

Generalized theory of first Townsend ionization coefficient in strong electric fields

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A one dimensional model of electron multiplication in a gas in strong electric fields is considered by using the Laplace transform technique. An approximate expression for the first Townsend ionization coefficient is derived and found to be in a good agreement with the experimental data for a variety of atomic and molecular gases.

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I. INTRODUCTION

Consider the conventional picture of two plane parallel electrodes in a gas. Assume that a uniform electric field E is applied between the electrodes and that the cathode is a stationary source of an electron current j_0 which is low enough so that one can neglect the space charge effects in the gap between the electrodes. If the value of E/p (here, p is the pressure of the gas) exceeds some minimum value, the current in the gap depends on the distance x from the cathode and grows faster with the increase of E/p . This effect was explained by Townsend as a result of electron multiplication due to ionization in the volume between the electrodes. Asymptotically (at large distances x) the electron current is given by

$$j(x) = j_0 e^{\alpha x}, \quad (1.1)$$

where α is the first Townsend ionization coefficient. Simple theoretical considerations show that

$$\frac{\alpha}{p} = f\left(\frac{E}{p}\right). \quad (1.2)$$

Many experimental curves illustrating this dependence can be found in the literature (for example, Brown,¹ Fletcher and Davies,² Kucukarpaci *et al.*³). Nonetheless, the problem of the theoretical determination of (1.2) has been found to be very complicated. In earlier investigations the Boltzmann equation was considered. The elastic and inelastic collisions of the electrons with atoms were taken into account; however, the term $v_x(\partial f/\partial x)$ (f being the electron velocity distribution function) and part of the collision integral describing the generation of new electrons in ionizing collisions were neglected in zero order. Using the method of Lorentz, the distribution function was written as $f(\mathbf{v}) = f_0(v) + f_1(v) \cos \theta$ where $\cos \theta$ is the angle between the electron velocity and the electric field. The assumption of smallness of f_1/f_0 permitted evaluation of $f_0(v)$ and $f_1(v)$ by applying the perturbation method to the reduced Boltzmann equation. Knowledge of $f_0(v)$ allowed one calculation of the average number of ionizations Z produced by an electron per second and $f_1(v)$ defined the drift velocity V_d . Finally, the Townsend coefficient α was found from the relation

$$Z = \alpha V_d. \quad (1.3)$$

Such a procedure is valid only for low values of E/p , when the current growth in the electric field is weak. In general,

however, one cannot neglect the term $v_x(\partial f/\partial x)$ when the current growth is significant, and of course the terms describing the new electrons created in ionizing collisions must be added to the collision integral in the Boltzmann equation. The Boltzmann equation becomes much more complicated and can be solved only numerically.³⁻⁸ The numerical methods involve either direct numerical solution of the Boltzmann equation or an application of the Monte-Carlo method. Such calculations yield as a rule a good agreement with the experimental results. Nevertheless, these comparisons were performed for relatively short range of values of E/p for some gases and involved the use of elaborate numerical techniques.

In this paper we present a different approach to the problem. Instead of solving the Boltzmann equation we proceed from the equation for the current flux³ and solve it by using the Laplace transform method. The solution is found for limiting cases of moderate and high values of E/p . And finally, an interpolating formula describing the Townsend coefficient α/p in a wide range of E/p is suggested. We will later compare this formula with the existing experimental data and show that it applies not only to wide range of values of E/p but also for a variety of atomic and molecular gases. Existence of such a universal analytic expression for α/p has obvious practical importance.

II. GENERAL THEORETICAL CONSIDERATIONS

Assume now that the electrons in the gap between the electrodes move only along the direction of the electric field and may lose their energy in inelastic collisions and ionizations. Let $\epsilon_{x,x}$ be the energy of an electron at a point x if this electron was created at a point x' . Assume also that all the new electrons in the gap are created as a result of ionization and initially have the same energy. Then the average energy $\bar{\epsilon}_{x,x}$ of an individual electron and the average number of ionizations $\bar{Q}_i dx$ in the interval $[x, x + dx]$ due to an electron created at the point x' , are functions only of the difference $x - x'$. Here, $Q_i = p Q_{i0}(\epsilon_{x,x})$, where p is the gas pressure in Torr and Q_{i0} the total ionization efficiency at normal conditions. The averaging is over all possible inelastic events in the interval $[x', x]$. As a result of ionizations the current in the gap will grow with x and one formally has

$$j(x) = j_0 P(x). \quad (2.1)$$

As was shown earlier⁹ the function $P(x)$ can be described by an integrodifferential equation

$$\frac{dP(x)}{dx} = - \int_0^x p(x')M(x-x')dx', \quad P(0) = 1, \quad (2.2)$$

where

$$M(x-x') = \frac{\partial}{\partial x'} \bar{Q}_i(x-x') = - \frac{d}{dy} \bar{Q}_i(y), \quad y = x-x', \quad (2.3)$$

Equation (2.2) can be solved by means of the Laplace transformation. In fact, define for any $\Psi(x)$,

$$\Psi_S = \int_0^\infty e^{-Sx} \Psi(x) dx, \quad (2.4)$$

where $\text{Re } S$ is assumed to be large enough for convergence. Then, on multiplying Eq. (2.2) by e^{-Sx} , integrating the resulting equation with respect to x , and using the well known theorem of the Laplace transformations we get

$$SP_S - 1 = -P_S M_S. \quad (2.5)$$

Here, we have from Eq. (2.3)

$$M_S = -S\bar{Q}_{iS}, \quad (2.6)$$

and, therefore, finally

$$P_S = \frac{1}{S(1 - \bar{Q}_{iS})}. \quad (2.7)$$

The actual x dependence of P can be now found by taking the inverse Laplace transformation of Eq. (2.7):

$$P(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dS \frac{e^{Sx}}{S(1 - \bar{Q}_{iS})} = \sum_k R_k, \quad (2.8)$$

where R_k is the sum of the residues of the function $[e^{Sx}/S(1 - \bar{Q}_{iS})] = \phi(S)$ in the complex S plane. In order to apply Eq. (2.8) all the poles of $\phi(S)$ must be found. One of them is $S = 0$. The others are defined by the equation

$$\bar{Q}_{iS} = 1. \quad (2.9)$$

By definition, the largest real part of the zeros of Eq. (2.9), which describes the asymptotic behavior of $P(x)$ at large x , has a meaning of the first Townsend ionization coefficient.

III. APPROXIMATE SOLUTION

A study of existing experimental data shows¹ that ionization cross section of various gases have the following characteristic dependence on the electron energy ϵ . The cross section grows linearly near the threshold of ionization, reaches a rather flat maximum at energies of about 100 eV and then slowly decrease as the energy increases. Simple esti-

mation shows that for the values of E/p usually used in the measurements of α/p , the average energies of individual electrons do not exceed 100 eV. Therefore, in the theory we can use an approximation for $Q_{i0}(\epsilon)$ which will describe the ionization efficiency up to the maximum. We choose the following approximate expression

$$Q_{i0}(\epsilon) = \frac{A(\epsilon - \xi)}{\epsilon_0 + \epsilon - \xi}, \quad \epsilon > \xi. \quad (3.1)$$

According to Eq. (3.1) near the threshold ($\epsilon = \xi$) the cross section grows linearly with ϵ and the slope of the curve is

$$a = \frac{A}{\epsilon_0}. \quad (3.2)$$

For $\epsilon \gg \epsilon_0 - \xi$

$$Q_{i0}(\epsilon) = A. \quad (3.3)$$

Thus, we limit our analysis only to the increasing with energy part of the ionization cross section. The case of larger values of E/p when one has to take into account the decrease of ionization cross section with energy was considered recently.¹⁰

In our approximation at large values of E/p

$$\bar{Q}_i(y) \approx pA. \quad (3.4)$$

The Laplace transformation of Eq. (3.4) gives

$$\bar{Q}_{iS} = \frac{pA}{S}. \quad (3.5)$$

In contrast, at low energies

$$Q_{i0}(\epsilon) = pa(\epsilon - \xi), \quad \epsilon > \xi, \quad (3.6)$$

and the averaging gives

$$\bar{Q}_i \approx pa\bar{\epsilon}. \quad (3.7)$$

We neglect here the energy losses in ionizations since the values of E/p are still assumed to be large enough so that $\bar{\epsilon} > \xi$. In addition the electrons in the gap at such values of E/p are created having an average energy of several electron volts,¹¹ which is close to the ionization threshold and therefore start ionizing the gas practically from the moment of their generation. The Laplace transformation of Eq. (3.7) gives

$$\bar{Q}_{iS} = pa\bar{\epsilon}_S \quad (3.8)$$

On the other hand, the energy conservation law for an individual electron in the gas moving along the field lines gives an additional equation for $\bar{\epsilon}$

$$\frac{d\bar{\epsilon}}{dy} = E - \bar{Q}(y)\xi, \quad (3.9)$$

TABLE I. Constants characterizing ionization and inelastic efficiencies in various gases.

	ξ (eV)	A (cm Torr) ⁻¹	ϵ_0 (eV)	a (cm Torr eV) ⁻¹	B (cm Torr) ⁻¹	k (cm Torr eV) ⁻¹
Hg	10	40	45	0.9	26	0.36
A	13.5	18	24	0.75	0	0.68
Ne	19	5.5	100	0.055	0.15	0.037
He	22	2	44	0.045	0	0.06
N ₂	11	20	80	0.24	5.5	0.05
H ₂	11	4.5	20	0.225	7.5	0

where $Q(\epsilon) = pQ_0(\epsilon)$ and $Q_0(\epsilon)$ is the total inelastic collision efficiency (which also includes ionization).

In Eq. (3.9) we assumed that in all inelastic collisions the electron loses the same amount of energy ξ . In order to further simplify the problem we now approximate the total inelastic collision efficiency in the same range of energy as it was done for $Q_0(\epsilon)$. A study of available experimental data shows that in this range in a good approximation $Q_0(\epsilon) = p[B + K(\epsilon - \xi)]$ so that we use

$$\bar{Q} \approx (B + K\bar{\epsilon})p. \quad (3.10)$$

then the Laplace transformation of Eq. (3.9) gives

$$S\bar{\epsilon}_S = \frac{E - Bp\xi}{S} - pk\xi\bar{\epsilon}_S. \quad (3.11)$$

Substituting $\bar{\epsilon}_S$ from Eq. (3.11) into Eq. (3.8) we finally arrive at

$$\bar{Q}_{iS} = \frac{pa(E - Bp\xi)}{S(S + pk\xi)} = \frac{a(E/p - B\xi)}{\frac{S}{p}\left(\frac{S}{p} - k\xi\right)}. \quad (3.12)$$

Equations (3.5) and (3.12) describe \bar{Q}_{iS} for the limiting cases of "low" and "large" values of E/p . The simplest interpolation formula which gives the same expressions for \bar{Q}_{iS} in the two limits is

$$\bar{Q}_{iS} = \frac{P}{S} \frac{A(E/p - B\xi)}{\epsilon_0(S/p + k\xi) + E/p - B\xi}. \quad (3.13)$$

And, finally, from Eq. (2.9) we get an expression for the first Townsend ionization coefficient

$$\frac{\alpha}{p} = \frac{S_{\max}}{p} = -\frac{1}{2} \left[k\xi + \frac{1}{\epsilon_0} \left(\frac{E}{p} - B\xi \right) \right] + \left[\frac{1}{4} \left[k\xi + \frac{1}{\epsilon_0} \left(\frac{E}{p} - B\xi \right) \right]^2 + a \left(\frac{E}{p} - B\xi \right) \right]^{1/2}. \quad (3.14)$$

IV. RESULTS AND DISCUSSION

In this section we compare our approximate expression for α/p with the existing experimental data for various gases.

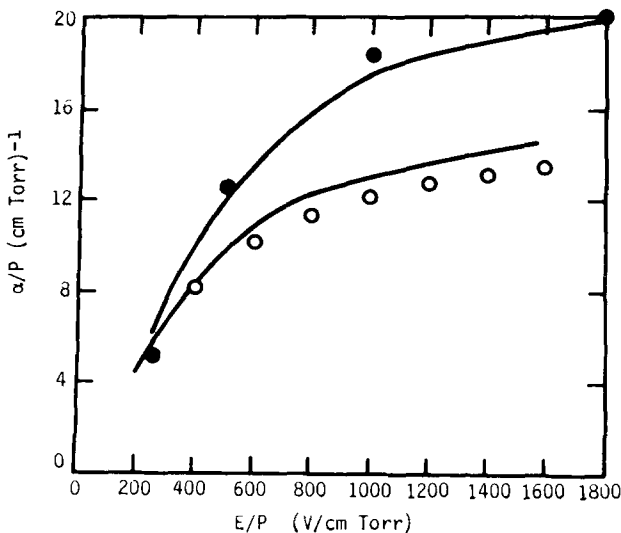


FIG. 1. First Townsend ionization coefficient vs E/p for Hg and A: solid lines—experimental results; ●—calculation for Hg; ○—calculation for A.

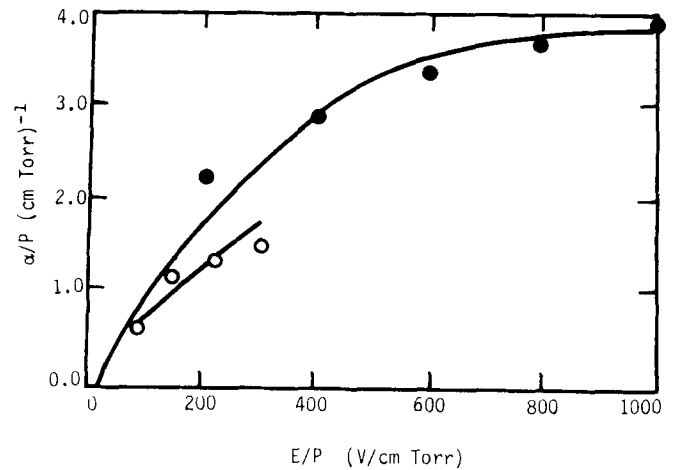


FIG. 2. First Townsend ionization coefficient vs E/p for Ne and He: solid lines—experimental results; ●—calculation for Ne; ○—calculation for He.

The constants A , ξ , B , k , ϵ_0 , characterizing the cross sections, were found as follows. We found A and ϵ_0 so that the slope $a = A/\epsilon_0$ at small energies would agree with the experimental data on ionization cross section and that Eq. (3.1) would give the best approximation of the ionization cross section up to the maximum. The following experimental results were explored: Hg—[Smith¹²; A. von Engel¹³]; A, Ne, and He—[Asundi and Kurepa¹⁴]; H₂ and N₂[Rapp and Englander-Golden¹⁵]. The constants B and k were also derived from the existing experimental measurements of the total inelastic cross sections: Hg, [A. von Engel¹⁶]; [Ne—Thomas¹⁷]; [A—Sakai *et al.*⁵]; He—[Maier-Leibnitz¹⁸]; H₂ [Brunet and Vincent⁷]; N₂ [Cartwright *et al.*¹⁹]. The inelastic energy loss ξ was assumed to be the half of the sum of the energy of the first electronically excited level and the ionization energy. We summarized all the necessary constants in Table I. In Figs. 1–3 we compare the results for $(\alpha/p) = f(E/p)$ obtained by applying our Eq. (3.14) with the existing experimental data. We see in the figures that in all cases the formula agrees with the experiment over a fairly wide range of values of E/p .

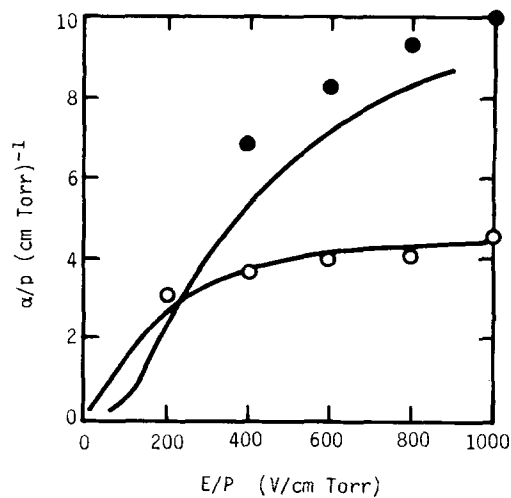


FIG. 3. First Townsend ionization coefficient vs E/p for N₂ and H₂: solid lines—experimental results; ●—calculation for N₂; ○—calculation for H₂.

It should be mentioned here that since the ionization cross sections are measured quite accurately for most of the gases, the constants a and ϵ_0 in Eq. (3.14) can be assumed to be relatively precise. In contrast the precision of our values for k and B is rather poor because of the lack of accurate experiments in the required range of electron energies. Nonetheless, the structure of the Eq. (3.14) for α/p is such that for $E/p > 2B\xi$ it is practically insensitive to the variation of k and B in large limits around the values presented in Table I. Clearly, for these values of E/p Eq. (3.14) is also insensitive to the accuracy of the values of ξ used. The good agreement between our approximate expression for (α/p) with the experiment illustrates the ability of a simple model such as that considered in this paper to correctly describe the α/p dependence on E/p in a very wide range of values of E/p . Figure 3 also shows that the formula is in a good agreement with the experiments in molecular gases. Here, there exists the possibility of rotational and vibrational excitation in addition to electronic excitation. Nevertheless, the energy losses in such collisions are two orders of magnitude smaller than in excitation of electronic levels and we neglected these processes in

the energy balance Eq. (3.5)

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