Toroidally linked mirrors

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Mirrors linked by toroidal segments may be stabilized by ponderomotive force produced by electromagnetic waves at the ion-cyclotron frequency. This may provide a simple alternative to tandem mirrors and bumpy tori. Approximate calculations of equilibrium, stability, and diffusion are given.

I. INTRODUCTION

We consider the possibility of stabilizing toroidally linked mirrors by the ponderomotive force of electromagnetic waves at the ion-cyclotron frequency. Ponderomotive stabilization has been demonstrated in a straight tandem mirror.[1] In tandem mirrors,[2,3] an axial electrostatic potential reduces the axial particle losses. Creating this potential is complicated, and trapped particle modes,[4] rotational instabilities, and other modes[5] may be excited in the process. A much more direct and simple way to eliminate end losses is to add toroidal segments linking the ends of two or more mirrors.

The toroidal mirror resembles a bumpy torus or ELMO bumpy square.[6] In these devices, stabilization is provided by hot electron rings. This method is limited by the requirement that the hot electrons must not respond to the fluid motion of the bulk plasma.[7,8] Toroidal mirrors do not rely on a hot electron population and are geometrically simpler (no bumps).

Equilibrium, stability, and particle diffusion are considered. Our equilibrium and stability calculations are based on the long thin magnetofluid ordering[9,10] including toroidal curvature[11] and ponderomotive force.[1,12–15] We assume low beta. In zero order, the pressure surfaces have concentric circular cross sections. In first order, we obtain the outward shift of the pressure surfaces, which is opposite to the inward shift in bumpy tori.

We consider stability in the short-wavelength limit. Equilibria are found to be stable for interchange modes, except for a small unstable region near the magnetic axis, which should flatten the pressure profile in the center. Ballooning modes in the toroidal sections do not impose a restrictive beta limit. Finite Larmor radius effects, of course, would be expected to be strongly stabilizing.

We then consider particle orbits and diffusion. The orbits are affected by the electrostatic potential, which we do not calculate, but estimate to be $e\phi \sim nkT$. The diffusion associated with the orbits is estimated using mappings.[16,17] The diffusion rate is typically comparable to that of neoclassical plateau regime, and scales approximately as the inverse square of the mirror ratio.

II. EQUILIBRIUM

The magnetic geometry consists of N mirror machines, $N>2$, connected at the ends by N toroidal segments, each of length $2\pi R/N$. Each mirror machine is divided into a central cell of length L_c and two plugs of length L_p. The cross sections are circular. The magnetic field strength in the toroidal segments is B_t, and in the central cells B_0, with mirror ratio $M = B_t/B_0 > 1$. The cross-section radius of the central cells is a; that of the toroidal sections, $a/M^{1/2}$, by flux conservation. The central cells are irradiated with ICRF (Ion-Cyclotron Radio Frequency radiation). The plugs are transition regions in which the magnetic field changes from B_0 to B_t.

A low beta, long thin equilibrium satisfies, in coordinates (r,θ,z), where the magnetic field is predominantly in the z direction,[10,12]

$$\mathbf{B}\cdot\nabla(J_z/B) + 2\boldsymbol{\kappa}\times\nabla p\cdot\hat{\mathbf{z}}/B + \nabla W\times\nabla n\cdot\hat{\mathbf{z}}/B = 0\,, \tag{1}$$

$$\mathbf{B}\cdot\nabla p = 0\,, \tag{2}$$

where $\mathbf{B}$ is the magnetic field, J_z is the current, $\boldsymbol{\kappa}$ is the curvature, p is the pressure, n is the density, and W is the ponderomotive potential, given by[12,14,15]

$$W = \frac{e^2}{4m_i\omega}\frac{|E_+|^2}{(\omega-\Omega)}\,, \tag{3}$$

where E_+ is the amplitude of left circularly polarized waves with frequency ω, m_i is the ion mass, and $\Omega = eB/m_ic$.

A vacuum field line in leading order satisfies

$$r = \sigma\rho\,, \tag{4}$$

where $\sigma = \sigma(z)$ and $\rho = $ const on a field line; the curvature $\boldsymbol{\kappa}$ is given by[10,11]

$$\boldsymbol{\kappa} = \nabla\left(\frac{1}{\sigma}\frac{d^2\sigma}{dz^2}\frac{r^2}{2} - \frac{r\cos\theta}{R}\right), \tag{5}$$

where the toroidal curvature $1/R$ vanishes outside the toroidal segments and $d\sigma/dz$ vanishes outside the plugs.

The magnetic field has the form

$$\mathbf{B} = \nabla\psi\times\hat{\mathbf{z}} + B_v\nabla\zeta\,,$$
$$\zeta = z - \frac{1}{\sigma}\frac{d\sigma}{dz}\frac{r^2}{2} + O\left(\frac{r}{R}\right),\quad B_v = \frac{B_0}{\sigma^2}\,, \tag{6}$$

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where r/R is assumed small, and B_0 is constant, by axial flux conservation.

In the absence of toroidal curvature, $1/R = 0$, we have $B = B_v(z)$, $\psi = 0$, and $p = p_0(r)$. A shear Alfvén displacement of the form

$$\boldsymbol{\xi} = \nabla u \times \hat{\mathbf{z}}/B_v \qquad (7)$$

produces perturbations of ψ, p, and n:

$$\psi_1 = (\mathbf{B}\cdot\nabla u)/B_v\,, \qquad (8)$$

$$p_1 = -\frac{1}{r}\frac{dp}{dr}\frac{\partial u}{\partial\theta}\frac{1}{B_v}\,, \qquad (9)$$

$$n_1 = -\frac{1}{r}\frac{dn}{dr}\frac{\partial u}{\partial\theta}\frac{1}{B_v}\,. \qquad (10)$$

Assuming that β is small, the displacement stream function u satisfies, substituting (8)–(10) into (1) and using (5),

$$\nabla_0^2\frac{\partial^2 u}{\partial z^2} + (S-K)\frac{\partial^2 u}{\partial\theta^2} = D\rho\sin\theta - D\left(\sin\theta\frac{\partial^2 u}{\partial\rho\,\partial\theta} + \frac{\cos\theta}{\rho}\frac{\partial^2 u}{\partial\theta^2}\right), \qquad (11)$$

where

$$\nabla_0^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\,,$$

$$K = -\frac{1}{\rho^2}\frac{\partial W}{\partial\rho}\frac{dn_0}{d\rho}\frac{\sigma^2}{B_0^2}\,, \qquad (12)$$

$$S = \frac{2\sigma^3}{\rho B_0^2}\frac{d^2\sigma}{dz^2}\frac{dp_0}{d\rho}\,, \qquad (13)$$

$$D = -\frac{2\sigma^3}{RB_0^2\rho}\frac{dp_0}{d\rho}\,. \qquad (14)$$

For small $D \ll S \sim K$ we can neglect the last terms on the right side of (11) proportional to Du. This approximation is roughly that $a/(RM^{3/2}) \ll (a/L_p)^2$ and is improved by having a mirror ratio $M \gg 1$.

We obtain an approximate solution of (11) for the case in which n_0, p_0, W_0 have parabolic profiles:

$$p_0 = \tfrac{1}{2}\beta B_0^2(1-\rho^2/a^2)\,, \quad W = W_0(\rho/a)^2\,. \qquad (15)$$

Expanding

$$u = [u_1(\rho/a) + u_3(\rho/a)^3 + \cdots]\sin\theta$$

gives a set of equations

$$\frac{8}{a^2}\frac{\partial^2}{\partial z^2}u_3 + (K-S)u_1 = D\,,$$

$$\left(\frac{n^2+4n+3}{a^2}\right)\frac{\partial^2}{\partial z^2}u_{n+2} + (S-K)u_n = 0\,, \quad n>1\,, \qquad (16)$$

where S and K depend only on z. The boundary condition $u=0, \rho=a$ requires

$$u_1 + u_3 + \cdots = 0\,. \qquad (17)$$

Because of the large coefficient (n^2+4n+3), the u_n converge rapidly; thus, we shall take only the first two terms u_1, u_3. Using the condition (17) in the $n=1$ equation [(16)] gives

$$\frac{8}{a^2}\frac{\partial^2 u_1}{\partial z^2} + (K-S)u_1 = D\,. \qquad (18)$$

An interchange-type solution can be obtained if $(K-S)L^2a^2 \ll 1$, where L is the length of the torus,

$$u_1 = \langle D\rangle/\langle K-S\rangle\,, \qquad (19)$$

where the bracket denotes a z average.

To calculate the shift and to compare with a straight mirror, introduce the constant λ,

$$\langle K\rangle = (\lambda+1)\langle S\rangle\,, \qquad (20)$$

where $\lambda = 0$ is the marginal stability criterion for interchange (flute) modes in a straight mirror. We estimate S by assuming that in a plug of length $L_p = z_2 - z_1$,

$$\sigma^2 = \tfrac{1}{2}(1+M^{-1}) + \tfrac{1}{2}(1-M^{-1})\cos[\pi(z-z_1)/L_p]\,,$$

where M is the mirror ratio; in the central cell $\sigma^2 = 1$, and in a toroidal segment $\sigma^2 = 1/M$.

The average curvature due to $2N$ plugs is

$$\left\langle\sigma^3\frac{d^2\sigma}{dz^2}\right\rangle = -\frac{3}{4}\left\langle\left(\frac{d\sigma^2}{dz}\right)^2\right\rangle = -\frac{3N}{8}(1-M^{-1})^2\frac{\pi^2}{L_pL}\,, \qquad (21)$$

where L is the total length of the magnetic axis, with N central cells of length L_c, $2N$ plugs of length L_p, and toroidal segments of length $2\pi R$.

The average curvature due to the toroidal segments is

$$\langle\sigma^3/R\rangle = 2\pi/(M^{3/2}L)\,.$$

Note that R cancels out of this expression.

From (7) and (19) the axis shift is given by

$$\frac{\Delta}{a} = \frac{16}{3\pi N}\frac{L_p}{a\lambda}\frac{M^{1/2}}{(M-1)^2}\,. \qquad (22)$$

The shift Δ depends only on L_p/a, N, and λ, but is independent of R/a. For $\lambda > 0$, necessary for stability, the axis shift is outward, not inward as in the Elmo Bumpy Torus.[6] The shift may be made small by choosing a large mirror ratio M, and increasing λ and N.

For example, letting $L_p = \pi a$, $\lambda = 1$, $M = 9$, $N = 2$ gives

$$\Delta/a = \tfrac{1}{8}\,.$$

This example shows that it is possible to obtain a small shift for reasonable parameters. To obtain the equilibrium, the rf energy density, or W, must be increased by a factor $1+\lambda$ over the value required for marginal stability to interchange modes in a mirror with no toroidal segments, $D = 0$.

III. STABILITY

We now consider interchange stability, including the toroidal curvature. Linearizing about the equilibrium and assuming

$$u = u_I(\rho,\theta,z)\exp(\mathrm{i}m\theta)\,,$$

with $m \gg 1$, we obtain from (11)

$$\frac{1}{\rho^2}\frac{\partial^2}{\partial z^2}u_I + (K-S)u_I = \frac{D}{\rho}u_I\,, \qquad (23)$$

where we have assumed u_I is localized about $\theta = 0$, where the toroidal curvature is most destabilizing.

Proceeding as before, the marginal stability condition is

$$\rho = \langle D\rangle/\lambda\langle S\rangle$$

or

$$\rho = \Delta .$$

Thus, for $\rho < \Delta$, the interchange modes are unstable, and we should expect that, dynamically, they should produce a flattening of the n, p profiles for $\rho < \Delta a$. Taking finite Larmor radius effects into account could stabilize this central region. In any case, outside a small central region, the interchange modes are completely stable.

To estimate ballooning stability, multiply (23) by u and integrate over z to obtain the variational integral

$$\delta W = \int dz\left[\left(\frac{\partial}{\partial z} u_2\right)^2 - (D\rho + S\rho^2 - K\rho^2)u_2^2\right],$$

which vanishes for marginally stable perturbations. Choose a trial function

$$u_2 = \cos(\pi z/2z_2), \quad |z| < z_2,$$
$$= 0, \quad |z| > z_2,$$

where $z_2 = \pi R/N + L_p$. This makes u_2 nonzero in a toroidal segment and the adjacent plugs, in which the curvature is destabilizing, while $u_2 = 0$ in the central cells to avoid the stabilizing ICRF. The most unstable field line is at $\rho = a$. Since we only want an estimate, we replace S by its average over the plug. The condition δW yields the beta limit

$$\beta = \frac{\chi^2}{8}\left[\frac{R}{N^2 M^{3/2} a}\left(\chi + \frac{\sin \pi\chi}{\pi}\right) + \frac{3}{16}\frac{\chi^2}{1-\chi}\left(1 - \frac{\sin \pi\chi}{\pi(1-\chi)}\right)\right]^{-1}, \tag{24}$$

$$\chi = [1 + (NL_p/\pi R)]^{-1}.$$

As an example, let $N = 2$, $L_p = 3\pi a/2$, $R = 3a$, $M = 9$. Then $\beta < 0.6$, which is not restrictive.

On the basis of magnetohydrodynamics (MHD) considerations, it appears quite feasible to link mirrors together with toroidal sections. We remark that although we have specialized to mirrors stabilized by rf, similar results apply to mirrors stabilized by minimum B cells. In this case, $\langle K \rangle$ represents the average positive curvature due to the minimum B cells, which must be increased by a factor $1 + \lambda$, $\lambda \sim 1$, to add toroidal segments.

IV. TRANSPORT

Turning now to diffusion, we limit our discussion to diffusion associated with particle orbits. As for microinstabilities, we observe that the mirror loss cone is filled, and the curvature driving trapped particle modes in the mirrors is favorable. Losses due to the waves at the ion-cyclotron frequency should be the same as in a straight mirror.

The ion drift velocity is given by

$$\mathbf{v}_D = [mv^2\boldsymbol{\kappa} + \nabla(e\phi - W)] \times \mathbf{z}/(eB), \tag{25}$$

where $v^2 = v_\parallel^2 + v_\perp^2$, and a low beta equilibrium has been assumed. Excluding the toroidal drift, the z-averaged drift velocity is given approximately by

$$\langle v_{0\theta} \rangle \approx \frac{v^2\delta}{v_i}\frac{\rho}{a^2}\left(\frac{\lambda\pi^2 a^2}{LL_p} + \alpha\right), \tag{26}$$

where

$$\alpha = \frac{a^2}{2\rho}\frac{d\phi}{d\rho}\frac{e}{T_i}, \quad \delta = \frac{m_i v_i}{eB_0}, \tag{27}$$

v_i is the ion thermal speed, and δ is the Larmor radius in the central cell. For $e\phi \sim T_e \sim T_i$, we have $\alpha \sim 1$; the drift is dominated by the electrostatic $\mathbf{E} \times \mathbf{B}$ rotation.

In the toroidal sections,

$$\frac{d\rho}{dt} = \frac{v^2\delta}{M^{1/2}v_i}\frac{\sin\theta}{R},$$

$$\frac{d\theta}{dt} = \frac{v^2\delta}{M^{1/2}v_i}\left(\frac{a^2}{\rho R}\cos\theta + \alpha\right).$$

The z velocity is given by

$$\frac{dz}{dt} = v_\parallel = \left(v^2 - \frac{v_{\perp m}^2}{\sigma^2}\right)^{1/2},$$

where $v^2 = v_{\perp m}^2 + v_{\parallel m}^2 = \text{const}$, and $v_{\perp m}, v_{\parallel m}$ are velocity components in the central cell.

Particles trapped in the mirrors do not experience toroidal drift or neoclassical diffusion. A particle is in the loss cone of the mirror if

$$v_{\parallel t} = [v_{\parallel m}^2 - (M-1)v_{\perp m}^2]^{1/2} > 0.$$

We shall neglect particles trapped in the toroidal segments; as long as $\alpha \neq 0$, they do not make large excursions. This differs from tokamaks, where α is replaced by $v_\parallel B_\theta/B_z$, which vanishes at the turning points of trapped orbits. We then approximate $v_\parallel$ as piecewise constant in the mirrors and toroidal sections. We also neglect α in the toroidal sections, since it is essential only for toroidally trapped orbits. These simplifications allow the orbits to be easily integrated, giving a mapping

$$x_{n+1} = x_n, \quad y_{n+1} = y_n + \xi, \quad \theta_{n+1} = \theta_n + \omega, \tag{28}$$

where

$$x_n = \rho_n \cos\theta_n, \quad y_n = \rho_n \sin\theta_n,$$

$$\xi = 2\pi(\delta/M^{1/2})(v^2/v_{\parallel t}v_i),$$

$$\omega = \frac{\delta v^2}{a^2 v_i}\frac{L}{v_{\parallel m}}\left(\alpha + \frac{\lambda\pi^2 a^2}{LL_p}\right).$$

In the special case $\omega = \text{const}$, the orbits are not confined, but diffuse outward. We have

$$\rho_{n+1}^2 = \rho_n^2 + 2\rho_n\xi\sin\theta_n + \xi^2.$$

The $\sin\theta$ term averages out, if ω is irrational, giving a diffusion rate per step

$$D_0 = \xi^2.$$

The diffusion coefficient is obtained by dividing D_0 by the transit time over which diffusion occurs,

$$\tau = 2\pi R_t/[v_{\parallel m}^2 - (M-1)v_{\perp m}^2]^{1/2},$$

multiplying by an assumed Maxwellian distribution function, and integrating over $v_{\perp m}, v_{\parallel m}$:

$$D = \frac{2\pi^{1/2}\delta^2 v_i}{M(M-1)R}\int_0^\infty \int_0^y \frac{[x^2/(M-1) + y^2]^2}{(y^2 - x^2)^{1/2}} \times \exp[-x^2/(M-1) - y^2]x\,dx\,dy,$$

where $x = (M-1)^{1/2}v_{\perp m}/v_i$, $y = v_{\parallel m}/v_i$. The integral is taken over particles in the mirror loss cone, $x<y$. There is no singularity at $x=y$, $v_{\|t}=0$. For large M, and assuming the integrand is significant only for $x \sim y \sim 1$, we approximate $x^2/(M-1) \approx 0$ and obtain

$$D = 2\pi^{1/2}\delta^2 v_i / M(M-1)R \,, \tag{29}$$

which resembles a neoclassical diffusion coefficient in the plateau regime. The diffusion is strongly reduced by $M(M-1)$, which is partly due to the smaller Larmor radius in the toroidal sections, and partly due to the loss cone volume.

The same result holds in the more general case in which α is a function of ρ. Linearizing and integrating the orbits gives a form of the standard map[18]

$$\rho_{n+1} = \rho_n + \xi \sin\theta_n \,,$$

$$\theta_{n+1} = \theta_n + \frac{d\omega}{d\rho}\rho_{n+1} + \omega_0 \,.$$

The stochasticity parameter[17–19] K is given by

$$K = \frac{d\omega}{d\rho}\xi \,.$$

If $d\alpha/d\rho \sim 1/a$, $\delta/a \ll 1$, then for most particles $K<1$, and the orbits are not stochastic. However, for barely untrapped particles with $v_{\|t} \approx 0$, $K>1$. For these particles, the diffusion per step is D_0, and, assuming they dominate the diffusion, we obtain (29).

To summarize, toroidally linked mirrors appear to be a reasonable alternative to tandem mirrors and bumpy squares. Although we have emphasized circular cross-section mirrors stabilized by ponderomotive force, essentially the same analysis applies for minimum B stabilized mirrors.

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