Renormalized geometric optics description of mode conversion in weakly inhomogeneous plasmas

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Conventional mode conversion theory in inhomogeneous plasmas starts with a local dispersion relation, which serves as a base for constructing a differential equation describing the waves in nearly degenerate plasma regions, where the mode conversion can take place. It will be shown, however, that the usual geometric optics perturbation scheme, which in the zero order leads to the aforementioned dispersion relation, predicts either rapid variations of the local wave vector and amplitude of the wave, or large first-order corrections to the amplitude in nearly degenerate plasma regions. A novel, general, renormalized perturbation scheme will be suggested in order to remove this singular behavior. The new method is formulated in terms of the conventional, general plasma dielectric tensor and yields two coupled, energy-conserving differential equations describing the mode conversion. Simple asymptotic solutions of these equations exist if the mode coupling is localized and weak. The method is applied to the classical problem of transformation of the extraordinary mode propagating in a cold magnetized plasma at small angles to the magnetic field. Generalization of the method to the case of an unreduced, multicomponent wave propagation problem is discussed.

I. INTRODUCTION

Conventional theories of mode conversion in inhomogeneous plasmas are based on the idea of association of a differential equation with a local plasma dispersion relation.[1] Asymptotic solutions of this differential equation describe possible wave transformations between normal modes of the plasma in regions where two or more modes have almost identical values of local wave vectors and frequencies.

The conventional approach had been successful in describing a variety of specific problems.[1-4] Nevertheless, a number of internal inconsistencies in the theory must be resolved before it can be safely used in general, usually for very complex situations characteristic of hot, magnetized, inhomogeneous plasmas. Indeed, as noted by other authors,[2,3] the conventional procedure can be ambiguous since many different differential equations can be associated with a given local dispersion relation. The most serious issue in this situation is that of energy conservation by the resulting wave equation.[5] The enforcement of energy conservation limits the number of possibilities for constructing the wave equation from the local dispersion relation, but does not completely remove the ambiguity in choosing the form or order of this equation. For example, a recent study[3] shows that in many cases two coupled differential equations, rather than one, are associated in a more natural way with the dispersion relation in mode conversion regions.

An additional internal difficulty in the theory is caused by the use of the local dispersion relation by itself. The reason is the fact that the dispersion relation results in zero order from the WKB (geometric optics) perturbation scheme which assumes slow variation of plasma and wave parameters on a typical local wavelength scale. This assumption, however, is violated in mode conversion regions, where the dispersion relation usually predicts fast variation of the local wave vector (see Sec. II). Moreover, in some cases, as will be shown in the next section, the amplitude of the wave may change rapidly with the distance in mode conversion regions, or, alternatively, the first-order corrections to the amplitude become large, which raises a question of the validity of the use of the zero-order result (the dispersion relation) in the theory.

Finally, the mode conversion theory has been frequently applied to situations where the dispersion relation could be reduced to a relatively simple, characteristic form. No theory exists so far that attempts to solve a general unreduced linear wave propagation problem including possible mode conversion. The first step toward a generalization of the conventional method was reported recently.[2] Nevertheless, the theory still was based on the use of a general local dispersion relation.

In the present article we attempt to resolve all the aforementioned difficulties in cases where the coupled modes are associated with two different eigenvalues of the local plasma dielectric tensor. In Sec. II we will review the formalism of the conventional geometric optics perturbation scheme and further discuss its failure to describe various mode conversion situations. Sections III and IV are devoted to the development of a new, general, geometric optics perturbation scheme, which avoids the limitations of the conventional approach. We will solve the ambiguity problem by employing the energy-conserving amplitude equation of the conventional geometric optics formalism. The limitations arising because of the above-mentioned singular behavior of the wave vector and the amplitude will be removed by renormalizing the amplitude equation prior to the application of the perturbation procedure. The new formalism is second order in nature, describing propagation of two modes with different polarizations in contrast to more standard, fourth-order mode conversion theories described in Ref. 1. In Sec. V, we will demonstrate the new method in the classical problem of interaction of normal modes propagating in an inhomogeneous, cold, magnetized plasma at small angles to the magnetic field. Finally, in Sec. VI, we will discuss a generalization of the renormalization method to the case of an unreduced,

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multi-component wave propagation problem that provides a way of treating the frequent situation of coupling between modes, both associated with a single eigenvalue of the dielectric tensor.

II. FAILURE OF THE CONVENTIONAL GEOMETRIC OPTICS FORMALISM IN MODE CONVERSION REGIONS

In this section we will discuss the problem of the applicability of the conventional geometric optics perturbation scheme in studying the mode conversion phenomenon. The discussion will be based on the general formalism of Ref. 6, some of the results of which will be repeated here for completeness.

Consider a weakly inhomogeneous, one-dimensional, stationary plasma, which supports a small amplitude electromagnetic wave that obeys the Maxwell equations:

$$c\nabla\times\mathbf{B}=4\pi\mathbf{J}+\frac{\partial\mathbf{E}}{\partial t},\qquad c\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{1}$$

We assume that the induced current $\mathbf{J}$ can be written in the following nonlocal, causal, linearized form:

$$\mathbf{J}(\mathbf{r},t)=\int d^3\mathbf{r}'\int_{-\infty}^{t}dt'\,\hat{\boldsymbol{\sigma}}\left(\mathbf{r}-\mathbf{r}',t-t';\frac{x+x'}{2}\right)\cdot\mathbf{E}(\mathbf{r}',t'), \tag{2}$$

where the conductivity kernel $\hat{\boldsymbol{\sigma}}$ varies rapidly with the difference arguments $\mathbf{r}-\mathbf{r}'$ and $t-t'$, and slowly (in a later described sense) with $(x+x')/2$. We seek eikonal-type solutions of (1),

$$\mathbf{E}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{a}(x)\exp[i(\psi(x)+k_y y+k_z z-\omega t)]\},\qquad \mathbf{B}(\mathbf{r},t)=\mathrm{Re}\{\mathbf{b}(x)\exp[i(\psi(x)+k_y y+k_z z-\omega t)]\}, \tag{3}$$

where k_y, k_z are constants and, if we define

$$k_x(x)=\frac{d\psi}{dx}, \tag{4}$$

then $\mathbf{a}$, $\mathbf{b}$, and k_x are slowly varying functions of x in a sense that there exists a small dimensionless parameter δ, such that

$$\frac{2\pi}{k_x}\left|\frac{d\ln A}{dx}\right|<\delta\ll 1, \tag{5}$$

where A represents the above-mentioned, slowly varying functions, as well as various plasma parameters, such as density, temperature, static magnetic fields, etc. The existence of the small parameter in the problem allows one to solve (1) by using a perturbation method. As a first step, one shows that to the first order in δ, the amplitude $\mathbf{a}$ of the electric field of the wave is described by the following equation[6]:

$$i\omega\boldsymbol{\epsilon}\cdot\mathbf{a}=\mathsf{L}\cdot\mathbf{a}, \tag{6}$$

where

$$\boldsymbol{\epsilon}(\mathbf{k},\omega;x)=\left(1-\frac{c^2k^2}{\omega^2}\right)\mathsf{I}+\frac{c^2}{\omega^2}\mathbf{k}\mathbf{k}+\frac{4\pi i}{\omega}\boldsymbol{\sigma}(\mathbf{k},\omega;x) \tag{7}$$

is the local plasma dielectric tensor,[7] with I the unit tensor, $\boldsymbol{\sigma}$ the conductivity tensor, and $\mathbf{k}=k_x\hat{e}_x+k_y\hat{e}_y+k_z\hat{e}_z$. The linear-tensor operator L in (6) is defined by

$$\mathsf{L}\cdot\mathbf{a}=-\frac{\partial\omega\boldsymbol{\epsilon}}{\partial k_x}\cdot\frac{d\mathbf{a}}{dx}-\frac{1}{2}\left[\frac{d}{dx}\left(\frac{\partial\omega\boldsymbol{\epsilon}}{\partial k_x}\right)\right]\cdot\mathbf{a}. \tag{8}$$

Equation (6) will provide a background for the discussion in this section as well as a basis for the renormalization approach developed in Secs. III and IV.

Note that the right-hand side of Eq. (6) is of the order of δ. Therefore, we can attempt to solve it by using the following natural perturbation procedure. We seek a solution for $\mathbf{a}$ in the form

$$\mathbf{a}=\mathbf{a}_0+\mathbf{a}_1+\mathbf{a}_2+\cdots, \tag{9}$$

where the terms $\mathbf{a}_n$, are assumed to be of the order of δ^n. Then Eq. (6) in various orders yields

$$i\omega\boldsymbol{\epsilon}\cdot\mathbf{a}_0=0, \tag{10}$$

$$i\omega\boldsymbol{\epsilon}\cdot\mathbf{a}_1=\mathsf{L}\cdot\mathbf{a}_0, \tag{11}$$

$$i\omega\boldsymbol{\epsilon}\cdot\mathbf{a}_2=\mathsf{L}\cdot\mathbf{a}_1, \tag{12}$$

$$\cdots.$$

In the rest of the discussion, for simplicity we assume that $\boldsymbol{\epsilon}$ is Hermitian, which excludes possible dissipation mechanisms. The case of a weak dissipation can be included in a fashion similar to that of Ref. 6.

It is convenient, at this stage, to express the dielectric tensor in the diagonal form

$$\boldsymbol{\epsilon}=\epsilon_1\hat{e}_1\hat{e}_1^*+\epsilon_2\hat{e}_2\hat{e}_2^*+\epsilon_3\hat{e}_3\hat{e}_3^*, \tag{13}$$

where the eigenvalues ϵ_n $(n=1,2,3)$ are real and the eigenvectors $\hat{e}_n$ are orthonormal ($\hat{e}_n^*\cdot\hat{e}_m=\delta_{nm}$). If, in addition, we write

$$\mathbf{a}_0=\alpha_1\hat{e}_1+\alpha_2\hat{e}_2+\alpha_3\hat{e}_3,\quad \mathbf{a}_1=\gamma_1\hat{e}_1+\gamma_2\hat{e}_2+\gamma_3\hat{e}_3,\quad \mathbf{a}_2=\beta_1\hat{e}_1+\beta_2\hat{e}_2+\beta_3\hat{e}_3, \tag{14}$$

the zero-order equation (10) then yields

$$\epsilon_1\alpha_1=\epsilon_2\alpha_2=\epsilon_3\alpha_3=0. \tag{15}$$

Therefore, a nontrivial solution for $\mathbf{a}_0$ exists only when at least one eigenvalue of $\boldsymbol{\epsilon}$ (say ϵ_1) vanishes. In this case, if $\epsilon_2,\epsilon_3\neq 0$, we have $\alpha_2=\alpha_3=0$ and $\alpha_1\neq 0$. The commonly used, invariant form of the local dispersion relation in this situation is

$$D(\mathbf{k},\omega;x)=\mathrm{Det}(\epsilon)=\epsilon_1\epsilon_2\epsilon_3=0. \tag{16}$$

Next, we proceed to the first-order equation, Eq. (11). By multiplying it by $\hat{e}_1^*$ and using the orthonormality of the eigenvectors we obtain:

$$-\hat{e}_1^*\cdot\mathsf{L}\cdot\mathbf{a}_0=\frac{\partial\omega\epsilon_1}{\partial k_x}\frac{d\alpha_1}{dx}+\alpha_1\left\{\hat{e}_1^*\cdot\frac{\partial\omega\boldsymbol{\epsilon}}{\partial k_x}\cdot\frac{d\hat{e}_1}{dx}+\frac{1}{2}\hat{e}_1^*\cdot\left[\frac{d}{dx}\left(\frac{\partial\omega\boldsymbol{\epsilon}}{\partial k_x}\right)\right]\cdot\hat{e}_1^*\right\}=0, \tag{17}$$

which, if k_x is found from Eq. (16), describes the spatial evolution of the zero-order amplitude α_1.

The necessary conditions for the validity of the described perturbation scheme are: (a) the slowness of variation of k_x as given by the dispersion relation (16); (b) the slowness of variation of α_1 as described by Eq. (17); (c) the relative smallness of the first-order correction $\mathbf{a}_1$ to the amplitude. Consider first condition (b). It follows from Eq. (17) that if (a) is satisfied, the slowness of the spatial variation of α_1 is guaranteed if $\partial\omega\epsilon_1/\partial k_x$ is large enough. If, on the contrary, $\partial\omega\epsilon_1/\partial k_x \to 0$, then condition (b) is violated and the perturbation scheme fails to describe the situation adequately. Such a failure is characteristic, for example, in cases when the dispersion relation (16) has two, almost identical, roots associated with one of the eigenvalues (ϵ_1 in our case). This typical mode conversion situation, therefore, cannot be described by the conventional geometric optics formalism. Let us also show that the situation $\partial\omega\epsilon_1/\partial k_x \to 0$ may be accompanied by a violation of condition (a). Consider, for this purpose, a point $\bar{x}$ in the plasma and let $D(\bar{k}_x,\omega;\bar{x}) = 0$. In the neighborhood of $\bar{x}$ and $\bar{k}_x$ the dispersion relation can be approximated by

$$\frac{\partial D}{\partial \bar{k}_x}(k_x - \bar{k}_x) + \frac{\partial D}{\partial \bar{x}}(x - \bar{x}) + \frac{1}{2}\frac{\partial^2 D}{\partial \bar{k}_x^2}(k_x - \bar{k}_x)^2 + \frac{1}{2}\frac{\partial^2 D}{\partial \bar{x}^2}(x - \bar{x})^2 + \frac{\partial^2 D}{\partial \bar{k}_x \partial \bar{x}}(k_x - \bar{k}_x)(x - \bar{x}) = 0. \tag{18}$$

If now, in addition to $D(\bar{k}_x,\omega;\bar{x})$, the coefficient $\partial D/\partial \bar{k}_x = \epsilon_2\epsilon_3\partial\epsilon_1/\partial\bar{k}_x$ in this equation also vanishes and at the same time $\partial D/\partial\bar{x} \neq 0$, then, asymptotically for $x \to \bar{x}$ we obtain a solution

$$k_x = \bar{k}_x \pm \left[-\frac{2(x-\bar{x})(\partial D/\partial\bar{x})}{(\partial^2 D/\partial\bar{k}_x^2)} \right]^{1/2} \tag{19}$$

and therefore dk_x/dx is singular at $\bar{x}$, thus violating condition (a).

Finally, consider condition (c). Assume that both (a) and (b) are satisfied. Then one can find the components γ_2 and γ_3 of $\mathbf{a}_1$ by multiplying the first-order equation (11) by $\hat{e}_2^*$ and $\hat{e}_3^*$, respectively, which results in

$$\gamma_2 = \hat{e}_2^* \cdot \mathsf{L} \cdot (\alpha_1\hat{e}_1)/\epsilon_2, \quad \gamma_3 = \hat{e}_3^* \cdot \mathsf{L} \cdot (\alpha_1\hat{e}_1)/\epsilon_3. \tag{20}$$

The spatial evolution of γ_1 can be found from the second-order equation (12) multiplied by $\hat{e}_1^*$:

$$\hat{e}_1^* \cdot \mathsf{L} \cdot (\gamma_1\hat{e}_1 + \gamma_2\hat{e}_2 + \gamma_3\hat{e}_3) = 0. \tag{21}$$

As follows from Eq. (20), a difficulty arises when in some plasma region ϵ_2 (or ϵ_3) becomes small. The perturbation scheme fails in this case, predicting large first-order corrections to the amplitude.[8] In addition to this, when $\epsilon_1 = 0$ and ϵ_2 becomes small, the eigenvectors $\hat{e}_1$ and $\hat{e}_2$ may rapidly vary with x since their evaluation in this case involves division by vanishing cofactors of $\boldsymbol{\epsilon}$, a procedure which is also ill-defined from the numerical point of view.

The next section deals with a renormalization of the geometric optics perturbation scheme on purpose to describe the latter situation of possible mode conversion when the interacting modes are associated with two different eigenvalues (ϵ_1 and ϵ_2) of the dielectric tensor. The previously mentioned possibility of having a double zero in the same eigenvalue of $\boldsymbol{\epsilon}$ will be discussed in Sec. VI.

III. RENORMALIZED PERTURBATION ANALYSIS

Consider a case where in Eq. (13) ϵ_1/ϵ_3 and ϵ_2/ϵ_3 are of the order of δ, or smaller. Prior to the perturbation analysis of the amplitude equation (6) in this case, it is convenient to introduce several new quantities and notations. First of all, we will abandon the use of the ill-defined eigenvectors $\hat{e}_1$ and $\hat{e}_2$ in the theory. Instead, since the eigenvector $\hat{e}_3$ is associated with the nonvanishing eigenvalue ϵ_3, and therefore is well defined and varies slowly with x, we will use $\hat{e}_3$ in defining a new set of orthonormal base vectors. Assume that $A^2 = |\hat{e}_3 \cdot \hat{e}_x|^2 \neq 0$. Then one of the choices is to use, instead of $\hat{e}_1$ and $\hat{e}_2$,

$$\hat{m}_1 = (\hat{e}_3^* \times \hat{e}_x)/(1 - A^2)^{1/2}, \qquad \hat{m}_2 = (\hat{e}_3^* \times \hat{m}_1^*). \tag{22}$$

Here, correct to the first order in δ, $\hat{e}_3$ can be found from

$$(\boldsymbol{\epsilon})^2 = \epsilon_1^2\hat{e}_1\hat{e}_1^* + \epsilon_2^2\hat{e}_2\hat{e}_2^* + \epsilon_3^2\hat{e}_3\hat{e}_3^* \simeq \epsilon_3^2\hat{e}_3\hat{e}_3^* \tag{23}$$

so that

$$\hat{e}_3 \simeq (\boldsymbol{\epsilon})^2 \cdot \hat{e}_x / A\epsilon_3^2, \tag{24}$$

where

$$\epsilon_3^2 \simeq \mathrm{Tr}(\boldsymbol{\epsilon})^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2. \tag{25}$$

In terms of the new base vectors,

$$\boldsymbol{\epsilon} = \boldsymbol{\Delta} + \epsilon_3\hat{e}_3 e_3^*, \tag{26}$$

where

$$\boldsymbol{\Delta} = \sum_{i,j=1}^{2} \epsilon_{ij}\,\hat{m}_i\hat{m}_j^* \tag{27}$$

and

$$\epsilon_{ij} = \hat{m}_i^* \cdot \boldsymbol{\epsilon} \cdot \hat{m}_j = \epsilon_1(\hat{m}_i^* \cdot \hat{e}_1)(\hat{e}_1^* \cdot \hat{m}_j) + \epsilon_2(\hat{m}_i^* \cdot \hat{e}_2)(\hat{e}_2^* \cdot \hat{m}_j)$$

are small and of order δ.

As a last preliminary step, we will now exploit the freedom in choosing the vectors $\hat{m}_1$ and $\hat{m}_2$. Consider, for this, the following tensor:

$$\mathsf{Q} = \sum_{i,j=1}^{2} \frac{\partial\epsilon_{ij}}{\partial k_x}\hat{m}_i\hat{m}_j^* \tag{28}$$

which in contrast to $\boldsymbol{\Delta}$, generally is of order $(\delta)^0$. Given $\hat{e}_3$, one can now always find a new set of basis vectors, $\hat{m}_1'$, $\hat{m}_2'$ orthogonal to $\hat{e}_3$, such that Q in terms of $\hat{m}_i'$ is diagonal. Indeed, let $\hat{m}_1$ and $\hat{m}_2$ be given by Eqs. (22), and define

$$\hat{m}_1' = \xi\hat{m}_1 + \sqrt{1-\xi^2}\,e^{i\phi}\hat{m}_2, \qquad \hat{m}_2' = \sqrt{1-\xi^2}\,\hat{m}_1 - \xi e^{i\phi}\hat{m}_2, \tag{29}$$

where $\mathrm{Im}\,\xi = 0$ and $|\xi| < 1$. Now, in terms of $\hat{m}_i'$,

$$Q_{12}' = Q_{21}'^* = \hat{m}_1'^* \cdot \mathsf{Q} \cdot \hat{m}_2' = \xi\sqrt{1-\xi^2}\,\frac{\partial(\epsilon_{11}-\epsilon_{22})}{\partial k_x} - \xi^2\frac{\partial\epsilon_{21}}{\partial k_x}e^{i\phi} + (1-\xi^2)\frac{\partial\epsilon_{12}}{\partial k_x}e^{-i\phi}. \tag{30}$$

Then, if we choose ϕ so that $\mathrm{Im}[(\partial\epsilon_{12}/\partial k_x)e^{-i\phi}] = 0$, there always exists $|\xi| < 1$ such that Q_{12}' and Q_{21}' vanish. In the rest of this paper we will always use the vectors $\hat{m}_i'$ defined as above, but will suppress the prime notation for simplicity. Thus, in the new notation,

$$\boldsymbol{\epsilon} = \sum_{i,j=1}^{2} \epsilon_{ij}\, \hat{m}_i \hat{m}_j^* + \epsilon_3 \hat{e}_3 \hat{e}_3^* \tag{31}$$

and

$$\mathsf{Q} = \frac{\partial \epsilon_{11}}{\partial k_x}\, \hat{m}_1 \hat{m}_1^* + \frac{\partial \epsilon_{22}}{\partial k_x}\, \hat{m}_2 \hat{m}_2^*. \tag{32}$$

Finally, we once again return to the basic equation (6) for the amplitude. As was shown in the last section, a straightforward perturbative solution of (6) is inapplicable in the case of interest, since it predicts large first-order corrections to the amplitude. In order to resolve the problem, let k_{x0} and x_0 be constant values in the neighborhood of k_x and x. These constant values will serve later as the crossing point of two decoupled geometric optics modes. The following renormalization procedure will be developed around this crossing point. Define

$$\boldsymbol{\epsilon}_0 = \boldsymbol{\epsilon}(\mathbf{k}_0,\omega;x_0) = \tilde{\boldsymbol{\epsilon}}_0 + \epsilon_{120} \hat{m}_{10} \hat{m}_{20}^* + \epsilon_{210} \hat{m}_{20} \hat{m}_{10}^*, \tag{33}$$

where the diagonal part

$$\tilde{\boldsymbol{\epsilon}}_0 = \epsilon_{110} \hat{m}_{10} \hat{m}_{10}^* + \epsilon_{220} \hat{m}_{20} \hat{m}_{20}^* + \epsilon_{30} \hat{e}_{30} \hat{e}_{30}^* \tag{34}$$

is used to "renormalize" the basic amplitude equation by rewriting it in the form

$$i\omega \tilde{\boldsymbol{\epsilon}}_0 \cdot \mathbf{a} = \tilde{\mathsf{L}} \cdot \mathbf{a}, \tag{35}$$

where

$$\tilde{\mathsf{L}} \cdot \mathbf{a} = \mathsf{L} \cdot \mathbf{a} - i\omega(\boldsymbol{\epsilon} - \tilde{\boldsymbol{\epsilon}}_0) \cdot \mathbf{a}. \tag{36}$$

Now we proceed to the perturbation analysis by seeking a solution of Eq. (35) in the form

$$\mathbf{a}(x) = \mathbf{a}_0(x) + \mathbf{a}_1(x) + \mathbf{a}_2(x) + \cdots, \tag{37}$$

where, as in the conventional scheme, the terms are ordered in increasing powers of δ. In contrast to the usual procedure, however, we now express $\mathbf{a}_i(x)$ in terms of constant vectors $\hat{m}_{10}$, $\hat{m}_{20}$, and $\hat{e}_{30}$:

$$\begin{aligned} \mathbf{a}_0(x) &= \alpha_1(x) \hat{m}_{10} + \alpha_2(x) \hat{m}_{20} + \alpha_3(x) \hat{e}_{30}, \\ \mathbf{a}_1(x) &= \gamma_1(x) \hat{m}_{10} + \gamma_2(x) \hat{m}_{20} + \gamma_3(x) \hat{e}_{30}, \\ \mathbf{a}_2(x) &= \beta_1(x) \hat{m}_{10} + \beta_2(x) \hat{m}_{20} + \beta_3(x) \hat{e}_{30}, \\ &\cdots . \end{aligned} \tag{38}$$

Then in zero order, (35) yields

$$\tilde{\boldsymbol{\epsilon}}_0 \cdot \mathbf{a}_0 = 0 \tag{39}$$

or, equivalently,

$$\alpha_1(x) \epsilon_{110} = \alpha_2(x) \epsilon_{220} = \alpha_3(x) \epsilon_{30} = 0. \tag{40}$$

If, in addition, we require $\epsilon_{110} = \epsilon_{220} = 0$ (and as before $\epsilon_{30} \neq 0$), then Eq. (40) yields

$$\alpha_1, \alpha_2 \neq 0, \quad \alpha_3 = 0, \tag{41}$$

so that $\mathbf{a}_0 = \alpha_1 \hat{m}_{10} + \alpha_2 \hat{m}_{20}$.

Given the initial k_x and x, the conditions $\epsilon_{110} = \epsilon_{220} = 0$ are sufficient for finding the values of k_{x0} and x_0. In fact, if $k_x = k_{x0} + q$ and $x = x_0 + s$, we can expand

$$\begin{aligned} \epsilon_{110} &= \epsilon_{11} - \frac{\partial \epsilon_{11}}{\partial k_x} q - \frac{\partial \epsilon_{11}}{\partial x} s = 0, \\ \epsilon_{220} &= \epsilon_{22} - \frac{\partial \epsilon_{22}}{\partial k_x} q - \frac{\partial \epsilon_{22}}{\partial k_x} s = 0, \end{aligned} \tag{42}$$

thus obtaining a definite solution for q and s, provided that

$$\begin{aligned} R &= \frac{\partial \epsilon_{11}}{\partial k_x} \frac{\partial \epsilon_{22}}{\partial x} - \frac{\partial \epsilon_{22}}{\partial k_x} \frac{\partial \epsilon_{11}}{\partial x} \\ &\simeq \frac{\partial \epsilon_{110}}{\partial k_{x0}} \frac{\partial \epsilon_{220}}{\partial x_0} - \frac{\partial \epsilon_{220}}{\partial k_{x0}} \frac{\partial \epsilon_{110}}{\partial x_0} \neq 0, \end{aligned} \tag{43}$$

which will be assumed to be the case in the following. This completes the use of the zero order equation. Note that Eqs. (42) and (43) yield the crossing point of two renormalized formally decoupled geometric optics modes, polarizations of which define [see Eq. (38)] the polarization of the electromagnetic wave in question.

In the first order, the renormalized equation (35) yields

$$i\omega \tilde{\boldsymbol{\epsilon}}_0 \cdot \mathbf{a}_1 = \tilde{\mathsf{L}} \cdot \mathbf{a}_0. \tag{44}$$

Multiplying (44) by $\hat{m}_{10}^*$ and $\hat{m}_{20}^*$, we have

$$\begin{aligned} &- \hat{m}_{10}^* \cdot \frac{\partial \omega \boldsymbol{\epsilon}}{\partial k_x} \cdot \left(\hat{m}_{10} \frac{d\alpha_1}{dx} + \hat{m}_{20} \frac{d\alpha_2}{dx} \right) \\ &\quad - \frac{1}{2} \left[\frac{d}{dx} \left(\hat{m}_{10}^* \cdot \frac{\partial \omega \boldsymbol{\epsilon}}{\partial k_x} \right) \right] \cdot (\alpha_1 \hat{m}_{10} + \alpha_2 \hat{m}_{20}) \\ &\quad = i\omega \hat{m}_{10}^* \cdot \boldsymbol{\epsilon} \cdot (\alpha_1 \hat{m}_{10} + \alpha_2 \hat{m}_{20}), \\ &- \hat{m}_{20}^* \cdot \frac{\partial \omega \boldsymbol{\epsilon}}{\partial k_x} \cdot \left(\hat{m}_{10} \frac{d\alpha_1}{dx} + \hat{m}_{20} \frac{d\alpha_2}{dx} \right) \\ &\quad - \frac{1}{2} \left[\frac{d}{dx} \left(\hat{m}_{20}^* \cdot \frac{\partial \omega \boldsymbol{\epsilon}}{\partial k_x} \right) \right] \cdot (\alpha_1 \hat{m}_{10} + \alpha_2 \hat{m}_{20}) \\ &\quad = i\omega \hat{m}_{20}^* \cdot \boldsymbol{\epsilon} \cdot (\alpha_1 \hat{m}_{10} + \alpha_2 \hat{m}_{20}). \end{aligned} \tag{45}$$

Assume that the values of k_{x0} and k_x as well as x_0 and x are close enough, so that, to lowest significant order, we can replace $\hat{m}_{i0}$ by $\hat{m}_i$ in the left-hand side of Eq. (45). Then, by employing the diagonal property (32) of

$$\mathsf{Q} \simeq \sum_{i,j=1}^{2} \hat{m}_i^* \cdot \frac{\partial \omega \boldsymbol{\epsilon}}{\partial k_x} \cdot \hat{m}_j,$$

Eqs. (45) can be rewritten as

$$\frac{d\alpha_1}{dx} \frac{\partial \omega \epsilon_{11}}{\partial k_x} + \frac{1}{2} \left[\frac{d}{dx} \left(\frac{\partial \omega \epsilon_{11}}{\partial k_x} \right) \right] \alpha_1 + i\omega \epsilon_{11}' \alpha_1 = -i\omega \epsilon_{12}' \alpha_2, \tag{46}$$

$$\frac{d\alpha_2}{dx} \frac{\partial \omega \epsilon_{22}}{\partial k_x} + \frac{1}{2} \left[\frac{d}{dx} \left(\frac{\partial \omega \epsilon_{22}}{\partial k_x} \right) \right] \alpha_2 + i\omega \epsilon_{22}' \alpha_2 = -i\omega \epsilon_{21}' \alpha_1, \tag{47}$$

where $\epsilon_{ij}' = \hat{m}_{i0}^* \cdot \boldsymbol{\epsilon} \cdot \hat{m}_{j0}$. If $\epsilon_{12}' = \epsilon_{21}' = 0$, Eqs. (46) and (47) reduce to the slow amplitude equations in degenerate plasmas, described in detail in Ref. 6. Consequently, the perturbation procedure described above is thus an expansion around the degenerate point k_0, k_{x0} of the renormalized dielectric tensor $\tilde{\epsilon}_0$. Equations (46) and (47) are the desired coupled equations describing slow spatial evolution of the amplitudes α_1 and α_2.

At this point we are able to check the internal consistency of the renormalized perturbation analysis, namely, to demonstrate the relative smallness of the first-order corrections to the amplitude [condition (c) of Sec. II]. Component γ_3 of the first-order correction $\mathbf{a}_1$ is found by multiplying the first-order equation (44) by $\hat{e}_{30}^*$, yielding:

$$\gamma_3 = -i\hat{e}_{30}^* \cdot \tilde{\mathsf{L}} \cdot \mathbf{a}_0/\omega\epsilon_{30} \sim \delta. \quad (48)$$

Then, the differential equations for γ_1 and γ_2 can be found by multiplying the second-order equation

$$i\omega\tilde{\epsilon}_0 \cdot \mathbf{a}_2 = \tilde{\mathsf{L}} \cdot \mathbf{a}_1 \quad (49)$$

by $\hat{m}_{10}^*$ and $\hat{m}_{20}^*$:

$$\hat{m}_{10}^* \cdot \tilde{\mathsf{L}} \cdot (\gamma_1 \hat{m}_{10} + \gamma_2 \hat{m}_{20} + \gamma_3 \hat{e}_{30}) = 0,$$
$$\hat{m}_{20}^* \cdot \tilde{\mathsf{L}} \cdot (\gamma_1 \hat{m}_{10} + \gamma_2 \hat{m}_{20} + \gamma_3 \hat{e}_{30}) = 0. \quad (50)$$

The last two equations have structures similar to that of Eqs. (46) and (47) for the zero-order amplitude and they predict small values for γ_1 and γ_2, if initially these values are small. Thus, in conclusion, the renormalized perturbation scheme is internally consistent, and in contrast to the usual geometric optics formalism in the case of interest, does not violate condition (c).

We proceed now to the question of the conservation of energy. Multiplying (46) by α_1^* and adding the resulting equation to its complex conjugate we obtain

$$\frac{d}{dx}\left(\alpha_1^2 \frac{\partial\omega\epsilon_{11}}{\partial k_x}\right) = -i\omega(\epsilon_{12}^* \alpha_2 \alpha_1^* - \epsilon_{21}' \alpha_2^* \alpha_1). \quad (51)$$

Similarly Eq. (47) yields

$$\frac{d}{dx}\left(\alpha_2^2 \frac{\partial\omega\epsilon_{22}}{\partial k_x}\right) = -i\omega(\epsilon_{21}' \alpha_1 \alpha_2^* - \epsilon_{12}' \alpha_1^* \alpha_2). \quad (52)$$

By adding the last two equations, we have

$$\frac{d}{dx}\left(\alpha_1^2 \frac{\partial\omega\epsilon_{11}}{\partial k_x} + \alpha_2^2 \frac{\partial\omega\epsilon_{22}}{\partial k_x}\right) = \frac{d}{dx}(\mathbf{a}_0^* \cdot \mathsf{Q} \cdot \mathbf{a}_0) = 0. \quad (53)$$

This equation describes the conservation of the usual, time-averaged energy flux, associated with the wave in the plasma. Equation (53) by itself is not surprising, since our initial amplitude equation (6) conserved the energy flux. Nevertheless, we can now interpret the situation, as a simultaneous propagation of two coupled geometric optics modes, characterized in the usual way[6] by time-averaged energy densities $U_i = (\alpha_i^2/16\pi)(\partial\omega\epsilon_{ii}/\partial\omega)$ $(i = 1,2)$ and x components of the group velocities given by

$$(v_{gi})_x = -\frac{\partial(\omega\,\epsilon_{ii})/\partial k_x}{\partial(\omega\,\epsilon_{ii})/\partial\omega}.$$

Small coefficients $\epsilon_{12} = \epsilon_{21}^*$ couple the two modes, at the same time conserving the total energy flux.

IV. ANALYSIS OF THE COUPLED MODE EQUATIONS

We intend now to further simplify our coupled mode equations (46) and (47). As a first step, we introduce a new variable $s = x - x_0$ instead of x and expand

$$\epsilon_{ii} = \epsilon_{ii0} + \frac{\partial\epsilon_{ii0}}{\partial k_0}(k - k_0) + \frac{\partial\epsilon_{ii0}}{\partial x_0} s$$
$$= \frac{\partial\epsilon_{ii0}}{\partial k_0}(k - k_0) + \frac{\partial\epsilon_{ii0}}{\partial x_0} s, \quad i = 1,2, \quad (54)$$

$$\epsilon_{ij} = \epsilon_{ij0} + \frac{\partial\epsilon_{ij0}}{\partial k_0}(k - k_0) + \frac{\partial\epsilon_{ij0}}{\partial x_0} s \simeq \epsilon_{ij0}, \quad i \neq j. \quad (55)$$

Here we used the previous assumptions ($\epsilon_{110} = \epsilon_{220} = \partial\epsilon_{120}/\partial k_0 = \partial\epsilon_{210}/\partial k_0 = 0$) and neglected the spatial variation of ϵ_{ij0} in Eq. (55). Using (54) and (55), we can rewrite Eqs. (46) and (47) in the form

$$\frac{d\alpha_1}{ds} G_1 + \frac{1}{2}\frac{dG_1}{ds}\alpha_1 + i[(k - k_0)G_1 + sH_1]\alpha_1 = -i\lambda\alpha_2,$$
$$\frac{d\alpha_2}{ds} G_2 + \frac{1}{2}\frac{dG_2}{ds}\alpha_2 + i[(k - k_0)G_2 + sH_2]\alpha_2 = -i\lambda^*\alpha_1, \quad (56)$$

where $G_i = \partial\epsilon_{ii}/\partial k \simeq \partial\epsilon_{ii0}/\partial k_0$, $H_i = \partial\epsilon_{ii}/\partial x \simeq \partial\epsilon_{ii0}/\partial x_0$, and $\lambda = \epsilon_{120}$. Further simplification is possible by defining $A_{1,2} = \sqrt{G_{1,2}}\,\alpha_{1,2} \exp[i\psi(s) - ik_{x0}s]$. In this notation, (56) becomes

$$\frac{dA_1}{ds} + isc_1 A_1 = -id_1 A_2, \quad (57)$$

$$\frac{dA_2}{ds} + isc_2 A_2 = -id_2 A_1, \quad (58)$$

where $d_1 = d_2^* = \lambda/(G_1 G_2)^{1/2}$, $c_1 = H_1/G_1$, and $c_2 = H_2/G_2$.

A similar set of coupled mode equations was recently suggested by Cairns and Lashmore-Davies,[3] who based their approach on the use of a simple characteristic form of the local dispersion relation. The choice from many other possibilities was made in Ref. 3 by guessing the existence of two equations, (in contrast to the conventional one), describing the mode conversion, and imposing conservation of the energy flux. The results of the present study support their successful guess, at least at this stage, for the case when the modes are associated with two different eigenvalues of the dielectric tensor. In addition, the present investigation prescribes a way of consistent derivation of the coupled mode equations from the Maxwell-kinetic equations in the most general circumstances, and provides a physical meaning to the polarizations $\hat{m}_1$ and $\hat{m}_2$ of the interacting modes (information missing completely in the theories based on local dispersion relations).

Simple, approximate, asymptotic solutions of Eqs. (57) and (58) exist in the case of weak coupling (where d_1 and d_2 are small enough). Indeed, assuming that solutions $A_1(-s_0)$ and $A_2(-s_0)$ at the point $-s_0 < 0$, far away from the coupling point ($s = 0$), are known, we can rewrite (57) and (58) as

$$A_1(s) = A_1(-s_0)e^{-ic_1(s^2 - s_0^2)/2} - id_1 e^{-ic_1 s^2/2} \times \int_{-s_0}^{s} e^{ic_1 s'^2/2} A_2(s')ds', \quad (59)$$

$$A_2(s) = A_2(-s_0)e^{-ic_2(s^2 - s_0^2)/2} - id_2 e^{-ic_2 s^2/2} \int_{-s_0}^{s} e^{ic_2 s'^2/2} A_1(s') ds'. \quad (60)$$

Now we can use iterations to find A_1 and A_2 to the desired accuracy. For example, assume that at $-s_0(\simeq -\infty)$ only mode A_1 is present and propagates in the positive s direction. Substitution of Eq. (60) [with $A_2(-s_0) = 0$] into (59) yields

$$A_1(s) = A_1(-s_0)e^{-ic_1(s^2 - s_0^2)/2} - d_1 d_2 e^{ic_1 s^2/2} \times \int_{-s_0}^{s} e^{i(c_1 - c_2)s'^2/2} ds' \int_{-s_0}^{s'} e^{ic_1 s''^2/2} A_1(s'') ds''. \quad (61)$$

Now, if $d_1 d_2$ is small enough, we can use as a first guess in the right-hand side of Eq. (61), $A_1(s'') \simeq A_1(-s_0) \times \exp[-ic_1(s''^2 - s_0^2)/2]$. Then, by iterating once,

$$A_1(s) \simeq A_1(-s_0)e^{-ic_1(s^2 - s_0^2)/2} \times \left(1 - d_1 d_2 \int_{-s_0}^{s} e^{i(c_1 - c_2)s'^2/2} ds' \times \int_{-s_0}^{s'} e^{-i(c_1 - c_2)s''^2/2} ds''\right). \quad (62)$$

Therefore, on using a new variable, $t = s\sqrt{|c_1 - c_2|/2}$, we obtain

$$A_1(+s_0) = A_1(-s_0)\left(1 - \frac{2d_1 d_2}{|c_1 - c_2|} \int_{-t_0}^{t_0} e^{it'^2\chi} dt' \times \int_{-t_0}^{t'} e^{-it''^2\chi} dt''\right), \quad (63)$$

where $\chi = \text{sign}\,(c_1 - c_2)$ and $t_0 = s_0\sqrt{|c_1 - c_2|/2}$. Finally, taking in (63) the asymptotic limit $t_0 \to \infty$, we obtain

$$A_1(+s_0) \to A_1(-s_0) \times \left(1 - \frac{d_1 d_2}{|c_1 - c_2|} \int_{-\infty}^{+\infty} e^{i\chi t'^2} dt' \int_{-\infty}^{+\infty} e^{-i\chi t''^2} dt''\right) = A_1(-s_0)\left(1 - \frac{\pi d_1 d_2}{|c_1 - c_2|}\right). \quad (64)$$

Thus, the transmission coefficient for mode A_1 is given by

$$T = 1 - \frac{2\pi d_1 d_2}{|c_1 - c_2|} \simeq \exp\left(-\frac{2\pi\lambda^2}{|R|}\right) \quad (65)$$

[for a definition of R, see Eq. (43)]. A similar result was also obtained in Ref. 3 by using a more sophisticated asymptotic analysis.

We now discuss an exceptional case when the secondary wave A_2, excited as the primary wave A_1 passes the coupling region, has a relatively small, or even vanishing group velocity. We will present an example of such a case in the next section. The validity of our expansion procedure needs to be reconsidered in such circumstances. In fact, when $G_2 = \partial\epsilon_{22}/\partial k_x \to 0$, the quantity $d_1 d_2$ in (61) becomes large. Nevertheless, as can be seen from (61), the mode coupling takes place in a region $\Delta s \sim (|c_1 - c_2|)^{-1/2}$, which becomes vanishingly small as $G_2 \to 0$. As a result, formally, the transmission coefficient remains finite when $G_2 \to 0$:

$$T = \exp(-2\pi\lambda^2/|G_1 H_2|). \quad (66)$$

Nevertheless, the case $G_2 = 0$ cannot be described by the theory, since a finite amplitude change occurs for a distance $\Delta s = 0$, thus violating the slow variation condition. The necessary requirement for the validity of our expansion scheme is the smallness of the relative change ζ of the amplitude A_1 in a distance Δs:

$$\zeta = \frac{d \ln A_1}{k\, dx} \sim \frac{\Delta A_1}{kA_1 \Delta s} \sim \frac{d_1 d_2}{k\sqrt{|c_2|}} \sim \frac{\lambda^2}{kG_1\sqrt{|G_2 H_2|}} \ll 1.$$

We require ζ to be of the order of δ, thus limiting the treatment to cases when $|G_2| > \lambda^4(k^2 G_1^2 H_2 \delta^2)$. This restriction is rather weak, because even the case $G_2 \sim \delta$ is allowed since $|\lambda|$ is of the order of δ.

V. MODE TRANSFORMATION IN A COLD MAGNETIZED PLASMA AT SMALL ANGLES TO THE MAGNETIC FIELD

This section presents an example of the theory just developed. Consider the classical case of a cold, magnetized plasma, characterized by a dielectric tensor of the form

$$\begin{aligned}
\epsilon_{xx} &= 1 - n^2 - v/(1-u), \\
\epsilon_{yy} &= 1 - n^2 - v(1 - u\sin^2\theta)/(1-u), \\
\epsilon_{zz} &= 1 - v(1 - u\cos^2\theta)/(1-u), \\
\epsilon_{xy} &= -\epsilon_{yx} = -iv\sqrt{u}\cos\theta/(1-u), \\
\epsilon_{xz} &= l - \epsilon_{zx} = iv\sqrt{u}\sin\theta/(1-u), \\
\epsilon_{yz} &= \epsilon_{zy} = uv\cos\theta\sin\theta/(1-u),
\end{aligned} \quad (67)$$

where we used the usual notations $n = ck/\omega$, $\mathbf{k} = k\hat{e}_z$, $u = \omega_c^2/\omega^2$, $v = \omega_p^2/\omega^2$ (ω_c and ω_p representing the electron cyclotron and plasma frequencies), and θ describes an angle between the direction of the magnetic field in the plasma (assumed to lie in the yz plane) and the z axis. We are interested in the case where the angle θ is a constant and sufficiently small, so that we can expand (67) in powers of θ and retain only the zero- and first-order terms in the expansion.

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_{yy} = 1 - n^2 - v/(1-u), \\
\epsilon_{zz} &= 1 - v, \\
\epsilon_{xy} &= -\epsilon_{yx} = -iv\sqrt{u}/(1-u), \\
\epsilon_{xz} &= -\epsilon_{zx} = iv\sqrt{u}\,\theta/(1-u), \\
\epsilon_{yz} &= \epsilon_{zy} = uv\theta/(1-u).
\end{aligned} \quad (68)$$

Now, it is convenient to introduce conventional base vectors:

$$\hat{e}_\pm = (\hat{e}_x \pm i\hat{e}_y)/\sqrt{2} \quad (69)$$

in terms of which (68) becomes

$$\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}} + \boldsymbol{\beta}, \quad (70)$$

where

$$\tilde{\boldsymbol{\epsilon}} = \epsilon_{++}\hat{e}_+\hat{e}_+ + \epsilon_{--}\hat{e}_-\hat{e}_- + \epsilon_{zz}\hat{e}_z\hat{e}_z, \quad (71)$$

with

$$\begin{aligned}
\epsilon_{++} &= 1 - n^2 - v/(1+\sqrt{u}), \\
\epsilon_{--} &= 1 - n^2 - v/(1-\sqrt{u}),
\end{aligned} \quad (72)$$

and the only nonzero elements of $\boldsymbol{\beta}$ are

$$\beta_{+z} = -\beta_{z+} = iv\sqrt{u}\,\theta/\sqrt{2}(1+\sqrt{u}),$$
$$\beta_{-z} = -\beta_{z-}\; iv\sqrt{u}\,\theta/\sqrt{2}(1-\sqrt{u}). \tag{73}$$

Note that in terms of the base vectors $\hat{e}_+$, $\hat{e}_-$, and $\hat{e}_z$, the dielectric tensor (70) already has the form necessary for the application of the renormalization analysis. Indeed, the coupling may take place between the three modes associated with the zeroes of the three eigenvalues, ϵ_{++}, ϵ_{--}, and ϵ_{zz}, of the diagonal tensor $\tilde{\epsilon}$. Moreover, the small tensor $\boldsymbol{\beta}$ is such that $\partial\beta_{ij}/\partial k = 0$ and therefore the tensor $Q \simeq \partial\epsilon/\partial k$ is diagonal. Thus, the vectors $\hat{e}_\pm$ and $\hat{e}_z$ may play the role of the base vectors $\hat{m}_1$, $\hat{m}_2$, and $\hat{e}_3$, described previously.

Consider a situation where $u = \text{const}$, $v = v(z)$, and a wave associated with $\epsilon_{++} = 0$ is propagating from a region with $v(z) < 1$ toward the region in the plasma where $v(z) > 1$. Note that as the wave passes the point z_0 at which $v(z_0) = 1$, an additional eigenvalue of ϵ (namely, $\epsilon_{zz} = 1 - v$) vanishes, providing the possibility of excitation of an additional (electrostatic) mode. Relevant polarizations in this case are $\hat{m}_1 = \hat{e}_+$, $\hat{m}_2 = \hat{e}_z$, and $\hat{e}_3 = \hat{e}_-$. Since the excited electrostatic mode has a vanishing group velocity, we encounter the situation discussed at the end of the last section. In practice, however, thermal effects, for example, can introduce a sufficient finite group velocity, thus removing the anomaly. Assuming that this is indeed the case, we can find the transmission coefficient from (66) by observing that, in the case of interest,

$$\lambda^2 = |\beta_{+z}|^2|_{v=1} = u\theta^2/2(1+\sqrt{u})^2,$$
$$G_1 = \frac{\partial\epsilon_{++}}{\partial k}\bigg|_{v=1} = \frac{-2cu^{1/4}}{\omega(1+\sqrt{u})^{1/2}},$$

and

$$H_2 = -\frac{dv}{dz}\bigg|_{v=1}.$$

Thus, we obtain

$$T = \exp\left[\frac{-\pi u^{3/4}\omega\theta^2}{2c(1+\sqrt{u})^{3/2}}\left(\left|\frac{dv}{dx}\right|_{v=1}\right)^{-1}\right]. \tag{74}$$

This result coincides with the classical expression derived by using variational[9] and phase integral[10] methods in analyzing the example considered in this section.

VI. INTERPRETATION OF THE INTERACTION OF MODES ASSOCIATED WITH A SINGLE EIGENVALUE OF THE DIELECTRIC TENSOR

Up to this point we studied the case where the interacting modes correspond to two distinct eigenvalues of the dielectric tensor and, therefore, are characterized by different polarizations. Frequently, however, the situation is different. As a simple example, consider again a cold, magnetized plasma described by (67). For $\mathbf{k}$ along the guide magnetic field ($\theta = 0$), the dielectric tensor becomes [see Eqs. (70)–(72)]

$$\epsilon = \epsilon_{++}\hat{e}_+\hat{e}_+ + \epsilon_{--}\hat{e}_-\hat{e}_- + \epsilon_{zz}\hat{e}_z\hat{e}_z. \tag{75}$$

Now consider the modes associated with the zeroes of $\epsilon_{--} = 1 - n^2 - \omega_p^2/\omega(\omega - \omega_c)$. The dispersion relation for these modes can be written as

$$(1-n^2)(\omega - \omega_c) = \omega_p^2/\omega \tag{76}$$

and describes, in the case of $\omega_p/\omega \ll 1$, a weak coupling between the right-hand circularly polarized vacuum electromagnetic mode ($n = 1$) and the electron cyclotron mode ($\omega = \omega_c$). The two modes in this example are associated with a single eigenvalue ϵ_{--} of the dielectric tensor.

Similar examples can be found in Ref. 3, where the relevant modes in the cases of electron and ion cyclotron heating schemes are reduced from the dielectric tensor of the following general form,

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xx} & \epsilon_{xy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}, \tag{77}$$

describing perpendicular propagation in hot nonrelativistic Maxwellian plasmas. The three eigenvalues of ϵ in this case are

$$\epsilon_{1,2} = (\epsilon_{xx} + \epsilon_{yy})/2 \pm \sqrt{[(\epsilon_{xx} + \epsilon_{yy})/2]^2 + \epsilon_{xy}^2}$$

and $\epsilon_3 = \epsilon_{zz}$. Since in hot plasmas all the elements of the dielectric tensor have resonant terms at electron and ion cyclotron frequency and its harmonics, the dispersion relations associated with any one of the eigenvalues ϵ_i ($i = 1,2,3$) will describe ordinary or extraordinary modes coupled to various cyclotron modes. For instance, the coupling of the extraordinary mode to the electron cyclotron modes at fundamental and first harmonics, as well as the interaction of the fast Alfvén wave and the ion cyclotron wave at fundamental frequency (examples considered in Ref. 3), both involve pairs of modes associated with either $\epsilon_1 = 0$ or $\epsilon_2 = 0$. Also, the electron cyclotron resonance heating by the ordinary wave at perpendicular propagation was interpreted in Ref. 5 as a mode conversion to the electron cyclotron mode. These two modes are defined as solutions of $\epsilon_3 = \epsilon_{zz} = 0$, and therefore are also associated with a single eigenvalue of the dielectric tensor and ordinary polarization of the electric field.

It seems rather surprising at this point that in the aforementioned examples the transmission coefficients determined in Refs. 3 and 5 by using Eq. (65) were found to be in an excellent agreement with the results of other, more refined, theories. Thus, the formula derived for one specific situation seems to be applicable in different, more frequently encountered, circumstances. This observation may have the following simple explanation.

The dimensionality (3×3) of the conventional dielectric tensor corresponds to the number of the unknown components of the electric field $\mathbf{E}$ of the wave in a reduced problem, which initially in many cases is characterized by a much larger number of unknowns. For example, a cold, magnetized plasma case is initially described by an unknown vector $\mathbf{Z} = \{E_x, E_y, E_z, B_x, B_y, B_z, v_{ex}, v_{ey}, v_{ez}, v_{ix}, v_{iy}, v_{iz}\}$ which is 12-dimensional and represents the electromagnetic field associated with the wave and perturbed velocities of electrons and ions, respectively. In order to treat the unreduced weakly inhomogeneous case, consider first an initial value problem in a homogeneous situation. As usual, in this case we perform a Fourier transformation in space and a Laplace transformation in time of the whole unreduced set of Maxwell-momentum equations. This yields an algebraic equation of the form

$$i\omega\mathscr{E}(\mathbf{k},\omega)\cdot\mathbf{Z}_{\mathbf{k},\omega}=\mathbf{Z}_{\mathbf{k}}(0), \tag{78}$$

where $\mathbf{Z}_{\mathbf{k},\omega}$ is the Fourier–Laplace image of $\mathbf{Z}$ and $\mathbf{Z}_{\mathbf{k}}(0)$ describes initial conditions at $t=0$. The usual procedure at this stage is to eliminate most of the unknowns in (78) and to reduce it to a set of three equations for $\mathbf{E}$. Instead of this, here we first write an integral, convolution-type equation for $\mathbf{Z}$:

$$\mathbf{Z}(\mathbf{r},t)=\mathbf{Z}(\mathbf{r},0)+\int d^3\mathbf{r}'\int_0^t dt' \times\hat{\mathbf{\Sigma}}(\mathbf{r}-\mathbf{r}',t-t')\cdot\mathbf{Z}(\mathbf{r}',t'), \tag{79}$$

where the Fourier–Laplace image of the kernel $\hat{\mathbf{\Sigma}}$ is given by

$$\mathbf{\Sigma}(\mathbf{k},\omega)=\mathsf{I}-\mathscr{E}(\mathbf{k},\omega). \tag{80}$$

The Fourier–Laplace transformation of Eq. (79) yields Eq. (78), and therefore Eq. (79) is an equivalent form of the linearized Maxwell-momentum equations.

The form of Eq. (79) provides a simple way of generalizing to a weakly inhomogeneous plasma case, namely, we assume that the inhomogeneous (one-dimensional) case can be modeled by

$$\mathbf{Z}(\mathbf{r},t)=\mathbf{Z}(\mathbf{r},0)+\int d^3\mathbf{r}'\int_0^t dt' \times\hat{\mathbf{\Sigma}}[\mathbf{r}-\mathbf{r}',t-t';(x+x')/2]\cdot\mathbf{Z}(\mathbf{r}',t'), \tag{81}$$

where, as previously, $\hat{\mathbf{\Sigma}}$ varies rapidly with its difference arguments $\mathbf{r}-\mathbf{r}'$, $t-t'$, and slowly with $(x+x')/2$. Now we consider a stationary case $[\mathbf{Z}\sim\exp(-i\omega t)]$, assume the existence of an infinitesimal positive imaginary part of ω, and write an asymptotic form of Eq. (81):

$$\mathbf{Z}(\mathbf{r},t)=\int d^3\mathbf{r}'\int_{-\infty}^t dt' \times\hat{\mathbf{\Sigma}}[\mathbf{r}-\mathbf{r}',t-t';(x+x')/2]\cdot\mathbf{Z}(\mathbf{r}',t'). \tag{82}$$

Next, similar to (3), we seek solutions of Eq. (82) in the form

$$\mathbf{Z}(\mathbf{r},t)=\mathrm{Re}\{\chi(x)\exp[i(\psi(x)+k_y y+k_z z-\omega t)]\}. \tag{83}$$

Then it can be shown that, correct to the first order in δ, Eq. (82) reduces to[11]:

$$\chi=\mathbf{\Sigma}\cdot\chi-i\left\{\frac{1}{2}\left[\frac{d}{dx}\left(\frac{\partial\mathbf{\Sigma}}{\partial k_x}\right)\right]\cdot\chi+\frac{\partial\mathbf{\Sigma}}{\partial k_x}\cdot\frac{d\chi}{dx}\right\}, \tag{84}$$

where

$$\mathbf{\Sigma}(\mathbf{k},\omega;x)=\int d^3r'\int_0^\infty dt \times\hat{\mathbf{\Sigma}}(\mathbf{r}',t';x)\exp[i(\omega t'-\mathbf{k}\cdot\mathbf{r}')]. \tag{85}$$

Simple algebraic manipulations allow us to rewrite (84) as

$$i\omega\mathscr{E}(\mathbf{k},\omega;x)\cdot\chi=\mathsf{L}'\cdot\chi, \tag{86}$$

where

$$\mathscr{E}(\mathbf{k},\omega;x)=\mathsf{I}-\mathbf{\Sigma}(\mathbf{k},\omega;x) \tag{87}$$

is the same as in the homogeneous plasma [see Eqs. (80) and (85)] with parameters everywhere the same as those characterizing our inhomogeneous plasma at point x. The operator L' in (86) is exactly the same as L in Eq. (8) with the substitution $\epsilon\rightarrow\mathscr{E}(\mathbf{k},\omega;x)$.

Although the unreduced cold plasma problem involves 12 unknowns, in matrix form (86) it looks similar to the conventional, reduced problem for $\mathbf{E}$ [see Eq. (6)]. Therefore Eq. (86) can be solved as follows. We apply the conventional geometric optics formalism, which in the zeroth order results in the requirement that at least one of 12 eigenvalues of $\mathscr{E}$ (say ϵ'_1) vanishes at all x. This dispersion relation can be used in the first order equation describing the slow variation of the amplitude $\chi(x)$. This straightforward procedure, as before, is only applicable if all the remaining eigenvalues of $\mathscr{E}$, $(\epsilon'_2,\epsilon'_3,\ldots,\epsilon'_{12})$ are relatively large. If, on the contrary, any pair of the eigenvalues of $\mathscr{E}$ (say ϵ'_1 and ϵ'_2) becomes simultaneously small, we first diagonalize $\mathscr{E}$ with respect to the remaining components, namely, we write

$$\mathscr{E}=\sum_{i=3}^{12}\epsilon'_i\hat{e}'_i\hat{e}'^{*}_i+\mathbf{\Delta} \tag{88}$$

and then introduce orthogonal base vectors $\hat{m}_1$ and $\hat{m}_2$, defined via $\hat{e}'_i$ $(i=3,\ldots,12)$ in a way similar to that of Sec. III. In terms of the resulting, slowly varying vectors $\hat{m}_1$ and $\hat{m}_2$,

$$\mathbf{\Delta}=\sum_{i,j=1}^{2}\epsilon_{ij}\,\hat{m}_i\hat{m}_j^{*},$$

and we can use the same renormalized perturbation analysis of (86) leading to the energy flux conserving Eqs. (57) and (58) and also to expression (65) for the transmission coefficient. The quantities A_1^2 and A_2^2 in (57) and (58) should again be interpreted as the time-averaged energy fluxes associated with the polarizations $\hat{m}_1$ and $\hat{m}_2$. The detailed interpretation of the latter, however, may be rather complex in the multi-component unreduced problem. Finally, because of the fact that in the conventional reduced problem

$$D=\epsilon_1\epsilon_2\epsilon_3\propto\prod_{i=1}^{12}\epsilon'_i,$$

it may frequently happen that a double root in a single eigenvalue of ϵ will appear as a result of the vanishing of two *different* eigenvalues of $\mathscr{E}$. This explains the rather general applicability of the coupled mode equations (57) and (58) and of the corresponding expression for the transmission coefficient.

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