Theory of electron multiplication in gases in strong weakly nonuniform electric fields

L. Friedland and Yu. M. Kagan
Center for Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem, Israel

(Received 9 June 1983; accepted for publication 1 November 1983)

A theory of avalanche ionization of gases in weakly nonuniform electric fields is developed. The generally nonlocal problem of the current growth is reduced to the solution of two ordinary differential equations. As an example the theory is applied to the case of the discharge between two concentric cylindrical electrodes.

I. INTRODUCTION

For many practical applications, for example, evaluation of breakdown voltages in nonplanar geometries, in the theories of the cathode fall in a glow discharge, etc., understanding of the current growth in gases due to ionization in nonuniform electrostatic fields, is of major importance. The theory for strong nonuniform fields, however, becomes both nonlinear and nonlocal so that complex numerical analysis is necessary in order to describe avalanche multiplication of electrons in such cases. Fortunately, in some applications the scalelength of variation of the field is much less than the characteristic ionization length of the gas. Then a WKB-type approximation can be introduced into the theory, reducing it to a solution of a set of ordinary differential equations. The present work is a generalization of our recent theories[1,2] of the current growth in strong, but uniform electric fields. For completeness, in the rest of this introduction, we give a brief description of the main results of Ref. 2, which will be used in the following sections. Consider a stationary one-dimensional current flow and let j_0 be the electron current density at the cathode ($x = 0$). As a result of ionization, the current density at $x > 0$ will be

$$j(x) = j_0 P(x) . \tag{1}$$

Assuming that all the electrons are moving along the field lines and that new electrons are created in ionizing collisions with the same energy, we have to solve the following integro-differential equation for $P(x)$:

$$\frac{dP(x)}{dx} = \int_0^x P(x')M(x,x')dx' , \quad P(0) = 1 , \tag{2}$$

where

$$M(x,x') = -\frac{\partial}{\partial x'}\overline{Q}_i(x,x') , \tag{3}$$

and $\overline{Q}_i(x,x') = N\overline{q}_i(x,x')$, where N is the concentration of the gas atoms and $\overline{q}_i(x,x')$ the average cross section for ionization by an electron which has been created at a point x' and passed the distance $x - x'$. In the case of a uniform electric field $q_i(x,x')$ and therefore $M(x,x')$ are functions of $x - x'$ and; thus, Eq. (2) can be conveniently solved by using the method of Laplace transformation. The asymptotic solution of Eq. (2) is found in the form

$$P(x) = e^{\alpha x} , \tag{4}$$

where α is the first Townsend ionization coefficient determined from equation

$$\alpha - M(\alpha) = 0 , \tag{5}$$

where $M(\alpha)$ is the Laplace transform of M. A simple approximate form for $M(\alpha)$ was found in Ref. 2:

$$M(\alpha) = p\frac{A(E/p - B\epsilon_1)}{\epsilon_0(\alpha/p + k\epsilon_1) + \dfrac{E}{p} - B\epsilon_1} , \tag{6}$$

where A, ϵ_0, B, k, ϵ_1 are constants characterizing the ionization and total inelastic collision cross sections and p the pressure of the gas. We used for the ionization efficiency $Q_{i0}(\epsilon)$ and the total inelastic collision efficiency $Q_0(\epsilon)$ at normal conditions the following approximations:

$$Q_{i0}(\epsilon) = A(\epsilon - \epsilon_1)/(\epsilon_0 + \epsilon - \epsilon_1) ; \quad Q_0(\epsilon) = B + k(\epsilon - \epsilon_1) ,$$

which are in good agreement with experimental data for a variety of atomic and molecular gases.[2]

We do not mention here other theoretical approaches, which were applied to the problem of the calculation of the first Townsend coefficient because they were discussed in our previous paper.[2] Most of them are numerical and involve either direct solution of the Boltzmann equation or an application of the Monte–Carlo method.

II. WEAKLY NONUNIFORM ELECTRIC FIELDS

In the case of a nonuniform field, Eqs. (1) and (2) remain valid. The function $M(x,x')$, however, depends not only on $x - x'$, but also on x itself, which makes the application of the Laplace method impossible. Nonetheless, for a weak nonuniformity we have solved Eq. (2) by using a method similar to the WKB approximation, conventionally applied to waves propagation in weakly inhomogeneous dielectrics.

Note, first, that the solution of Eq. (2) for $P(x)$ is identical to the solution of the equation

$$\frac{dP(x)}{dx} = \int_{-\infty}^x P(x')M(x,x')dx' , \tag{7}$$

subject to initial conditions

$$P(0) = 1 , \quad \left.\frac{dP}{dx}\right|_{x=0} = 0 . \tag{8}$$

We assumed in Eq. (7) that $M(x,x')$ is suitably continued into the region $x' < 0$ and that the solution of Eq. (7) exists. Next, we write Eq. (7) in the form

$$\frac{dP}{dx} = \int_{-\infty}^{x} P(x') M\left(\frac{x+x'}{2}, x-x'\right) dx' . \tag{9}$$

In the case of weakly nonuniform electric fields, we can assume that the dependence of M in Eq. (9) on the first argument is weak. Then, on defining $y = x - x'$, we rewrite Eq. (9) as

$$\frac{dP}{dx} = \int_{0}^{\infty} P(x-y) M\left(x - \frac{y}{2}, y\right) dy , \tag{10}$$

and seek solution of this equation in the form

$$P(x) = a(x) e^{\psi(x)} , \tag{11}$$

where if we define

$$\alpha(x) = \frac{d\psi(x)}{dx} , \tag{12}$$

it is assumed that the relative changes of both $a(x)$ and $\alpha(x)$ on an average mean free path λ_i for ionization are small, namely,

$$\lambda_i \frac{d \ln a}{dx} \ll 1 ; \quad \lambda_i \frac{d \ln \alpha}{dx} \ll 1 .$$

These conditions can be fulfilled only if the electric field $E(x)$ is weakly non-linear in the same sense

$$\lambda_i \frac{d \ln E(x)}{dx} \ll 1 . \tag{13}$$

Expanding now, P and M in Eq. (10) (in the first argument) in powers of y we get, corrected to the first order,

$$[\alpha - M(\alpha,x)] a(x) + \frac{da}{dx}\left[1 - \frac{\partial M(\alpha,x)}{\partial \alpha}\right] - \frac{1}{2} a \frac{\partial^2 M}{\partial \alpha \partial x} = 0 , \tag{14}$$

where

$$M(\alpha,x) = \int_0^{\infty} M(x,y) e^{-\alpha y} dy . \tag{15}$$

Note that $M(\alpha,x)$ would be the Laplace transform of M in the case of a discharge in a uniform electric field with the strength E everywhere the same as that at the point x in our nonuniform case. Thus, according to Eq. (6), we can write

$$M(\alpha,x) = p \frac{A\left[\frac{E(x)}{p} - B\epsilon_1\right]}{\epsilon_0\left(\frac{\alpha}{p} + \kappa\epsilon_1\right) + \frac{E(x)}{p} - B\epsilon_1} . \tag{16}$$

Due to Eq. (13), the terms involving spatial derivatives in Eq. (14) are small and we can solve it by perturbation method. In the zero order, we have

$$\alpha - M(\alpha,x) = 0 , \tag{17}$$

which is identical to Eq. (5) except with the electric field in $M(\alpha,x)$ depending on x. The use of Eq. (16) then provides two solutions:

$$\alpha_{1,2}(x) = -\{pk\epsilon_1 + [E(x) - pB\epsilon_1]/\epsilon_0\}/2 \pm \sqrt{\{pk\epsilon_1 + [E(x) - pB\epsilon_1]/\epsilon_0\}^2/4 + pA[E(x) - pB\epsilon_1]/\epsilon_0} . \tag{18}$$

Clearly, Eq. (18) is an ordinary differential equation for ψ [see Eq. (11)].

$$\frac{d\psi}{dx} = \alpha , \quad \psi|_{x=0} = 0 .$$

When $\alpha(x)$ is known the amplitude $a(x)$ in Eq. (11) can be found from the first order equation in our perturbation scheme:

$$\frac{da}{dx}\left(1 - \frac{\partial M}{\partial \alpha}\right) - \frac{1}{2} a \frac{\partial^2 M}{\partial \alpha \partial x} = 0 , \tag{19}$$

or

$$\frac{d(\ln a^2)}{dx} = \frac{\dfrac{\partial^2 M}{\partial \alpha \partial x}}{1 - \dfrac{\partial M}{\partial \alpha}} , \quad a(0) = 1 . \tag{20}$$

The two solutions of Eqs. (18) and (20) for $\psi_{1,2}$ and $a_{1,2}$ corresponding to $\alpha_1(x)$ and $\alpha_2(x)$, respectively, can be now used in constructing the final solution of Eq. (7):

$$P(x) = \beta_1 a_1(x) e^{\psi_1(x)} + \beta_2 a_2(x) e^{\psi_2(x)} , \tag{21}$$

where β_1 and β_2 are chosen to satisfy

$$\begin{aligned} &\beta_1 + \beta_2 = 1 , \\ &\beta_1 \alpha_1(0) + \beta_2 \alpha_2(0) = 0 , \end{aligned} \tag{22}$$

in consistence with initial conditions Eq. (8). Equation (22) yields

$$\beta_1 = -\frac{\alpha_2(0)}{\alpha_1(0) - \alpha_2(0)} , \quad \beta_2 = \frac{\alpha_1(0)}{\alpha_1(0) - \alpha_2(0)} . \tag{23}$$

Since, according to Eq. (18), $\alpha_1 > 0$ and $\alpha_2 < 0$ and $|\alpha_2| > \alpha_1$, we have $\psi_1 > 0$, $\psi_2 < 0$ and $|\psi_2| > \psi_1$. Thus, the second term in Eq. (21) is important only at small distances from the cathode.

III. CURRENT GROWTH BETWEEN TWO CONCENTRIC CYLINDRICAL ELECTRODES

In the case of two concentric cylindrical electrodes with inner and outer radii r_1 and r_2, respectively, the coordinate x is measured from the surface of the cathode along the radius. Let the cathode be the inner cylinder. Then the value of ψ on the anode will be

$$\psi(r_1,r_2) = \int_0^{r_2 - r_1} \alpha(r_1 + x) dx = \int_{r_1}^{r_2} \alpha(y) dy . \tag{24}$$

In the case of opposite polarity, when the outer cylinder becomes a cathode

$$\psi(r_2,r_1) = \int_0^{r_2 - r_1} \alpha(r_2 - x) dx = \int_{r_1}^{r_2} \alpha(y) dy . \tag{25}$$

Thus $\psi(r_2,r_1) = \psi(r_1,r_2)$. Similar considerations, based on Eq. (20) for the amplitude, show that the amplitudes $a(r_1,r_2)$ and $a(r_2,r_1)$, corresponding to the inner cylinder being cathode, or anode, respectively, are equal and

$$\ln a^2(r_1,r_2) = \ln a^2(r_2,r_1) = \int_{r_1}^{r_2} \phi(y)dy, \quad (26)$$

where

$$\phi(x) = \frac{\dfrac{\partial^2 M}{\partial\alpha\partial x}}{1 - \dfrac{\partial M}{\partial\alpha}}. \quad (27)$$

Thus, finally, the current at the anode, when the inner cylinder serves as a cathode is

$$\frac{j(r_2)}{j_0} = \beta_1(r_1)a_1(r_1,r_2)e^{\psi_1(r_1,r_2)} + \beta_2(r_1)a_2(r_1,r_2)e^{\psi_2(r_1,r_2)}, \quad (28)$$

where

$$\beta_1(r_1) = -\frac{\alpha_2(r_1)}{\alpha_1(r_1)-\alpha_2(r_1)}, \quad \beta_2(r_1) = \frac{\alpha_1(r_1)}{\alpha_1(r_1)-\alpha_2(r_1)}. \quad (29)$$

Similarly, with reversed polarity, the current to the anode (the inner cylinder) will be

$$\frac{j(r_1)}{j_0} = \beta_1(r_2)a_1(r_2,r_1)e^{\psi_1(r_2,r_1)} + \beta_2(r_2)a_2(r_2,r_1)e^{\psi_2(r_2,r_1)}, \quad (30)$$

with

$$\beta_1(r_2) = -\frac{\alpha_2(r_2)}{\alpha_1(r_2)-\alpha_2(r_2)}, \quad \beta_2(r_2) = \frac{\alpha_1(r_2)}{\alpha_1(r_2)-\alpha_2(r_2)}. \quad (31)$$

Comparison between Eqs. (28) and (30) shows that reversing the polarity results generally in different reduced currents j/j_0 to the anode and the difference is due only to different values of the coefficients $\beta_1(r_1)$ and $\beta_2(r_1)$ compared to $\beta_1(r_2)$ and $\beta_2(r_2)$. These values are determined by the local electric fields on the inner and outer cylinders, respectively.

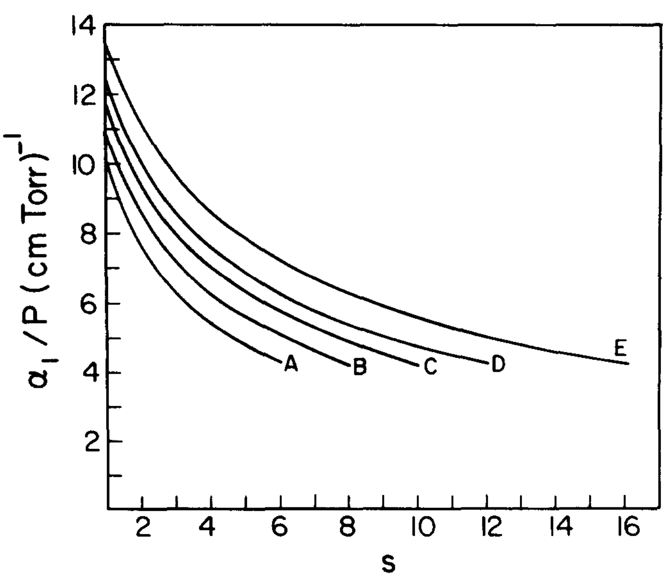

FIG. 1. α_1/p vs s for different values of reduced electric field C on the cathode: A: 600 V/cm Torr; B: 800 V/cm Torr; C: 1000 V/cm Torr; D: 1200 V/cm Torr; E: 1600 V/cm Torr.

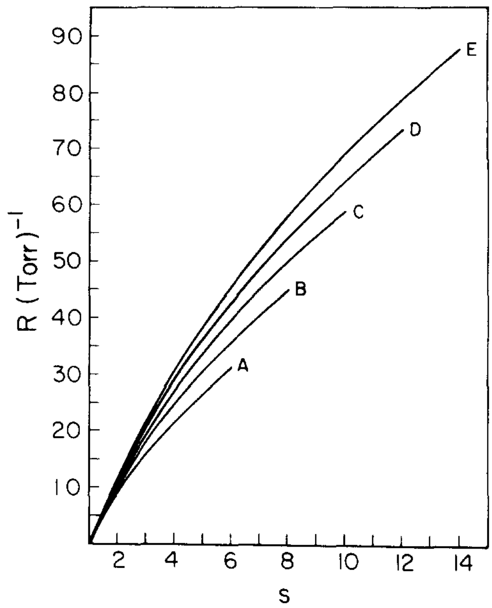

FIG. 2. $R = \int_1^s \alpha_1(s')/p\,ds'$ vs s for different values of reduced electric field on the cathode: A: 600 V/cm Torr; B: 800 V/cm Torr; C: 1000 V/cm Torr; D: 1200 V/cm Torr; E: 1400 V/cm Torr.

In the rest of this section we consider, as an example, the case of argon, where the constants in Eq. (18) are $\epsilon_1 = 13.5$ eV, $A = 18$ (cm Torr)$^{-1}$, $\epsilon_0 = 24$ eV, and $k_0 = 0.68$ (cm Torr eV)$^{-1}$. Let $C = E(r_1)/p$ be defined on the surface of the inner cylinder. Then with $s = r/r_1$ $(1 < s < r_2/r_1)$, we have

$$\frac{E(s)}{p} = \frac{C}{s}. \quad (32)$$

Equations (24) and (26) now can be written as

$$\psi(s) = pr_1 \int_1^s \frac{\alpha(s')}{p} ds', \quad (33)$$

and

$$\ln a^2 = \frac{1}{r_1} \int_1^s F(s')ds', \quad (34)$$

where

$$\frac{\alpha_{1,2}(s)}{p} = -\frac{1}{2}\left(9.18 + 0.042\frac{C}{s}\right) \pm \sqrt{\frac{1}{4}\left(9.18 + 0.042\frac{C}{s}\right)^2 + 0.75\frac{C}{s}}, \quad (35)$$

and

$$F(s) = \frac{\left(\dfrac{\alpha}{p} + 9.18 - 0.042\dfrac{C}{s}\right)}{\left(\dfrac{\alpha}{p} + 9.18 + 0.042\dfrac{C}{s}\right)} \times \frac{0.75\dfrac{C}{s^2}}{\left(\dfrac{\alpha}{p} + 9.18 + 0.042\dfrac{C}{s}\right)^2 + 0.75\dfrac{C}{s}}. \quad (36)$$

The functions α_1/p, $R = \int_1^s \alpha_1(s')/p\,ds'$ and $T = \int_1^s F_1(s')ds'$ vs s for different values of $C = 600$, 800, 1000, 1200, and 1600 V/cm Torr, are shown in Figs. 1–3. These results can

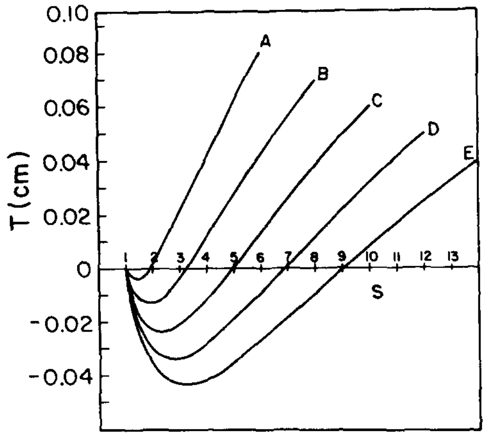

FIG. 3. $T = \int_1^s F_1(s')ds'$ vs s for different values of reduced electric field on the cathode: A: 600 V/cm Torr; B: 800 V/cm Torr; C: 1000 V/cm Torr; D: 1200 V/cm Torr; E: 1400 V/cm Torr.

be used in finding the current to the anode. For example, let the inner cylinder to be a cathode and consider the case $p = 0.6$ Torr, $C = 1000$ V/cm Torr, $r_1 = 0.1$ cm, and $r_2 = 1$ cm.

It is easy to show that in this case the condition Eq. (13) is fulfilled. Then on the cathode $(s = 1)\alpha_1/p = 11.95$ (cm Torr)$^{-1}$ and $\alpha_2/p = -62.79$ (cm Torr)$^{-1}$. Because of large and negative values of α_2 in the vicinity of the cathode, we will neglect the second term in Eq. (28) for the current. On using the results shown in Figs. 2 and 3 we get on the anode $(s = 10)\exp(\psi_1) = \exp(pr_1R) = 32$, and $\ln a_1^2 = T/R = 0.6(a_1 = 1.35)$. According to Eq. (29), in our case, $\beta_1(r_1) = 0.84$ and therefore $j(r_2)/j_0 \simeq \beta_1(r_1) \cdot a_1 \exp(\psi_1) = 36.3$. At the same potential difference, but with reversed polarity, $\beta_1(r_2) = 0.81$, and therefore $j(r_1)/j_0 = 34.8$. Thus, in this example, the reduced current j/j_0 is almost independent on polarity. However, the breakdown voltage may depend on the polarity because of possible different values of the second Townsend coefficient γ at the breakdown conditions on the inner and outer electrodes.

[1]Ju. M. Kagan and L. Friedland, Contributed paper, VI ESCAMPIG 1982 (Oxford), p. 76.
[2]L. Friedland and Ju. M. Kagan, J. Appl. Phys. **54**, 4947 (1983).