost. We will demonstrate such a possibility in the example in Sec. IV.

In the present study we will show that in contrast to the abovementioned coupling scheme, the method of Refs. 3 and 4, which suggests a single, second-order equation for the amplitude, describes a different singular property of the geometric optics perturbation scheme. The situation is characteristic of the case when only one eigenvalue (say ϵ_1) of the unreduced problem vanishes along the ray but at the same time $\partial\epsilon_1/\partial k_x \to 0$, x being the direction of the inhomogeneity. At these points the conventional geometric optics amplitude equation also predicts singular solutions. A new renormalized perturbation procedure will be suggested in this case, yielding a nonsingular, second-order differential equation for the amplitude that is a consistent counterpart of the theory of Refs. 3 and 4.

The presentation will be as follows. Section II reviews the basic eikonal expansion scheme of the conventional geometric optics theory and discusses singular properties of the expansions. In Sec. III, a renormalized expansion scheme will be developed in regions where ϵ_1 and $\partial\epsilon_1/\partial k$ become $O(\delta)$, leading to a second-order, nonsingular, amplitude equation. The problem of energy conservation will also be considered in Sec. III. Finally, in Sec. IV we will discuss the details of the application of the theory and present an example of an unreduced problem that possesses the aforementioned different types of geometric optics singularities.

II. SINGULARITIES OF THE EIKONAL EXPANSION SCHEME

Consider the unreduced linearized problem of propagation of a small amplitude multicomponent perturbation $\mathbf{Z}$ in a weakly inhomogeneous plasma. For a cold magnetized plasma, for example, the components of vector $\mathbf{Z}$ represent the electric and magnetic fields, associated with the perturbation and perturbed electron and ion velocities. In order to treat the inhomogeneous case, consider first an initial value problem in a homogeneous situation. As usual in this case, we perform a Fourier transformation in space and a Laplace transformation in time of the Maxwell-kinetic equations underlying the problem. This, conventionally, yields an algebraic equation

$$i\omega \mathsf{E}(\mathbf{k},\omega)\cdot \mathbf{Z}_{k\omega} = \mathbf{Z}_{\mathbf{k}}(0), \tag{1}$$

where $\mathbf{Z}_{\mathbf{k},\omega}$ is the Fourier–Laplace image of $\mathbf{Z}$, $\mathbf{Z}_{\mathbf{k}}(0)$ describes initial conditions at $t=0$, and tensor E is the multicomponent counterpart of the conventional plasma dielectric tensor in the reduced problem. We now use (1) to write $\mathbf{Z}$ as a convolution integral:

$$\mathbf{Z}(\mathbf{r},t) = Z(\mathbf{r},0) + \int d^3\mathbf{r}' \int_0^t dt' \times \hat{\mathbf{\Sigma}}(\mathbf{r}-\mathbf{r}',t-t')\cdot \mathbf{Z}(\mathbf{r}',t'), \tag{2}$$

where the Fourier–Laplace image of $\hat{\mathbf{\Sigma}}$ is

$$\mathbf{\Sigma}(\mathbf{k},\omega) = \mathsf{I} - \mathsf{E}(\mathbf{k},\omega) \tag{3}$$

and I is the unit matrix. The Fourier–Laplace transformation of (2) yields Eq. (1), and thus Eq. (2) is an equivalent form of linearized Maxwell-kinetic equations.

Now we return to a weakly inhomogeneous case and, by analogy with (2), assume that it can be modeled (for the inhomogeneity in the x direction) by[6]

$$\mathbf{Z}(\mathbf{r},t) = \mathbf{Z}(\mathbf{r},0) + \int d^3\mathbf{r}' \int_0^t dt' \times \hat{\mathbf{\Sigma}}[\mathbf{r}-\mathbf{r}',t-t';(x+x')/2]\cdot\mathbf{Z}(\mathbf{r}',t'), \tag{4}$$

where $\hat{\mathbf{\Sigma}}$ varies rapidly with its difference arguments $\mathbf{r}-\mathbf{r}'$, $t-t'$, and slowly with $(x+x')/2$, describing the inhomogeneity. Next, we consider a stationary case $[\mathbf{Z}\sim\exp(i\omega t)]$, assume the existence of an infinitesimal, positive imaginary part in ω, and write an asymptotic form of Eq. (4):

$$\mathbf{Z}(\mathbf{r},t) = \int d^3r' \int_{-\infty}^t dt' \times \hat{\mathbf{\Sigma}}[\mathbf{r}-\mathbf{r}',t-t';(x+x')/2]\cdot\mathbf{Z}(\mathbf{r}',t'). \tag{5}$$

We now seek an eikonal-type solution of (5):

$$\mathbf{Z}(\mathbf{r},t) = \mathrm{Re}\{\chi(x)\exp\{i[\psi(x)+k_y y+k_z z-\omega t]\}\}, \tag{6}$$

where k_y and k_z are constants and, if we define

$$k_x = \frac{d\psi}{dx}, \tag{7}$$

then χ and k_x are slowly varying functions of x in a sense that there exists a small nondimensional parameter δ such that

$$\frac{2\pi}{k_x}\left|\frac{d\ln A}{dx}\right| < \delta \ll 1, \tag{8}$$

where A describes either $|\chi|$ or $|k_x|$. We also assume that any scalar parameter characterizing the background plasma, such as density, temperature, magnetostatic fields, etc., also satisfies inequality (8).

Substitution of (6) into (5) yields

$$\chi(x)\exp[i\psi(x)] = \int_{-\infty}^{+\infty} dx'\, \hat{\mathbf{\Sigma}}\left[k_y,k_z,\omega,x-x';\frac{(x+x')}{2}\right]\cdot\chi(x')\exp[i\psi(x')], \tag{9}$$

where we have defined the Fourier–Laplace transform

$$\hat{\mathbf{\Sigma}}[k_y,k_z,\omega,x-x';(x+x')/2] = \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{t} dt' \times\exp\{i[k_y(y-y')+k_z(z-z')-\omega(t-t')]\} \times\hat{\mathbf{\Sigma}}[\mathbf{r}-\mathbf{r}',t-t';(x+x')/2]. \tag{10}$$

Next, we introduce a new variable $s=x-x'$ in (9) and expand all the functions in the integrand in powers of s:

$$\begin{aligned}\chi(x)\exp[i\psi(x)] &= \int_{-\infty}^{+\infty} ds\,\hat{\mathbf{\Sigma}}\left(s;x-\frac{s}{2}\right)\cdot\chi(x-s)\exp[i\psi(x-s)]\\ &= \int_{-\infty}^{+\infty} ds\left(\hat{\mathbf{\Sigma}} - \frac{s}{2}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial x} + \frac{s^2}{8}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial x^2}+\cdots\right)\cdot\left(\chi - s\frac{d\chi}{dx}+\frac{s^2}{2}\frac{d^2\chi}{dx^2}+\cdots\right)\\ &\quad\times\exp\left[i\left(\psi - sk_x + \frac{s^2}{2}\frac{dk_x}{dx} - \frac{s^3}{6}\frac{d^2k_x}{dx^2}+\cdots\right)\right]\\ &\simeq e^{i\psi}\int_{-\infty}^{+\infty} ds\left[\hat{\mathbf{\Sigma}}\cdot\chi - s\left(\hat{\mathbf{\Sigma}}\cdot\frac{d\chi}{dx}+\frac{1}{2}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial x}\cdot\chi\right)+\frac{s^2}{2}\left(\frac{1}{4}\frac{\partial^2\hat{\mathbf{\Sigma}}}{\partial x^2}\cdot\chi+\frac{\partial\hat{\mathbf{\Sigma}}}{\partial x}\cdot\frac{d\chi}{dx}+\hat{\mathbf{\Sigma}}\cdot\frac{d^2\chi}{dx^2}+i\hat{\mathbf{\Sigma}}\cdot\chi\frac{dk_x}{dx}\right)\right.\\ &\quad\left.-i\frac{s^3}{2}\left(\hat{\mathbf{\Sigma}}\cdot\frac{d\chi}{dx}\frac{dk_x}{dx}+\frac{1}{2}\frac{\partial\hat{\mathbf{\Sigma}}}{\partial x}\cdot\chi\frac{dk_x}{dx}+\frac{1}{3}\frac{d^2k_x}{dx^2}\hat{\mathbf{\Sigma}}\cdot\chi\right)\right]e^{-isk_x},\end{aligned} \tag{11}$$

where we have neglected the terms of $O(\delta^3)$ and higher. Note that the spatial derivatives of $\hat{\mathbf{\Sigma}}$ in (11) are taken with respect to the slow variation argument x, holding s fixed. Now we rewrite (11) in an alternative form:

$$\begin{aligned}\chi &= \hat{\mathbf{\Sigma}}\cdot\chi - i\left[\frac{\partial\mathbf{\Sigma}}{\partial k_x}\cdot\frac{d\chi}{dx}+\frac{1}{2}\frac{d}{dx}\left(\frac{\partial\mathbf{\Sigma}}{\partial k_x}\right)\cdot\chi\right] - \frac{1}{2}\frac{d}{dx}\left(\frac{\partial^2\mathbf{\Sigma}}{\partial k_x^2}\cdot\frac{d\chi}{dx}\right)\\ &\quad-\frac{1}{8}\left(\frac{\partial^4\mathbf{\Sigma}}{\partial k_x^2\,\partial x^2}+2\frac{dk_x}{dx}\frac{\partial^4\mathbf{\Sigma}}{\partial k_x^3\,\partial x}+\frac{8}{6}\frac{d^2k_x}{dx^2}\frac{\partial^3\mathbf{\Sigma}}{\partial k_x^3}\right)\cdot\chi,\end{aligned} \tag{12}$$

where

$$\mathbf{\Sigma}(\mathbf{k},\omega;x) = \int_{-\infty}^{+\infty} ds\, e^{-isk_x}\hat{\mathbf{\Sigma}}(k_y,k_z,\omega,s;x). \tag{13}$$

Finally, define a tensor

$$\mathsf{E}(\mathbf{k},\omega;x) = \mathsf{I} - \mathbf{\Sigma}(\mathbf{k},\omega;x) \tag{14}$$

and write (12), correct to the first order in δ, as

$$\mathsf{E}\cdot\chi = i\left[\frac{\partial\mathsf{E}}{\partial k_x}\cdot\frac{d\chi}{dx}+\frac{1}{2}\frac{d}{dx}\left(\frac{\partial\mathsf{E}}{\partial k_x}\right)\cdot\chi\right] + \frac{1}{2}\frac{\partial^2\mathsf{E}}{\partial k_x^2}\cdot\frac{d^2\chi}{dx^2} = \mathsf{L}(\chi). \tag{15}$$

Note that $\mathsf{E}(k,\omega;x)$ is the local analog of $\mathsf{E}(k,\omega)$ in the homogeneous plasma [see Eq. (3)] and formally describes a homogeneous case with parameters everywhere as in our real inhomogeneous plasma at point x. Note also that in contrast to Ref. 6, we have retained in tensorial differential operator L in Eq. (15) an additional (the last) term; that is, we have assumed at this point that $d^2\chi/dx^2$ is of $O(\delta)$. The importance of this term will become clear later. Equation (15) is the desired general amplitude equation, the solution of which will be discussed in the following.

First, we note that the right-hand side of (15) is of $O(\delta)$. This suggests a solution by means of a perturbation technique. We write χ as

$$\chi = \chi_0 + \chi_1 + \chi_2 + \cdots, \quad (16)$$

where the terms are ordered in increasing powers of δ. Then, in various orders,

$$\mathsf{E}\cdot\chi_0 = 0, \quad (17)$$

$$\mathsf{E}\cdot\chi_1 = \mathsf{L}(\chi_0), \quad (18)$$

$$\mathsf{E}\cdot\chi_2 = \mathsf{L}(\chi_1), \quad (19)$$

$$\cdots .$$

At this point, we assume that E is Hermitian (the anti-Hermitian case can be treated by methods similar to those used in Ref. 7). Then we can express E as

$$\mathsf{E} = \sum_{i=1}^{n} \epsilon_i \hat{e}_i \hat{e}_i^*, \quad (20)$$

where the eigenvalues ϵ_i are real and the eigenvectors $\hat{e}_i$ are in general complex but orthonormal ($\hat{e}_i^*\cdot\hat{e}_j = \delta_{ij}$). We thus write

$$\chi_0 = \sum_{i=1}^{n}\alpha_i\hat{e}_i, \quad \chi_1 = \sum_{i=1}^{n}\gamma_i\hat{e}_i, \quad \chi_2 = \sum_{i=1}^{n}\beta_i\hat{e}_i. \quad (21)$$

Substitution of the first expression in (21) into (17) yields

$$\epsilon_1\alpha_1 = \epsilon_2\alpha_2 = \epsilon_2\alpha_3 = \cdots = \epsilon_n\alpha_n = 0, \quad (22)$$

which has a nontrivial solution only if at least one of the eigenvalues of E (say, ϵ_1) vanishes. Assume that the rest of the eigenvalues are not zero. Then (22) gives

$$\epsilon_1 = 0, \quad (23)$$

$$\alpha_1 \neq 0, \quad \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0. \quad (24)$$

Equation (23) is the dispersion relation that defines, for given ω, the function $k = k(x)$ to be used later in the first-order equation.

Next, we proceed to the first-order equation [Eq. (18)], which becomes

$$\mathsf{E}\cdot\left(\sum_{i=1}^{n}\gamma_i\hat{e}_i\right) = \mathsf{L}(\alpha_1\hat{e}_1). \quad (25)$$

Multiplication of Eq. (25) by $\hat{e}_i^*(i = 1,\cdots,n)$ yields

$$e_1^*\cdot\left\{i\left[\frac{\partial\mathsf{E}}{\partial k_x}\cdot\frac{d}{dx}(\alpha_1\hat{e}_1) + \frac{\alpha_1}{2}\frac{d}{dx}\left(\frac{\partial\mathsf{E}}{\partial k_x}\right)\cdot\hat{e}_1\right] + \frac{1}{2}\frac{\partial^2\mathsf{E}}{\partial {k_x}^2}\cdot\frac{d^2}{dx^2}(\alpha_1\hat{e}_1)\right\} = 0, \quad (26)$$

$$\gamma_i = \frac{1}{\epsilon_i}\hat{e}_i^*\cdot\mathsf{L}(\alpha_1\hat{e}_1); \quad i > 2. \quad (27)$$

Equation (26) describes the evolution of zero-order amplitude α_1. When the latter is found, Eqs. (27) yield first-order correction γ_i: ($i > 2$) to the amplitude. The remaining unknown first-order correction γ_1 is found by multiplying Eq. (19) by $\hat{e}_1^*$, yielding

$$\hat{e}_1^*\cdot L\left(\sum_{i=1}^{n}\gamma_i\hat{e}_i\right) = 0. \quad (28)$$

This completes the perturbation analysis to the lowest significant (first) order in δ.

Let us now discuss the validity of the perturbation procedure just described. First, we use the orthonormality of the eigenvectors $\hat{e}_i$ and rewrite the zero-order amplitude equation (26) as

$$\frac{\partial\epsilon_1}{\partial k_x}\frac{d\alpha_1}{dx} + \alpha_1\left\{\hat{e}_1^*\cdot\frac{\partial\mathsf{E}}{\partial k_x}\cdot\frac{d\hat{e}_1}{dx} + \frac{1}{2}\hat{e}_1^*\cdot\left[\frac{d}{dx}\left(\frac{\partial\mathsf{E}}{\partial k_x}\right)\right]\cdot\hat{e}_1\right\} - \frac{i}{2}e_1^*\cdot\frac{\partial^2\mathsf{E}}{\partial k_x^2}\cdot\frac{d^2}{dx^2}(\alpha_1\hat{e}_1) = 0. \quad (29)$$

Note that if $\partial\epsilon_1/\partial k_x$ is of $O(\delta^0)$ we can neglect the last term in (29). Indeed, by neglecting this term, we obtain

$$\frac{d\alpha_1}{dx} = -\frac{\alpha_1}{\partial\epsilon_1/\partial k_x}\left\{\hat{e}_1^*\cdot\frac{\partial\mathsf{E}}{\partial k_x}\cdot\frac{d\hat{e}_1}{dx} + \frac{1}{2}\hat{e}_1^*\cdot\left[\frac{d}{dx}\left(\frac{\partial\mathsf{E}}{\partial k_x}\right)\right]\cdot\hat{e}_1\right\}. \quad (30)$$

Differentiation of (30) yields $d^2\alpha_1/dx^2 \sim 0(\delta^2)$, confirming the validity of the approximation.

Equation (30) is the well-known first-order amplitude equation of the conventional geometric optics perturbation scheme.[7] The slowness of variation of α_1 [which is guaranteed by (30) provided that $\partial\epsilon_1/\partial k_x \sim O(\delta^0)$] is not the only necessary condition for the successful perturbation expansion. Another necessary requirement is the relative smallness of first-order corrections γ_i ($i = 1,\ldots,n$) as given by Eqs. (27) and (28). Equations (27) guarantee the smallness of γ_i for $i > 2$, as long as all the corresponding eigenvalues ϵ_i ($i > 2$) are of $O(\delta^0)$. In this case, Eq. (28) will in general keep γ_1 small, if at the boundary of the region of interest this correction is of $O(\delta)$. The problem of convergence thus arises only if, in addition to ϵ_1, another eigenvalue of E (say ϵ_2) becomes of $O(\delta)$. Then γ_2 becomes large [see Eq. (27)] and the perturbation scheme fails. This problem was considered in detail in Ref. 6. A renormalized perturbation scheme was suggested in this case, providing a convergent expansion for $\chi[\gamma_i \sim O(\delta)]$. In contrast to (30), the method of Ref. 6 yields two coupled, first-order energy conserving equations for the zero-order amplitudes of the modes associated with eigenvalues ϵ_1 and ϵ_2.

In the present study we consider a different situation where only one eigenvalue of E(ϵ_1) vanishes at a time, but in some region $\partial\epsilon_1/\partial k_x$ becomes of $O(\delta)$. One cannot use Eq. (30) in this case since it predicts a rapid variation of α_1 with x. A natural solution to the problem seems to be using Eq. (29) instead of (30). Nevertheless, the situation of $\partial\epsilon_1/\partial k_x \to 0$ usually also leads to a rapid variation of k_x, as given by the conventional dispersion relation (23). Thus, Eq. (30) still has singular properties and does not lead to the resolution of the problem. In the next section we will develop a new, renormalized perturbation scheme that resolves these difficulties.

III. RENORMALIZED PERTURBATION EXPANSION

Assume that the conventional geometric optics perturbation scheme is used in solving a wave propagation problem in a weakly inhomogeneous plasma. As described in the previous section, the procedure involves an integration of the reduced amplitude equation (30) with $k_x = k_x(x)$ given by dispersion relation (23). Assume now that at a certain point of integration x' (with corresponding $k_x = k'_x$), $\partial\epsilon_1/\partial k_x$ becomes of order δ. We introduce, in this case, additional constant values x_0 and k_{x0} in the neighborhood of x' and k'_x. The following renormalization (reordering) of general amplitude equation (15) will be based on the expansion around x_0 and k_{x0}, while their precise values will be chosen later in such a way that the final amplitude equation will have its most simple form.

We define

$$\tilde{\mathsf{E}} = \mathsf{E} + \mathbf{\Delta}, \tag{31}$$

where

$$\mathbf{\Delta} = \mathsf{E}(k_{x0}, x_0) - \mathsf{E}(k_{x0}, x) \tag{32}$$

and "renormalize" Eq. (15) by rewriting it in the form

$$\tilde{\mathsf{E}}\cdot\chi = \mathsf{L}(\chi) - \mathsf{L}'(\chi) + \mathbf{\Delta}\cdot\chi. \tag{33}$$

Here, operators L and L′ are the same as L in (15) with E replaced by $\tilde{\mathsf{E}}$ and $\mathbf{\Delta}$, respectively. Note that by construction $\mathbf{\Delta}$ does not depend on k_x and therefore L′ in (33) vanishes. Note also that by expanding in (32) we have

$$\mathbf{\Delta} \simeq -\frac{\partial \mathsf{E}_0}{\partial x_0}(x - x_0), \tag{34}$$

where $\mathsf{E}_0 = \mathsf{E}(k_{x0}, x_0)$ and, therefore, $\mathbf{\Delta}$ is of $O(\delta)$. Furthermore, (31) yields

$$\mathsf{E} = \mathsf{E}_0 + \frac{\partial \mathsf{E}_0}{\partial k_{x0}}(k_x - k_{x0}) + \frac{1}{2}\frac{\partial \mathsf{E}_0}{\partial k^2_{x0}}(k_x - k_{x0})^2 + \cdots$$
$$\approx \mathsf{E}(k_x, x_0). \tag{35}$$

At this point we solve Eq. (33) by means of a perturbation expansion. Namely, we again use (16) for χ, which after substitution into (33), gives in various orders

$$\tilde{\mathsf{E}}\cdot\chi_0 = 0, \tag{36}$$
$$\tilde{\mathsf{E}}\cdot\chi_1 = \mathsf{L}(\chi_0) + \mathbf{\Delta}\cdot\chi_0, \tag{37}$$
$$\tilde{\mathsf{E}}\cdot\chi_2 = \mathsf{L}(\chi_1) + \mathbf{\Delta}\cdot\chi_1, \tag{38}$$
$$\cdots .$$

The zero-order equation (36) differs slightly from the conventional zero-order equation (17) (first-order correction $\mathbf{\Delta}$ is added to E). The right-hand side of first-order equation (37), in contrast, includes a large additional term as compared to the conventional equation (18). These differences in the renormalized equations will allow one to remove the singularity of the conventional perturbation scheme.

The solution of system (36)–(38) proceeds along the lines of the usual perturbation scheme. We write $\tilde{\mathsf{E}}$ in terms of its eigenvalues and eigenvectors,

$$\tilde{\mathsf{E}} = \tilde{\epsilon}_1\hat{\tilde{e}}_1\hat{\tilde{e}}_1^* + \tilde{\epsilon}_2\hat{\tilde{e}}_2\hat{\tilde{e}}_2^* + \tilde{\epsilon}_3\hat{\tilde{e}}_3\hat{\tilde{e}}_3^* + \cdots + \tilde{\epsilon}_n\hat{\tilde{e}}_n\hat{\tilde{e}}_n^*, \tag{39}$$

and use representations (21) for $\chi_0, \chi_1, \chi_2, \ldots$ with $\hat{e}_i$ replaced by $\hat{\tilde{e}}_i$. Then, in the zero order, Eq. (36) gives

$$\alpha_1 \neq 0, \quad \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0 \tag{40}$$

and, as before, $\tilde{\epsilon}_2, \tilde{\epsilon}_3, \ldots, \tilde{\epsilon}_n \neq 0$, while [see Eq. (35)]

$$\tilde{\epsilon}_1 \simeq \epsilon_1(k_x, x_0)$$
$$\simeq \epsilon_{10} + \frac{\partial\epsilon_{10}}{k_{x0}}(k_x - k_{x0}) + \frac{1}{2}\frac{\partial^2\epsilon_{10}}{\partial k^2_{x0}}(k_x - k_{x0})^2 = 0, \tag{41}$$

where ϵ_{10} is evaluated at x_0 and k_{x0}.

Now we require that ϵ_{10} and $\partial\epsilon_{10}/\partial k_{x0}$ in Eq. (41) vanish simultaneously. These two conditions uniquely define the values of x_0 and k_{x0}, which were introduced at the beginning of this section. Again using x' and k'_x (the starting point of the renormalization procedure), we have to lowest significant order

$$\epsilon_{10} \simeq \frac{\partial\epsilon_1}{\partial k'_x}(k_{x0} - k'_x) + \frac{\partial\epsilon_1}{\partial x'}(x'_0 - x')$$
$$+ \frac{1}{2}\frac{\partial^2\epsilon_1}{\partial k'^2_x}(k_{x0} - k')^2 = 0, \tag{42}$$

$$\frac{\partial\epsilon_{10}}{\partial k_{x0}} \simeq \frac{\partial\epsilon_1}{\partial k'_x} + \frac{\partial^2\epsilon_1}{\partial k'^2_x}(k_{x0} - k'_x) = 0. \tag{43}$$

Here we set $\epsilon_1(x', k'_x) = 0$ and assume that $\partial\epsilon_1/\partial k'_x$ is of $O(\delta)$. The last two equations yield the desired values

$$k_{x0} = k'_x - \frac{\partial\epsilon_1/\partial k'_x}{\partial^2\epsilon_1/\partial k'^2_x}, \tag{44}$$

$$x_0 = x' + \frac{1}{2}\frac{(\partial\epsilon_1/\partial k'_x)^2}{\partial^2\epsilon_1/\partial k'^2_x\,\partial\epsilon_1/\partial x'}. \tag{45}$$

Note that, as expected, both $k_{x0} - k'_x$ and $x_0 - x'$ are of $O(\delta)$.

Now, we again consider dispersion relation (41), which gives for *all* x in the region of interest (in the neighborhood of x_0)

$$k_x(x) = k_{x0} = \text{const.} \tag{46}$$

Also, all the eigenvectors $\hat{\tilde{e}}_i = \hat{e}_i|_{x = x_0, k_x = k_{x0}} = \hat{e}_{i0}$ are then constants. With this in mind, we proceed to first-order equation (37) and by substituting $\chi_0 = \alpha_1(x)\hat{e}_{10}$, rewrite it in the form

$$\tilde{\mathsf{E}}\cdot\chi_1 = i\left[\frac{\partial(\tilde{\mathsf{E}}\cdot\hat{e}_{10})}{\partial k_x}\frac{d\alpha_1}{dx} + \frac{1}{2}\frac{d}{dx}\left(\frac{\partial(\tilde{\mathsf{E}}\cdot\hat{e}_{10})}{\partial k_x}\right)\right]$$
$$+ \frac{1}{2}\frac{\partial^2(\tilde{\mathsf{E}}\cdot\hat{e}_{10})}{\partial k^2_x}\frac{d^2\alpha_1}{dx^2} + \mathbf{\Delta}\cdot\hat{e}_{10}\alpha_1. \tag{47}$$

Multiplication of this equation by $\hat{e}^*_{10}$ finally yields

$$\frac{d^2\alpha_1}{dx^2} + Q(x)\alpha_1 = 0, \tag{48}$$

where

$$Q(x) = 2\hat{e}^*_{10}\cdot\mathbf{\Delta}(x)\cdot\hat{e}_{10}\left(\frac{\partial^2\epsilon_{10}}{\partial k^2_{x0}}\right)^{-1}$$
$$\simeq -2\epsilon_1(k_{x0}, x)\left(\frac{\partial^2\epsilon_{10}}{\partial k^2_{x0}}\right)^{-1}. \tag{49}$$

In deriving Eqs. (48) and (49), we let $\hat{e}^*_{10}\cdot\mathsf{E} = \epsilon_{10}\hat{e}_{10} = 0$, approximate $\tilde{\mathsf{E}}\cdot\hat{e}_{10} \simeq \tilde{\mathsf{E}}\cdot\hat{e}_1 = \tilde{\epsilon}_1$ in the right-hand side of (47), employ Eq. (34) for $\mathbf{\Delta}$, and use the following relations obtained from dispersion relation (41):

$$\frac{\partial \tilde{\epsilon}_1}{\partial k_x} = \frac{\partial \epsilon_{10}}{\partial k_{x0}} + \frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}(k_x - k_{x0}) = 0, \tag{50}$$

$$\frac{d}{dx}\left(\frac{\partial \tilde{\epsilon}_1}{\partial k_x}\right) = \frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\frac{dk_x}{dx} = 0, \tag{51}$$

$$\frac{\partial^2 \tilde{\epsilon}_1}{\partial k_x^2} = \frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}. \tag{52}$$

Equation (48) is the desired nonsingular amplitude equation. It has the form suggested in Ref. 4. Thus the renormalization procedure described here can be viewed as a self-consistent derivation of this equation in the most general circumstances of a multidimensional unreduced wave propagation problem. A special case of (48) is obtained if we retain only the leading term in expansion (34) for $\boldsymbol{\Delta}$. Then

$$Q(x) = -2(x - x_0)\frac{\partial \epsilon_{10}}{\partial x_0}\left(\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\right)^{-1} \tag{53}$$

and Eq. (47) assumes a well known form, describing wave propagation near caustics.[8] Other possibilities are also represented by (49). For instance, if $\partial \epsilon_{10}/\partial x_0$ in (34) vanishes, one has to go to the next order in the expansion, which gives

$$Q(x) = -(x - x_0)^2\frac{\partial^2 \epsilon_{10}}{\partial x_0^2}\left(\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\right)^{-1}. \tag{54}$$

Note that in this case we also go to higher order with respect to $(x_0 - x')$ in (42), which replaces (45) by

$$(x_0 - x')^2 = \frac{1}{2}\left(\frac{\partial \epsilon_1}{\partial k_x'}\right)^2\left(\frac{\partial^2 \epsilon_1}{\partial k_x'^2}\frac{\partial^2 \epsilon_1}{\partial x'^2}\right)^{-1}. \tag{55}$$

Situations described by $\partial \epsilon_{10}/\partial x_0 = 0$ may occur, for example, if one of the plasma parameters such as density, temperature, etc., reaches its extremum at x_0.

Finally, we discuss the problem of matching of the solutions of Eq. (48) to the conventional geometric optics modes, outside singular regions. This problem is tightly related to the question of conservation of energy. The energy conservation is naturally implied in all the described geometric optics expansion schemes if the initial unreduced system of equations conserves energy. This happens when tensor E is Hermitian. Indeed, by multiplying Eq. (12) by $\boldsymbol{\chi}^*$ from the left and taking the imaginary part of the resulting equation, in the case of Hermitian E, we obtain

$$\frac{d}{dx}\left(\boldsymbol{\chi}^*\cdot\frac{\partial \mathsf{E}}{\partial k_x}\cdot\boldsymbol{\chi}\right) + \frac{1}{2}\,\mathrm{Im}\left(\boldsymbol{\chi}^*\cdot\frac{\partial^2 \mathsf{E}}{\partial k_x^2}\cdot\frac{d^2\boldsymbol{\chi}}{dx^2} + \boldsymbol{\chi}^*\cdot\frac{\partial^3 \mathsf{E}}{\partial k_x\,\partial x}\cdot\frac{d\boldsymbol{\chi}}{dx}\right) = 0, \tag{56}$$

or

$$J = \boldsymbol{\chi}^*\cdot\frac{\partial \mathsf{E}}{\partial k_x}\cdot\boldsymbol{\chi} + \frac{1}{2}\,\mathrm{Im}\left(\boldsymbol{\chi}^*\cdot\frac{\partial^2 \mathsf{E}}{\partial k_x^2}\cdot\frac{d\boldsymbol{\chi}}{dx}\right) = \text{const}. \tag{57}$$

The last equation describes conservation of the time-averaged energy flux J in the entire domain of interest. In plasma regions, where one can apply the conventional geometric optics formalism, we neglect the small second term in (57) and obtain the usual energy flux conservation formula[7]:

$$J(x) = \sum|\alpha_1|^2\frac{\partial \epsilon_1}{\partial k_x} = \sum\frac{\partial \epsilon_1}{\partial \omega}|\alpha_1|^2\frac{\partial \omega}{\partial k_x} = \text{const}, \tag{58}$$

where the summation is over all possible excited geometric optics modes in the region of interest. In singular regions of the type described in the present work, when $\partial \epsilon_1/\partial k_x$ for one of the modes becomes of $O(\delta)$, the terms of (59) are of the same order. In this case the renormalization procedure yields a new expression for the flux

$$J(x) = |\alpha_1|^2\frac{\partial \epsilon_{10}}{\partial k_{x0}} + \frac{1}{2}\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\,\mathrm{Im}\left(\alpha_1^*\frac{d\alpha_1}{dx}\right) = \frac{1}{2}\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\,\mathrm{Im}\left(\alpha_1^*\frac{d\alpha_1}{dx}\right) = \text{const}. \tag{59}$$

Nevertheless, Eq. (56) ensures the continuity of the flux at the boundary of singular regions and the matching of the solutions of (48) onto the relevant geometric optics modes. The latter are defined by asymptotic WKB solutions of Eq. (48) on the boundaries of the singular region:

$$\alpha_+ = GY_{1+} + HY_{2+}, \quad \alpha_- = EY_{-1} + FY_{2-}, \tag{60}$$

where G, H, E, and F are constants, $-$ and $+$ denote asymptotic boundary values $x' < x_0$ and $x'' > x_0$, respectively,

$$Y_{1,2}(x) = |Q|^{-1/4}\exp\left(\pm i\mu\int_{x_0}^{x}|Q|^{1/2}dx\right), \tag{61}$$

and $\mu = 1$ or i for Q positive or negative, respectively. Note that functions $Y_{1,2}$ describe two possible geometric optics modes outside the singular region where by expanding around x_0 and k_{x0} we have

$$\epsilon_1(k,x) = \epsilon_{10} + \epsilon_{1k_{x0}}(k - k_{k0}) + \tfrac{1}{2}\epsilon_{1k_xk_{x0}}(k - k_{x0})^2 - \epsilon_{1k_xk_{x0}}Q(x) = 0, \tag{62}$$

so that

$$k - k_{x0} = \pm\mu|Q|^{1/2} \tag{63}$$

in accordance with (61). Thus the corresponding values of $\partial \epsilon_1/\partial k_x$ in Eq. (58) for the two modes are

$$\frac{\partial \epsilon_1}{\partial k_x} = \epsilon_{1k_xk_{x0}}(k - k_{x0}) = \pm\mu\epsilon_{1k_xk_{x0}}|Q|^{1/2}. \tag{64}$$

Substitution of (60) into (59) yields

$$J = \frac{1}{2}\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}(|G|^2 - |H|^2) = \frac{1}{2}\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}(|E|^2 - |F|^2), \tag{65}$$

when $Q > 0$ on both sides of the singular point x_0. This is the case described by Eq. (54) when $(\partial^2\epsilon_{10}/\partial x_0^2)/(\partial^2\epsilon_{10}/\partial k_{x0}^2) < 0$. We denote this case as case (i). If, in contrast, Q is given by (53), characteristic to caustic regions [case (ii)] where $Q > 0$ for $x < x_0$ and $Q < 0$ for $x > x_0$, we have

$$J = \frac{1}{2}\frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}(|G|^2 - |H|^2) = \frac{\partial^2 \epsilon_{10}}{\partial k_{x0}^2}\,\mathrm{Im}(E^*F). \tag{66}$$

Connection formulas (65) and (66) allow evaluation of the geometric optics amplitudes on the boundary of the singular regions. For example, in case (i) the amplitude of the incident wave in (58) is given, according to (65), by

$$|\alpha_1|^2_{\rm in} = \frac{(\partial^2\epsilon_{10}/\partial k^2_{x0})\,|G|^2}{2\,\partial\epsilon_1/\partial k_x} = \frac{|G|^2}{2Q^{1/2}}. \quad (67)$$

Similar expressions connect amplitudes of the remaining geometric optics modes on the boundary, with coefficients $|H|^2$, $|E|^2|$, and $|F|^2$. On the other hand, G, H, E, and F are not independent since Eq. (48), being second order in nature, has a general solution defined by only two constants. The interrelations between G, H, E, and F describe connections between the wave amplitudes in various channels (incident, reflected, transmitted, or mode converted) on the boundary of singular regions and depend on the concrete form of Q. Detailed study of these interrelations in case (i) can be found in Ref. 4 and in applications to lower-hybrid waves in Ref. 8. Solution in the vicinity of caustics [case (ii)] can be found elsewhere.[9]

IV. APPLICATIONS OF THE THEORY

This section implements and illustrates the renormalization technique just developed in solving a concrete, unreduced wave propagation problem within geometric optics approximation. First, we discuss a convenient way of distinguishing the described mode conversion situation from the case when the coupled modes are associated with two different eigenvalues of E, which requires a different renormalization procedure.[6] The distinction is possible without evaluating eigenvalues ϵ_i of E explicitly. Indeed, the determinant of E is $D' = \Pi^n_{i=1}\epsilon_i$, while the sum of the $(n-1)$-st order diagonal cofactors of E is given by $F' = \Sigma^n_{m=1}\Pi_{i\neq m}\epsilon_i$. Thus the necessary and sufficient conditions for two eigenvalues of E to be simultaneously of $O(\delta)$ are $D'\sim\delta^2$ and $F'\sim\delta$. This fact allows us to conveniently diagnose the proximity of different types of singular regions, while using the conventional geometric optics formalism[7] (solution of amplitude equation (30) along the rays defined by $D'[k(x),x]=0$) and to apply an appropriate renormalization procedure in two possible cases:

(a) $\partial D'/\partial k_x \sim \partial\epsilon_1/\partial k_x$ becomes small, but F remains of $O(\delta^0)$;

(b) F' becomes of $O(\delta)$. The following example illustrates this method.

Consider the case of a small amplitude, high-frequency, transverse electromagnetic wave propagating in a cold plasma along a guide magnetic field $\mathbf{B}_0 = B_0\hat{e}_z$. Assuming a one-dimensional model with slow variation of B_0 with z, namely, $B_0 = B_0(z)$ (so-called plane parallel stratification), we can describe electromagnetic perturbations $\mathbf{E}(z,t)$ and $\mathbf{B}(z,t)$ by the system of linearized Maxwell equations

$$c\hat{e}_z\times\frac{\partial\mathbf{B}}{\partial z} = \frac{\partial\mathbf{E}}{\partial T} - 4\pi e N_0\mathbf{V},$$
$$c\hat{e}_z\times\frac{\partial\mathbf{E}}{\partial z} = -\frac{\partial\mathbf{B}}{\partial t}, \quad (68)$$

where $\mathbf{E}\cdot\hat{e}_z = \mathbf{B}\cdot\hat{e}_z = 0$, N_0 in the unperturbed electron density and electron velocity perturbation $\mathbf{V}$ ($\mathbf{V}\cdot\hat{e}_z = 0$) is described by the momentum equation

$$\frac{d\mathbf{V}}{dt} = -\frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{V}}{c}\times\mathbf{B}_0\right). \quad (69)$$

We now seek solutions of Eqs. (68) and (69) in the form

$$\mathbf{E}(z,t) = \mathrm{Re}[(mc^2/e)\mathbf{a}(z)e^{-i\omega t}],$$
$$\mathbf{B}(z,t) = \mathrm{Re}[(mc^2/e)\mathbf{b}(z)e^{-i\omega t}], \quad (70)$$
$$\mathbf{V}(z,t) = \mathrm{Re}[c\mathbf{v}(z)e^{-i\omega t}].$$

The amplitudes $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v}$ then obey

$$\hat{e}_z\times\frac{\partial\mathbf{b}}{\partial z} = -i\omega'\mathbf{a} - \omega_p'^2\mathbf{v},$$
$$\hat{e}_z\times\frac{\partial\mathbf{a}}{\partial z} = i\omega\mathbf{b}, \quad (71)$$
$$-i\omega'\mathbf{v} = -\mathbf{a} - \omega_c'\mathbf{v}\times\hat{e}_z,$$

where

$$\omega' = \frac{\omega}{c}, \quad \omega_c' = \frac{\omega_c}{c} = \frac{eB_0}{mc^2}, \quad \omega_p' = \frac{\omega_p}{c} = \left(\frac{4\pi e^2 N_0}{mc^2}\right)^{1/2};$$

finally, by introducing

$$\tilde{a} = a_x + ia_y, \quad \tilde{b} = b_x + ib_y, \quad \tilde{v} = v_x + iv_y, \quad (72)$$

we rewrite (71) as

$$\frac{\partial\tilde{b}}{\partial z} = -\omega'\tilde{a} - \omega_p'^2\tilde{v}, \quad (73)$$

$$\frac{\partial\tilde{a}}{\partial z} = \omega'\tilde{b}, \quad (74)$$

$$0 = i(\omega' - \omega_c')\tilde{v} - \tilde{a}. \quad (75)$$

Equations (73)–(75) comprise a third-order system, describing an unreduced, three-component ($\tilde{a}$, $\tilde{b}$, and $\tilde{v}$) wave propagation problem. The algebraic form of (75) does not reduce the third-order nature of the system and number of possible modes, which can be seen by adding a small streaming velocity $\mathbf{V}_0 = V_0\hat{e}_z$ to the electrons, leading to an additional term $V_0(\partial\tilde{v}/\partial z)$ in the left-hand side of Eq. (75) for $\tilde{v}$.

In a homogeneous situation Eqs. (73)–(75) yield [see Eq. (1)]

$$\mathsf{E} = \frac{1}{c}\begin{pmatrix} 1 & -in & -i\sqrt{h} \\ in & 1 & 0 \\ i\sqrt{h} & 0 & 1-\sqrt{u} \end{pmatrix}, \quad (76)$$

with the wave amplitude defined by

$$\chi = \begin{pmatrix} -i\tilde{a} \\ i\tilde{b} \\ \omega_p'\tilde{v} \end{pmatrix} \quad (77)$$

and $n = ck/\omega$, $h = \omega_p^2/\omega^2$, $u = w_c^2/\omega^2$. The determinant of E is

$$D' = (1/c^3)[(1-n^2)(1-\sqrt{u}) - h^2], \quad (78)$$

and yields the dispersion relation

$$n^2 = 1 - h^2/(1-\sqrt{u}) \quad (79)$$

describing, of course, the whistler mode in the plasma. Next, we find the sum of the second-order diagonal cofactors of E:

$$F' = (1/c^2)[2(1-\sqrt{u}) + 1 - n^2 - h^2] \quad (80)$$

or, on using (79)

$$F' = (1/c^2)[2(1-\sqrt{u}) + h^2\sqrt{u}/(1-\sqrt{u})]. \quad (81)$$

At this point we can localize the singularities of the geometric optics approximation in the given case. For example, if h is of $O(\delta)$, then F' also becomes of $O(\delta)$ in the vicinity of the cyclotron resonance, where $1-u=(\omega-\omega_c)/\omega$ is of $O(\delta)$. In this case two eigenvalues of E become simultaneously small and the renormalization procedure of Ref. 6 should be applied. The origin of the coupling in this case can be seen by rewriting dispersion relation (79) in the form

$$(1-n^2)\,(\omega-\omega_c) = \omega h^2, \quad (82)$$

thus the coupling may be interpreted as taking place between the "vacuum" electromagnetic mode ($n^2 \simeq 1$) and the cyclotron mode ($\omega \simeq \omega_c$), carried by the electrons. If, initially, the energy of the wave is confined to the electromagnetic mode, substantial electron "heating" near the resonance may take place in this low-density plasma case, provided the magnetic field gradient is sufficiently small.[6] This example also illustrates the importance of studying the *unreduced* problem. Indeed, the two coupled modes belong to the *same* eigenvalue of the conventional reduced dielectric tensor,[10] thus making it difficult to diagnose and resolve the singularity.

It is easily seen from Eq. (81) that the described case of simultaneously small h and $1-u$ is the only case when F' becomes of $O(\delta)$. Thus at large plasma densities $[h \sim 0(\delta^0)]$, one can only encounter a singular situation of type (b), where only one eigenvalue of E (say ϵ_1) vanishes at a time and, in addition,

$$\frac{\partial \epsilon_1}{\partial k} = \frac{\partial D'}{\partial k}\Big/F' = \frac{2n(1-\sqrt{u})}{c^2\omega F'} = O(\delta). \quad (83)$$

Condition (83) can be satisfied, for example, by a wave approaching the electron-cyclotron resonance ($u=1$) from the higher magnetic field side ($1-u \to -0$), in which case $k_0^2 \to \infty$. Moreover, an additional eigenvalue of E becomes singular at the resonance since, as follows from (81), $F' = \epsilon_2\epsilon_3 \to -\infty$. Also, the proximity to the resonance is usually accompanied by the strong Landau damping and nonlinear effects, both leading to an irreversible heating of the electrons. Consideration of all these phenomena is out of the scope of the present work. Instead, we will discuss now another possibility of satisfying condition (83), by approaching the cutoff ($n \to 0$). This is the case when a wave, propagating towards the cyclotron resonance from the lower magnetic field side ($u<1$) passes point x_c, where $1-u(x_c) = h^2(x_c)$. In this situation the values of k_0 and x_0, in the theory developed, are 0 and x_c, respectively. Then Eq. (35) yields in the neighborhood of x_0

$$Q(x) = -2\frac{\partial D'/\partial x}{\partial^2 D'/k_x^2}\bigg|_{x_0,k_0}(x-x_0)$$

$$= -\frac{\omega}{c^2\omega_p^2}\left(\frac{1}{\omega}\frac{d\omega_c}{dx} + \frac{d\omega_p^2}{dx}\right)\bigg|_{x_0}(x-x_0). \quad (84)$$

Here we used relations

$$\frac{\partial \epsilon_{10}}{\partial x_0} = \left(\frac{\partial D'/\partial x}{F'}\right)\bigg|_{x_0,k_0},$$

$$\frac{\partial^2 \epsilon_{10}}{\partial k_0^2} = \left(\frac{\partial^2 D'/\partial k^2}{F'}\right)\bigg|_{x_0,k_0}.$$

Substitution of (84) into (48) yields a caustic-type equation for the wave amplitude, which predicts a full reflection of the wave. The cyclotron resonance is thus inaccessible and no electron heating takes place, in contrast to previously mentioned cases of low density $[h \sim O(\delta)]$ for a wave propagating from the higher magnetic field side. A different situation may occur if $\partial\epsilon_{10}/\partial x_0 = 0$ at $x = x_c$, namely,

$$\left(\frac{1}{\omega}\frac{d\omega_c}{dx} + \frac{d\omega_p^2}{dx}\right)_{x_c} = 0.$$

Then Q must be calculated from (54), and consequently a part of the wave energy may still propagate toward the cyclotron resonance. If the resonant situation then occurs in the low-density plasma, where h is of $O(\delta)$, so that the condition $F' \sim O(\delta)$ is satisfied, we may again have a substantial mode conversion and subsequent electron heating. These accessibility properties of waves in the case considered are similar to those described in Ref. 4 in application to the lower-hybrid resonance heating.

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