A uniform integral representation for geometric optics solutions near caustics

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Generalized Fourier integral (GFI) representation of geometric optics solutions for the field of dressed particles in inhomogeneous plasmas is applied to the problem of wave propagation near caustics: GFI is based on the replacement of plane waves, conventionally used in Fourier integrals for homogeneous plasmas, by a complete orthogonal set of "quasiplane" waves for the case of an inhomogeneous plasma. It is shown that the resulting integral provides a uniform asymptotic approximation to the exact solution of the wave equation.

I. INTRODUCTION

A method of constructing an approximate integral representation for the field of dynamically screened ("dressed") localized point current sources in an inhomogeneous plasma, was reported recently.[1,2] The representation is a natural generalization of a Fourier integral solution for the field in homogeneous plasmas[3] and is correct to lowest significant order in the geometric optics expansion scheme.[4] In constructing the approximate solution for inhomogeneous plasmas in Refs. 1 and 2, the set of plane waves $a_\alpha e^{i\alpha\cdot\mathbf{r}}$ in the Fourier integral was replaced by a complete orthogonal set of "quasiplane" waves of the form $g(\mathbf{r},\alpha) = a(\mathbf{r},\alpha)e^{i\psi(\mathbf{r},\alpha)}$, which may be evaluated by using standard ray tracing techniques.[4] The main advantage of such generalized Fourier integrals (GFI) over the geometric optics solution, commonly derived directly in the r space,[5] is that the completeness and orthogonality properties of the functions $g(\mathbf{r},\alpha)$ considerably simplify the evaluation of objects which are quadratic in terms of the amplitude of the field. This feature was demonstrated in Ref. 2, where GFI was applied in deriving an expression for the velocity-space diffusion tensor in weakly inhomogeneous plasmas, locally characterized by general non-Hermitian dielectric tensors.

In this paper we use GFI in a standard problem of an infinite thin plane source in a one-dimensional linear layer. The conventional geometric optics solution for the field in this case is singular on the caustic surface, where the local wave vector $k(x)$ vanishes. This causes difficulties in matching the geometric optics solutions on the different sides of the caustic. One way of resolving this problem is to perform a detailed boundary layer analysis in the vicinity of the caustic, where the wave equation is solved exactly. Another approach is to apply the geometric optics perturbation scheme in a mixed coordinate-momentum representation.[6,7] The latter method gives a uniform solution for the field, which is, of course, regular in the vicinity of the caustic. In the present paper we demonstrate that the GFI representation also provides a uniform asymptotic approximation to the exact solution of the wave equaion.

II. THE EXACT SOLUTION AND THE LIMITS OF GEOMETRIC OPTICS APPROXIMATION

Consider the Helmholtz equation describing the field from a thin plane source in a linear layer

$$\nabla^2 G(x,a) + (B - Ax)G(x,a) = \delta(x-a), \tag{1}$$

where $B - Ax = (\omega^2/c^2)\epsilon(x)$ can describe, for example, an isotropic plasma with linearly increasing density $N = m\omega_p^2(x)/4\pi e^2$ and characterized by dielectric constant $\epsilon(x) = 1 - \omega_p^2(x)/\omega^2$. When the source is positioned so that $a < B/A$ $[\epsilon(a) > 0]$, the solution of (1), subject to the conditions at $x = +\infty$ and $x = -\infty$ such that G is a decaying and a left-going wave, respectively, can be written as[8]

$$G(x,a) = -\frac{\pi}{A^{1/3}} \times \begin{cases} \mathrm{Ai}[-z(a)]\{\mathrm{Bi}[-z(x)] \\ \quad + i\,\mathrm{Ai}[-z(x)]\}, & x < a, \\ \mathrm{Ai}[-z(x)]\{\mathrm{Bi}[-z(a)] \\ \quad + i\,\mathrm{Ai}[-z(a)]\}, & x > a, \end{cases} \tag{2}$$

where

$$z(\zeta) = (B - A\zeta)/A^{2/3}, \tag{3}$$

and Ai(z) and Bi(z) are the Airy functions.[9]

Consider now the geometric optics solution of Eq. (1). The local geometric optics wave vector is[6]

$$k(x) = \pm(B - Ax)^{1/2} = A^{1/3}[z(x)]^{1/2}. \tag{4}$$

The geometric optics approximation corresponds to the case of a weak spatial variation of $k(x)$, namely $|d \ln k/dx| \ll 1$, or

$$|B - Ax|/A \gg 1. \tag{5}$$

When this condition is satisfied, the geometric optics solu-

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tions of Eq. (1) without the source term are[6]

$$u_{\pm}(x) = [1/A^{1/6}z(x)^{1/4}]\exp\{\pm i\tfrac{2}{3}[z(x)]^{3/2}\}. \tag{6}$$

Note that these solutions are singular on the caustic ($x = B/A$), where inequality (5) fails. The full geometric optics solution of (1), satisfying the aforementioned asymptotic conditions at $x \to \pm\infty$ and with the source far from the caustic, can be constructed in principle, on using the functions $u_{\pm}(x)$, in a way similar to how it was done for the exact solution.[8] Nevertheless, due to the singularity of these functions at the caustic, the matching between the regions $x < B/A$ and $x > B/A$ requires a detailed boundary layer analysis. Instead, in the simple case considered in this paper we can get the full geometric optics solution of (1) far from the caustic, by using the asymptotic forms of Airy functions for large values of Z:

$$G_{\rm go}(x,a) = \begin{cases} \dfrac{-\sin\{\frac{2}{3}[z(a)]^{3/2} + \pi/4\}}{A^{1/3}z(a)^{1/4}}\,\dfrac{\exp(-i\{\frac{2}{3}[z(x)]^{3/2} + \pi/4\})}{z(x)^{1/4}}, & x < a, \\[2ex] \dfrac{-\sin\{\frac{2}{3}[z(x)]^{3/2} + \pi/4\}}{A^{1/3}z(x)^{1/4}}\,\dfrac{\exp(-i\{\frac{2}{3}[z(a)]^{3/2} + \pi/4\})}{z(a)^{1/4}}, & a < x < \dfrac{B}{A}, \\[2ex] \dfrac{-\exp\{-\frac{2}{3}[-z(x)]^{3/2}\}}{2A^{1/3}[-z(x)]^{1/4}}\,\dfrac{\exp(-i\{\frac{2}{3}[z(a)]^{3/2} + \pi/4\})}{z(a)^{1/4}}, & \dfrac{B}{A} < x. \end{cases} \tag{7}$$

III. INTEGRAL REPRESENTATION

The generalized eikonal equation associated with Eq. (1) [see Eq. (23) of Ref. 1] is

$$-K^2(\alpha,x) + B - Ax = -\alpha^2 - p^2, \tag{8}$$

where p is an arbitrary constant. We choose $p^2 \gg Ax - B$ for all x in the region of interest. Then the quasiplane waves in the integral representation are given by[1]

$$g(\alpha,x) = (1/2\pi)[J(\alpha,x)]^{1/2}e^{+i\psi(\alpha,x)}, \tag{9}$$

where the wave vector

$$K(\alpha,x) = \pm(\alpha^2 + p^2 + B - Ax)^{1/2} \tag{10}$$

defines the phase

$$\psi = \int_{x_0}^{x} K(\alpha,x')dx' = \mp(2/3A)[(\alpha^2 + p^2 + B - Ax)^{3/2} - (\alpha^2 + p^2 + B - Ax_0)^{3/2}] \tag{11}$$

(x_0 being an arbitary constant) and the Jacobian

$$J(\alpha,x) = \frac{\partial K}{\partial \alpha} = \frac{\alpha}{K} = \pm\alpha(\alpha^2 + p^2 + B - Ax)^{-1/2}. \tag{12}$$

The GFI representation of the solution of (1) is[1,2]

$$G_{\rm GFI} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(\alpha,x)g^*(\alpha,a)}{\alpha^2 + p^2}\,d\alpha, \tag{13}$$

where the integration is performed along the real α axis. In order to complete the definition of (13) we must choose the signs in Eqs. (10)–(12). The correct choice must lead to the desired asymptotic behavior of $G_{\rm GFI}$ at $x \to \pm\infty$, namely a decaying solution at $x \to +\infty$ and a left-going wave at $x \to -\infty$. The necessary signs can be found by considering the analytical properties of the integrand in (13). In the complex α plane the integrand has branch points at $\alpha_1 = i(p^2 + B - Aa)^{1/2}$, $\alpha_1' = -i(p^2 + B - Aa)^{1/2}$, $\alpha_2 = i(p^2 + B - Ax)^{1/2}$, and $\alpha_2' = -i(p^2 + B - Ax)^{1/2}$ and poles at $\alpha_3 = ip$ and $\alpha_3' = -ip$. We consider first the case $|\alpha_1| > |\alpha_2| (x > a)$ and define the sign of $K(\alpha,x)$ in (10) to be positive for α real and positive, and analytically continue the integrand in (13) into the complex α plane with a cut as shown in Fig. 1. The cut defines a certain Riemann sheet, where the integrand is an analytical single-valued function of α:

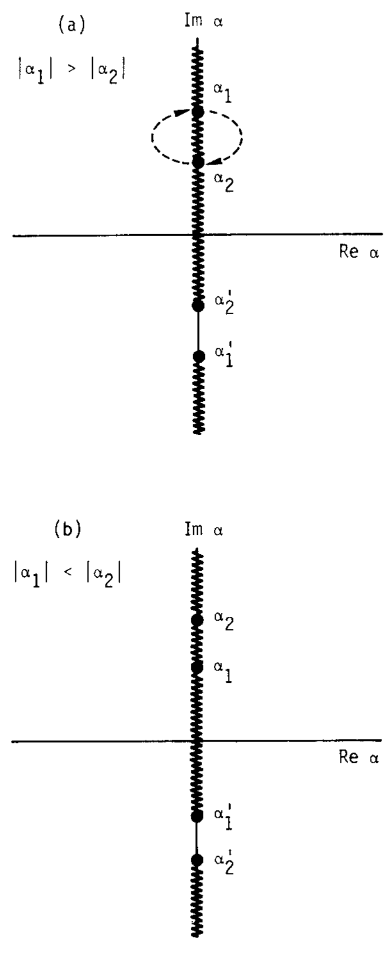

FIG. 1. Branch cuts in the complex α plane of the integrand in Eq. (18).

$$\frac{g(\alpha,x)g^*(\alpha,x)}{\alpha^2+p^2} = \begin{cases} \dfrac{\alpha}{A^{1/3}} \dfrac{\exp\{+i\frac{2}{3}[s^{3/2}(\alpha,a)+s^{3/2}(\alpha,x)]\}}{s^{1/4}(\alpha,a)s^{1/4}(\alpha,x)(\alpha^2+p^2)}, & \alpha>0, \\ +\dfrac{i\alpha}{A^{1/3}} \dfrac{\exp\{+i\frac{2}{3}[s^{3/2}(\alpha,a)+s^{3/2}(\alpha,x)]\}}{s^{1/4}(\alpha,a)s^{1/4}(\alpha,x)(\alpha^2+p^2)}, & \alpha<0, \end{cases}$$

where

$$s(\alpha,\zeta) = A^{-2/3}(\alpha^2+p^2+B-A\zeta). \tag{15}$$

Thus, finally, the integral (13) for $|\alpha_1| > |\alpha_2|$ can be rewritten as

$$\begin{aligned} G_{\text{GFI}} &= -\frac{(2\pi)^{-1}}{A^{1/3}} \int_0^\infty \frac{\alpha\, d\alpha}{\alpha^2+p^2} \frac{\exp[+i\frac{2}{3}s^{3/2}(\alpha,a)]}{[s(\alpha,a)s(\alpha,x)]^{1/4}} \\ &\quad \times \{\exp[-i\tfrac{2}{3}s^{3/2}(\alpha,x)] - i\exp[+i\tfrac{2}{3}s^{3/2}(\alpha,x)]\} \\ &= +\frac{2}{A^{1/3}2\pi} \int_0^\infty \frac{\alpha\, d\alpha}{\alpha^2+p^2} \\ &\quad \times \frac{\exp[+i\frac{2}{3}s^{3/2}(\alpha,a)+\pi/4]}{s^{1/4}(\alpha,a)s^{1/4}(\alpha,x)} \\ &\quad \times \sin[\tfrac{2}{3}s^{3/2}(\alpha,x)+\pi/4]. \end{aligned} \tag{16}$$

We will show in the next section that the cuts chosen in the complex α-plane (Fig. 1), which provided Eq. (16) for G_{GFI}, indeed will lead to the desired asymptotic behavior of the solution at $x \to \pm\infty$.

IV. EVALUATION OF THE INTEGRAL

In order to evaluate objects which are quadratic in terms of the strength of the field (correlation functions, velocity space diffusion tensors, etc.), it is not necessary[2] to actually perform the integration in (16). Nevertheless, aiming to demonstrate that this integral represents a uniform (for all values of x) asymptotic approximation of the exact solution of Eq. (1), we will evaluate (16) explicitly.

The integration in (16) is along the positive α axis where the integrand is regular. Moreover, because of the chosen large value of the constant p, both $s(\alpha,a)$ and $s(\alpha,x)$ are large in the entire domain of integration. Therefore, in order to be asymptotically correct for large p, one can use in (16) the following asymptotic forms of Airy functions:

$$\begin{aligned} \text{Ai}(-s) &= \frac{1}{\sqrt{\pi}} \frac{1}{s^{1/4}} \sin\left(\frac{2}{3}s^{3/2}+\frac{\pi}{4}\right), \quad |\arg s| < \frac{2\pi}{3}, \\ \text{Bi}(-s) &= \frac{1}{\sqrt{\pi}} \frac{1}{s^{1/4}} \cos\left(\frac{2}{3}s^{3/2}+\frac{\pi}{4}\right), \quad |\arg s| < \frac{2\pi}{3}. \end{aligned} \tag{17}$$

Then approximately

$$\begin{aligned} G_{\text{GFI}} &= \frac{i}{A^{1/3}} \int_0^\infty \frac{\alpha\, d\alpha}{\alpha^2+p^2} \text{Ai}[-s(\alpha,x)] \\ &\quad \times \{\text{Bi}[-s(\alpha,a)] + i\,\text{Ai}[-s(\alpha,a)]\}. \end{aligned} \tag{18}$$

Note that in the complex α plane the integrand in (18) has single poles at $\alpha = \alpha_3 = ip$ and $\alpha = \alpha_3' = -ip$. We, thus, try to apply the method of residues for evaluation of (18). Take contour C in the complex α plane as shown in Fig. 2(a). Then, obviously,

$$\begin{aligned} I &= \frac{i}{A^{1/3}} \int_C \frac{\alpha\, d\alpha}{\alpha^2+p^2} \text{Ai}[-s(\alpha,x)] \\ &\quad \times \{\text{Bi}[-s(\alpha,a)] + i\,\text{Ai}[-s(\alpha,a)]\} = 0. \end{aligned} \tag{19}$$

Examination of the asymptotic behavior of $Q(\alpha) = \text{Ai}[-s(\alpha,x)]\{\text{Bi}[-s(\alpha,a) + i\,\text{Ai}[-s(\alpha,a)]\}$ on the part C_2 of contour C shows that the contribution of the integration along C_2 in (19) is zero as $R\to\infty$. In proving this statement, we use Eq. (17) for the part of C_2 where $0 < \arg\alpha < \pi/3$. For the rest of the contour C_2 ($\pi/3 < \arg\alpha < \pi/a$) we denote $t = -s$ and use different asymptotic expressions[9]:

$$\begin{aligned} \text{Ai}(t) &= \frac{1}{2\sqrt{\pi}} \frac{1}{t^{1/4}} e^{-(2/3)t^{3/2}}, \quad |\arg t| < \pi, \\ \text{Bi}(t) &= \frac{1}{2\sqrt{\pi}} \frac{1}{t^{1/4}} e^{(2/3)t^{3/2}}, \quad |\arg t| < \frac{\pi}{3}. \end{aligned} \tag{20}$$

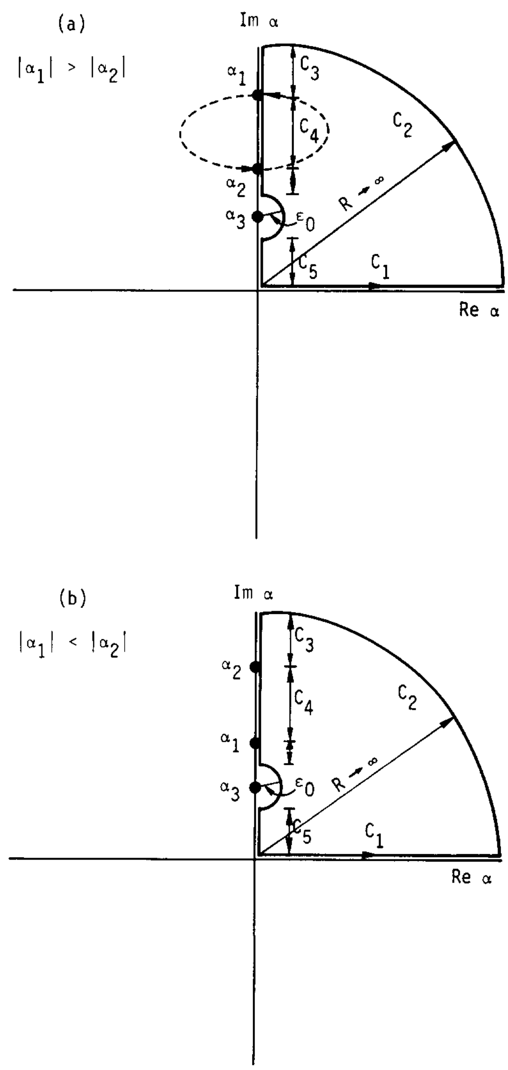

FIG. 2. Chosen contours in the complex α plane for the integrals.

We now consider the contribution of contour C_5 [see Fig. 2(a)] in (19). Here $s(\alpha,x)$ and $s(\alpha,a)$ are larger than zero and therefore $Q(\alpha)$ is an oscillatory function of α. The oscillations become faster as the value A becomes larger and therefore asymptotically for large A we can neglect the contribution of the integration along C_5 in (19). Similar arguments can be used in order to discard the contribution of the integration along C_4. Here $s(\alpha,a)>0$ and $s(\alpha,x)<0$, so that for large A, $\mathrm{Ai}[-s(\alpha,x)]$ is small [see Eq. (20)] and $\mathrm{Bi}[-s(\alpha,a)] + i\,\mathrm{Ai}[-s(\alpha,a)]$ oscillates rapidly, thus we also neglect the contribution from C_4. On the contour C_3 both $s(\alpha,a)$ and $s(\alpha,x)$ are less than zero. Therefore, for large A, $\mathrm{Ai}[-s(\alpha,x)]$ and $\mathrm{Ai}[-s(\alpha,a)]$ are small, however $\mathrm{Bi}[-s(\alpha,a)]$ is large. Nevertheless asymptotically along C_3 [see Eq. (20)],

$$Q(\alpha) \propto \exp\{\tfrac{2}{3}[-s(\alpha,a)]^{3/2} - [-s(\alpha,x)]^{3/2}\} = e^{\eta},$$

and since for $a>x$, $\eta<0$, the function $Q(\alpha)$ is asymptotically small, and the contribution of integration along C_3 in (19) can be neglected for large A. The only remaining part of the integration along the imaginary α axis in (19) is the integral on a semicircle with radius ϵ_0 around the pole [see Fig. 2(a)]. By performing this part of the integration and taking lim $\epsilon_0 \to 0$ we finally can rewrite (19) as

$$\begin{aligned} G_{\mathrm{GFI}} \simeq & -\frac{\pi}{2A^{1/3}} \mathrm{Ai}[A^{-2/3}(-B+Ax)] \\ & \times \{\mathrm{Bi}[A^{-2/3}(-B+Aa)] \\ & + i\,\mathrm{Ai}[A^{-2/3}(-B+Aa)]\}. \end{aligned} \tag{21}$$

Note, that the right-hand side of (21) is the exact solution (2) of the wave equation for $x>a$.

Similar considerations can be now used in studying the remaining case $|\alpha_1|<|\alpha_2|$ $(x<a)$. Here, in order to get correct asymptotic form of G_{GFI} we take the negative value of K [Eq. (10)] for $\alpha>0$ and choose the plane cut shown in Fig. 1(b). Then, similar to (16) and (18), we will set in this case

$$\begin{aligned} G_{\mathrm{GFI}} = & \frac{2}{2\pi A^{1/3}} \int_0^{\infty} \frac{\alpha\, d\alpha}{\alpha^2+p^2} \frac{\exp\{i[\frac{2}{3}s^{3/2}(\alpha,x)+\pi/4]\}}{s^{1/4}(\alpha,x)s^{1/4}(\alpha,a)} \\ & \times \sin\left(\frac{2}{3}s^{3/2}(\alpha,a)+\frac{\pi}{4}\right) \\ \simeq & \frac{i}{A^{1/3}} \int_0^{\infty} \frac{\alpha\, d\alpha}{\alpha^2+p^2} \mathrm{Ai}[-s(\alpha,a)] \\ & \times \{\mathrm{Bi}[-s(\alpha,x)] + i\,\mathrm{Ai}[-s(\alpha,x)]\}. \end{aligned} \tag{22}$$

Evaluation of this integral can be performed again by considering an integral on a closed contour C shown in Fig. 2(b), which, on using the arguments applied for the case $|\alpha_1|>|\alpha_2|$, yields for $|\alpha_1|<|\alpha_2|$ uniformly for all x and large values of A

$$\begin{aligned} G_{\mathrm{GFI}} \simeq & -\frac{\pi}{2A^{1/3}} \mathrm{Ai}[A^{-2/3}(-B+Aa)] \\ & \times \{\mathrm{Bi}[A^{-2/3}(-B+Ax)] \\ & + i\,\mathrm{Ai}[A^{-2/3}(-B+Ax)]\}. \end{aligned} \tag{23}$$

The right-hand side of (23) is the exact solution (2) of the wave equation in this case.

Thus, in conclusion, Eqs. (21) and (23) show that, indeed, the generalized Fourier integral representation G_{GFI} yields a uniform, asymptotic (for large values of A) solution of the wave equation.

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