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Geometric optics in plasmas characterized by non-Hermitian dielectric tensors

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This paper presents a generalization of the theory of geometric optics in plasmas where the local dielectric tensor $\epsilon(\vec{k},\omega;\vec{r},t)$ is not almost Hermitian, as heretofore assumed. It is shown that for general ϵ one can construct the formalism so that the new theory is characterized by the same equations determining the rays, and the equation for the amplitude of the wave along the rays is unmodified in structure. The theory uses the quasidispersion relation $\det\{\epsilon(\vec{k},\omega+i\nu;\vec{r},t)\cdot[\epsilon(\vec{k},\omega-i\nu;\vec{r},t)]^*\}=0$ to find the complex roots $\omega+i\nu$, where when the approximation of short wavelength compared with the scale length underlying geometric optics holds, the real part ω serves to generate the rays via $\dot{\vec{r}}=\omega_{\vec{k}}$, and the imaginary part $i\nu$ enters the transport equation for the amplitude.

I. INTRODUCTION

The geometric-optics approximation is currently widely used in studying electromagnetic phenomena in inhomogeneous plasmas of various types and dimensions.[1] Most of the studies are based on ray tracing although the general theory,[2,3] which uses the properties of the local dielectric tensor $\underline{\epsilon}(\vec{k},\omega;\vec{r},t)$ of the plasma, also allows one to find the amplitude of the electromagnetic field along the rays. The theory developed to date, however, is limited to cases where the tensor $\underline{\epsilon}$ is Hermitian or "almost" Hermitian, namely, it can be written as $\underline{\epsilon}=\underline{\epsilon}_A+\underline{\epsilon}_H$, where $\underline{\epsilon}_H$ is Hermitian and formally $\underline{\epsilon}_A \ll \underline{\epsilon}_H$. This restriction on the type of the dielectric tensor was imposed in order to provide a dispersion relation $D(\vec{k},\omega;\vec{r},t)=\det\underline{\epsilon}_H=0$ with a real solution $\omega=\omega(\vec{k};\vec{r},t)$ for real $\vec{k}$. This solution plays the role of a Hamiltonian in constructing the rays of geometric optics via $\dot{\vec{r}}=\omega_{\vec{k}}$, along which $\vec{k}$ and the amplitude of the electric field are found by integrating ordinary differential equations, e.g., $\dot{\vec{k}}=-\omega_{\vec{r}}$, etc. The anti-Hermitian part $\underline{\epsilon}_A$ then contributes only to the equation for the amplitude of the electromagnetic field and usually leads to a weak energy dissipation along the rays.

There are, however, cases where $\underline{\epsilon}$ has a large anti-Hermitian part. This situation is characteristic of a hot magnetized plasma in regions where the frequency of the wave approaches the local cyclotron frequency. The use of only the Hermitian part of $\underline{\epsilon}$ here is not justified. The determinant of the dielectric tensor is complex and therefore also cannot directly provide a real Hamiltonian for the ray equations. The problem in this case can be solved by reordering the terms in the expression for the determinant of $\underline{\epsilon}$ so that it can be written in the form $D=A(D_0+iD_1)$, where D_0 and D_1 are real and $D_1 \ll D_0$.[4] Then D_0 is used in the ray equations and the small correction D_1 serves to determine the transport of energy along the rays. This way of constructing the real Hamiltonian is nontrivial in the general case, and involves a study of all the terms in the expression for the determinant. Moreover the possibility of such a reordering is not clear *a priori*.

We propose in Sec. II of this paper a more simple and general method of constructing the real Hamiltonian for the ray equations in non-Hermitian plasmas. The method does not require the study of the determinant itself. In Sec. III we will derive the general transport equation for the amplitude of the electric field of the wave along the rays.

II. THE RAY EQUATIONS

Consider a weakly inhomogeneous and time-dependent plasma in which there propagates a small amplitude electromagnetic wave described by the Maxwell equations

$$c\vec{\nabla}\times\vec{B}=4\pi\vec{J}+\frac{\partial\vec{E}}{\partial t},\qquad c\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}. \tag{1}$$

Assume that the current density $\vec{J}$ in (1) can be written in the following form:

$$\vec{J}(\vec{r},t)=\int d^3r''\int_{-\infty}^{t}dt''\,\hat{\underline{\sigma}}[\vec{r}-\vec{r}'',t-t'';\tfrac{1}{2}(\vec{r}+\vec{r}''),\tfrac{1}{2}(t+t'')]\times\vec{E}(\vec{r}'',t''). \tag{2}$$

In analogy to the case of a homogeneous plasma in our inhomogeneous time-dependent case, we seek a solution of (1) in the form

$$\vec{E}(\vec{r},t)=\vec{a}(\vec{r},t)e^{i\Psi(\vec{r},t)},$$
$$\vec{B}(\vec{r},t)=\vec{b}(\vec{r},t)e^{i\Psi(\vec{r},t)}, \tag{3}$$

where if one defines

$$\vec{k}(\vec{r},t)=\nabla\Psi, \quad \omega(\vec{r},t)=-\frac{\partial\Psi}{\partial t}, \tag{4}$$

then the fractional changes in $\hat{\underline{\sigma}}$, $\vec{k}$, ω, $\vec{a}$ and $\vec{b}$, when their arguments $\vec{r}$ and t change by $|\Delta\vec{r}| = 2\pi/|\vec{k}|$ and $\Delta t = 2\pi/\omega$, respectively, are small and characterized by a small dimensionless parameter δ. We assume here that Ψ is real, which in turn leads to the real values of $\vec{k}$ and ω.

On using (3) and defining $\vec{r}'=\vec{r}-\vec{r}''$ and $t'=t-t''$, one can rewrite (2) in the form

$$\vec{J}(\vec{r},t)=\int d^3r'\int_0^t dt\hat{\underline{\sigma}}(\vec{r}',t';\vec{r}-\tfrac{1}{2}\vec{r}',t-\tfrac{1}{2}t')$$
$$\times\vec{a}(\vec{r}-\vec{r}',t-t')$$
$$\times\exp[i\Psi(\vec{r}-\vec{r}',t-t')]. \tag{5}$$

Then assuming that $\hat{\underline{\sigma}}$ is large only for small values of $\vec{r}'$ and t', expanding $\hat{\underline{\sigma}}$, $\vec{a}$, and Ψ in (5) in powers of $\vec{r}'$ and t', and leaving only the zero and the first-order terms in the expansion, one gets[2]

$$\vec{J}(\vec{r},t)=\underline{\sigma}\cdot\vec{a}e^{i\Psi}+K(\vec{a})e^{i\Psi}, \tag{6}$$

where

$$K(\vec{a})=-i\tfrac{1}{2}[\vec{\nabla}\cdot(\nabla_{\vec{k}}\underline{\sigma})]\cdot\vec{a}-i[(\nabla\vec{a})^T\cdot\vec{\nabla}_{\vec{k}}]\underline{\sigma}^T$$
$$+i\tfrac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\underline{\sigma}}{\partial\omega}\right)\right]\cdot\vec{a}+i\frac{\partial\underline{\sigma}}{\partial\omega}\frac{\partial\vec{a}}{\partial t} \tag{7}$$

and

$$\underline{\sigma}(\vec{k},\omega;\vec{r},t)=\int d^3r'\int_0^\infty dt'\hat{\underline{\sigma}}(\vec{r}',t';\vec{r},t)$$
$$\times\exp[i(\omega t'-\vec{k}\cdot\vec{r}')] \tag{8}$$

is the conductivity of the homogeneous plasma which has everywhere the same parameters as those characterizing our inhomogeneous plasma at the point $\vec{r}$ and time t. In contrast to Ref. 2 we are not assuming here that the conductivity tensor can be divided into a large anti-Hermitian and small Hermitian parts and allow $\underline{\sigma}$ to be arbitrarily non-Hermitian.

We use expressions (3) for the fields in the Maxwell equations (1) to get

$$i(c\vec{k}\times\vec{b}+\omega\vec{a}+4\pi i\underline{\sigma}\cdot\vec{a})=4\pi K(\vec{a})-c\vec{\nabla}\times\vec{b}+\frac{\partial\vec{a}}{\partial t}, \tag{9}$$

$$i(c\vec{k}\times\vec{a}-\omega\vec{b})=-c\vec{\nabla}\times\vec{a}-\frac{\partial\vec{b}}{\partial t}. \tag{10}$$

One can find from (10):

$$\vec{b}=\frac{c}{\omega}\vec{k}\times\vec{a}-\frac{ic}{\omega}\vec{\nabla}\times\vec{a}-\frac{i}{\omega}\frac{\partial\vec{b}}{\partial t}, \tag{11}$$

which on insertion into (9) results in

$$\omega\underline{\epsilon}\cdot\vec{a}=-4\pi iK(\vec{a})+\frac{ic^2}{\omega}\vec{k}\times(\vec{\nabla}\times\vec{a})-i\frac{\partial\vec{a}}{\partial t}$$
$$+ic\vec{\nabla}\times\vec{b}+\frac{ic}{\omega}\vec{k}\times\frac{\partial\vec{b}}{\partial t}, \tag{12}$$

where $\underline{\epsilon}$ is the local dielectric tensor of the plasma

$$\underline{\epsilon}=I\left(1-\frac{c^2k^2}{\omega^2}\right)+\frac{c^2}{\omega^2}\vec{k}\vec{k}+\frac{4\pi i}{\omega}\underline{\sigma}. \tag{13}$$

The form of (12), where on the left there are only small quantities of the order of δ, suggests that one writes

$$\vec{a}=\vec{a}_0+\vec{a}_1+\vec{a}_2+\cdots,$$
$$\vec{b}=\vec{b}_0+\vec{b}_1+\vec{b}_2+\cdots, \tag{14}$$

where the terms in the expansion are ordered in descending powers of δ. Then on equating terms of equal order in Eq. (12) one gets the following zero- and first-order equations:

$$\omega\underline{\epsilon}\cdot\vec{a}_0=0, \tag{15}$$

$$i\omega\underline{\epsilon}\cdot\vec{a}_1=4\pi K(\vec{a}_0)-\frac{c^2}{\omega}\vec{k}\times(\vec{\nabla}\times\vec{a}_0)+\frac{\partial\vec{a}_0}{\partial t}$$
$$-c\vec{\nabla}\times\vec{b}_0-\frac{c}{\omega}k\times\frac{\partial\vec{b}_0}{\partial t}. \tag{16}$$

By substituting the zero-order result

$$\vec{b}_0=c\vec{k}\times\vec{a}_0/\omega \tag{17}$$

obtained from Eq. (11), one can rewrite (16) in the form

$$i\omega\underline{\epsilon}\cdot\vec{a}_1=\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\frac{\partial\vec{a}_0}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\right)\right]\cdot\vec{a}_0$$
$$-[(\nabla\vec{a}_0)^T\cdot\nabla_{\vec{k}}]\cdot\omega\underline{\epsilon}^T-\tfrac{1}{2}[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega\epsilon)]\cdot\vec{a}_0. \tag{18}$$

Equation (15) has a nontrivial solution only if

$$D=\det(\underline{\epsilon})=0. \tag{19}$$

If, however, the dielectric tensor is non-Hermitian, Eq. (19) in general cannot have a real solution $\omega=\omega(\vec{k};\vec{r},t)$ for real $\vec{k}$, since D is then a complex function and both its real and imaginary parts must simultaneously vanish, which gives two not necessarily consistent dispersion relations for ω and $\vec{k}$. Thus, in order to be consistent we must modify our zero-order dispersion relation. This can be done in the following way. Let us add the quantity

$$\Delta=i\nu\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{a} \tag{20}$$

to both sides of Eq. (12). We assume that ν in

(20) is real and of order δ. Then Eq. (12) becomes

$$\left[\omega\underline{\epsilon}+i\nu\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\right]\cdot\vec{a}\simeq(\Omega\underline{\tilde{\epsilon}})\cdot\vec{a}$$

$$=i\nu\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{a}-4\pi iK(\vec{a})+\frac{i\vec{c}^2}{\omega}\vec{k}\times(\vec{\nabla}\times\vec{a})-i\frac{\partial\vec{a}}{\partial t}$$

$$+ic\nabla\times\vec{b}+\frac{i\vec{c}}{\omega}\vec{k}\times\frac{\partial\vec{b}}{\partial t}, \tag{21}$$

where we define

$$\Omega=\omega+i\nu \tag{22}$$

and correct to first order in δ

$$\underline{\tilde{\epsilon}}=\epsilon(\vec{k},\Omega;\vec{r},t). \tag{23}$$

Now Eqs. (15) and (18) can be replaced by

$$(\Omega\underline{\tilde{\epsilon}})\cdot\vec{a}_0=0, \tag{24}$$

$$i(\Omega\underline{\tilde{\epsilon}})\cdot\vec{a}_1=-\nu\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{a}_0+\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\frac{\partial\vec{a}_0}{\partial t}+\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\right)\right]\cdot\vec{a}_0$$
$$-[(\nabla\vec{a}_0)^T\cdot\nabla_{\vec{k}}]\cdot\omega\epsilon^T-\tfrac{1}{2}[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\cdot\vec{a}_0. \tag{25}$$

Note that the only difference between Eqs. (15) and (24) is that in the latter we are formally allowing the frequency Ω to have a small imaginary part $i\nu$. This assumption introduces a new term $-\nu(\partial(\omega\underline{\epsilon})/\partial\omega)\cdot\vec{a}_0$ in the first-order equation (25) as compared to (18).

The modified dispersion relation is therefore

$$\tilde{D}=\det(\underline{\tilde{\epsilon}})=D(\vec{k},\Omega;\vec{r},t). \tag{26}$$

On separating real and imaginary parts in (26), one gets a set of two equations for two quantities ω and ν, and the aforementioned inconsistency is removed. One can also use the smallness of ν and derive a real dispersion relation for ω and $\vec{k}$, independent of ν. This can be conveniently done in the following way. Let us rewrite (26) in the form

$$D(\Omega)=D_0(\Omega)+iD_1(\Omega), \tag{27}$$

where D_0 and D_1 are the real and imaginary parts of the determinant $D(\omega)$. For simplicity the arguments $\vec{k}$, $\vec{r}$, and t will not be indicated explicitly. On multiplying (27) by

$$D^H(\Omega)=D_0(\Omega)-iD_1(\Omega), \tag{28}$$

one gets

$$\Phi(\Omega)=DD^H=D_0^2(\Omega)+D_1^2(\Omega)=0. \tag{29}$$

Correct to the second order in ν, this equation can be written

$$\Phi(\Omega)=\Phi(\omega)+i\nu\Phi_\omega(\omega)-\tfrac{1}{2}\nu^2\Phi_{\omega\omega}(\omega)=0. \tag{30}$$

On solving (30) for ν we have

$$\nu=-\frac{i\Phi_\omega(\omega)}{\Phi_{\omega\omega}(\omega)}\pm\left[-\left(\frac{\Phi_\omega(\omega)}{\Phi_{\omega\omega}(\omega)}\right)^2+\frac{2\Phi(\omega)}{\Phi_{\omega\omega}(\omega)}\right]^{1/2}. \tag{31}$$

Therefore, since $\Phi(\omega)\geq 0$, ν will be real if and only if

$$\Phi_\omega(\omega)=0 \tag{32}$$

and

$$\Phi_{\omega\omega}(\omega)>0. \tag{33}$$

In this case

$$\nu=\pm\left(\frac{2\Phi(\omega)}{\Phi_{\omega\omega}(\omega)}\right)^{1/2}. \tag{34}$$

The plus and minus signs in this solution correspond to the zeros $\Omega=\omega+i\nu$ of $D^H(\Omega)$ and $D(\Omega)$, respectively. Note that the real parts of the zeros are identical. Note also that Eqs. (32) and (33) define a minimum of the function $\Phi(x)$ at the point $x=\omega$.

In addition to an expression (34) for the correction ν, which is necessary if one considers the first-order equation (25) for the amplitude, we now have a real dispersion relation (32) for real objects ω and $\vec{k}$. Assuming that Eq. (32) has a solution $\omega=\omega(\vec{k};\vec{r},t)$, we introduce the group velocity $\vec{V}_g=\omega_{\vec{k}}$ and define trajectories via

$$\frac{d\vec{r}}{dt}=\vec{V}_g=\omega_{\vec{k}}. \tag{35}$$

Then, since on cross differentiation of Eq. (4) one has

$$\nabla\omega+\frac{\partial\vec{k}}{\partial t}=\omega_{\vec{r}}+(\nabla\vec{k})\cdot\omega_{\vec{k}}+\frac{\partial\vec{k}}{\partial t}$$
$$=\omega_{\vec{r}}+\omega_{\vec{k}}\cdot(\nabla\vec{k})+\frac{\partial\vec{k}}{\partial t}=0, \tag{36}$$

one gets

$$\frac{d\vec{k}}{dt}=-\omega_{\vec{r}}. \tag{37}$$

Equations (35) and (37) together with

$$\frac{d\omega}{dt}=\omega_t+\omega_{\vec{k}}\cdot\frac{d\vec{k}}{dt}+\omega_{\vec{r}}\cdot\frac{d\vec{r}}{dt}=\omega_t, \tag{38}$$

form a set of first-order ordinary differential equations, which are commonly known as the ray equations. On implicit differentation of the dispersion relation (32) one can also write the ray equations in the form

$$\frac{d\vec{r}}{dt}=-\frac{\Phi_{\omega\vec{k}}}{\Phi_{\omega\omega}},\quad \frac{d\vec{k}}{dt}=\frac{\Phi_{\omega\vec{r}}}{\Phi_{\omega\omega}},\quad \frac{d\omega}{dt}=-\frac{\Phi_{\omega t}}{\Phi_{\omega\omega}}, \tag{39}$$

which does not require explicit knowledge of the solution $\omega=\omega(\vec{k},\vec{r},t)$ and is therefore more convenient for computations. By solving Eqs. (39) with the appropriate initial conditions, one can simultaneously determine ν from Eq. (34). The smallness of ν in comparison to ω will then give an estimate of whether the geometric-optics approximation can be applied in a given case.

III. THE TRANSPORT EQUATION

We proceed now to the derivation of the transport equation for the amplitude $\vec{a}_0$ of the electric field along the rays defined by Eq. (39). This can be done by using the first-order equation (25). To this end we use the singular value decomposition of $\tilde{\underline{\epsilon}}$.[5] As is well known one can always find two sets of orthonormal, in general complex vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3(\vec{u}_i^H\cdot\vec{u}_j=\delta_{ij})$ and $\vec{v}_1,\vec{v}_2,\vec{v}_3(\vec{v}_i^H\cdot\vec{v}_j=\delta_{ij})$, in terms of which $\tilde{\underline{\epsilon}}$ can be expressed as

$$\tilde{\underline{\epsilon}}=\epsilon_1\vec{v}_1\vec{u}_1^H+\epsilon_2\vec{v}_2\vec{u}_2^H+\epsilon_3\vec{v}_3\vec{u}_3^H\ , \tag{40}$$

where the $\epsilon_i(i=1-3)$ are real singular values of $\tilde{\underline{\epsilon}}$ and the superscript H denotes transpose complex conjugate. Note that the vectors $\vec{u}_i$ and $\vec{v}_i$ satisfy the equations

$$\begin{aligned}&\tilde{\underline{\epsilon}}\cdot\vec{u}_i=\epsilon_i\vec{v}_i\ ,\\&\tilde{\underline{\epsilon}}^H\cdot\vec{v}_i=\epsilon_i\vec{u}_i\ ,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&(\tilde{\underline{\epsilon}}^H\cdot\tilde{\underline{\epsilon}})\cdot\vec{u}_i=\epsilon_i^2\vec{u}_i\ ,\\&(\tilde{\underline{\epsilon}}\cdot\tilde{\underline{\epsilon}}^H)\cdot\vec{v}_i=\epsilon_i^2\vec{v}_i\ ,\end{aligned} \tag{42}$$

and therefore, $\vec{u}_i$ and $\vec{v}_i$ are the orthonormal eigenvectors of the Hermitian matrices $\tilde{\underline{\epsilon}}^H\cdot\tilde{\underline{\epsilon}}$ and $\tilde{\underline{\epsilon}}\cdot\tilde{\underline{\epsilon}}^H$, respectively. The quantities $\epsilon_i^2(i=1-3)$ are the three eigenvalues of these matrices.

One can express the amplitude $\vec{a}_0$ of the electric field of the wave in terms of the base vectors $\vec{u}_i$:

$$\vec{a}_0=\alpha_1\vec{u}_1+\alpha_2\vec{u}_2+\alpha_3\vec{u}_3\ . \tag{43}$$

Then on using (40), Eq. (24) becomes

$$\alpha_1\epsilon_1\vec{v}_1+\alpha_2\epsilon_2\vec{v}_2+\alpha_3\epsilon_3\vec{v}_3=0\ . \tag{44}$$

According to (26),

$$\det(\tilde{\underline{\epsilon}}^H\cdot\tilde{\underline{\epsilon}})=\epsilon_1^2\epsilon_2^2\epsilon_3^2=0\ , \tag{45}$$

and therefore, at least one of ϵ_i vanishes. In the rest of the paper we will assume that $\epsilon_1=0$ and $\epsilon_2,\epsilon_3\neq 0$. The theory for the degenerate case, when more than one eigenvalue of $\tilde{\underline{\epsilon}}\cdot\tilde{\underline{\epsilon}}^H$ vanish simultaneously, can be developed in a fashion similar to that used in Refs. 2 and 3.

It follows from (44) that $\alpha_i\epsilon_i=0$, since the vectors $\vec{v}_i$ are orthonormal and therefore linearly independent. Thus, $\alpha_2=\alpha_3=0$ or $\vec{a}_0=\alpha_1\vec{u}_1$. Then, on multiplying (25) by $\vec{v}_1^H$ from the left and using the fact that $\vec{v}_1^H\cdot\tilde{\underline{\epsilon}}=0$, one gets

$$\vec{v}_1^H\cdot\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{u}_1\frac{\partial\alpha_1}{\partial t}-\vec{v}_1^H\cdot[\nabla\alpha_1\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\cdot\vec{u}_1=\beta\alpha_1\ , \tag{46}$$

where

$$\beta=\vec{v}_1^H\cdot\left\{\nu\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\vec{u}_1-\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\frac{\partial\vec{u}_1}{\partial t}-\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\right)\right]\cdot\vec{u}_1+[(\nabla\vec{u}_1)^T\cdot\nabla_{\vec{k}}]\cdot\omega\underline{\epsilon}^T+\tfrac{1}{2}[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\cdot\vec{u}_1\right\}\ . \tag{47}$$

On varying $\vec{v}_1^H\cdot(\Omega\tilde{\underline{\epsilon}})\cdot\vec{u}_1=0$ with respect to $\vec{k}$ holding $\vec{r}$ and t fixed, one obtains

$$\delta\omega\vec{v}_1^H\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\vec{u}_1+\vec{v}_1^H\cdot[\delta\vec{k}\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\cdot\vec{u}_1\simeq 0\ , \tag{48}$$

and therefore (46) can be written as

$$\vec{v}_1^H\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\vec{u}_1\frac{d\alpha_1}{dt}=\beta\alpha_1\ , \tag{49}$$

where the time derivative is taken along the ray $(d/dt=\partial/\partial t+\omega_{\vec{k}}\cdot\vec{\nabla})$. By solving Eq. (49) in parallel with the ray equations, one finds the amplitude of the electric field of the wave along the ray and gets a full geometric-optics solution of the problem.

Equation (49) for the amplitude has the same structure as the one previously derived for the Hermitian case.[2] The only difference is that when the dielectric tensor is Hermitian the base vectors $\vec{u}_i$ and $\vec{v}_i$ are identical, while in the non-Hermitian case the two sets of the base vectors are, in general, different. This lack of symmetry in the non-Hermitian case leads to a new effect related to possible dissipation of electromagnetic energy in the plasma. We illustrate this effect by constructing an analog to Pointing's theorem. To this end let us formally consider a dual plasma with the conductivity tensor $\underline{\sigma}^H$. We denote the geometric-optics solution for the electric field of a wave in the dual plasma by

$$E'(\vec{r},t)=\vec{a}'(\vec{r},t)e^{-i\Psi'(\vec{r},t)} \tag{50}$$

and define

$$\vec{k}'=\nabla\Psi',\ \omega'=-\frac{\partial\Psi'}{\partial t}\ . \tag{51}$$

Then, similar to Eqs. (24) and (25), one can write the equations for the zero- and the first-order parts of the amplitude:

$$(\Omega'\underline{\tilde{\epsilon}}'^{H})\cdot\vec{a}_0'=0\,, \tag{52}$$

$$-i(\Omega'\underline{\tilde{\epsilon}}'^{H})\cdot\vec{a}_1'=\nu'\frac{\partial\omega'\underline{\epsilon}'^{H}}{\partial\omega'}\cdot\vec{a}_0+\frac{\partial\omega'\underline{\epsilon}'^{H}}{\partial\omega'}\cdot\frac{\partial\vec{a}_0'}{\partial t}+\frac{1}{2}\left(\frac{\partial}{\partial t}\frac{\partial\omega'\underline{\epsilon}'^{H}}{\partial\omega'}\right)\cdot\vec{a}_0'-[(\nabla\vec{a}_0')^T\cdot\nabla_{\vec{k}'}]\cdot\omega'\underline{\epsilon}'^{*}-\tfrac{1}{2}[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega'\underline{\epsilon}'^{H})]\cdot\vec{a}_0', \tag{53}$$

where

$$\Omega'=\omega'+i\nu'\,, \tag{54}$$

$\underline{\epsilon}'^{H}=\underline{\epsilon}^{H}(\vec{k}',\omega';\vec{r},t)$ and $\underline{\tilde{\epsilon}}'^{H}=[\underline{\epsilon}(\vec{k}',\Omega'^{*};\vec{r},t)]^{H}$.

Equation (54) implies

$$D^{H}(\vec{k}',\Omega';\vec{r},t)=0\,, \tag{55}$$

where D^H is the quantity already defined in (28). Therefore, the dispersion relation for the real quantities ω' and $\vec{k}'$ is exactly the same [Eq. (32)] as for ω and $\vec{k}$ in our real plasma. Thus the rays in the plasmas are identical if one starts the ray tracing with the same initial conditions in both cases. Then one has $\vec{k}'=\vec{k}$, $\omega'=\omega$, and $\Psi'=\Psi$ along the ray. In contrast $\nu'=-\nu$, as follows from (34). Therefore, Eq. (53) can be written in the form

$$i\vec{a}_1'^{H}\cdot(\Omega\underline{\tilde{\epsilon}})=-\nu\vec{a}_0'^{H}\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\frac{\partial\vec{a}_0'^{H}}{\partial t}+\tfrac{1}{2}\vec{a}_0'^{H}\cdot\left[\frac{\partial}{\partial t}\left(\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\right)\right]-[(\nabla\vec{a}_0'^{H})^T\cdot\nabla_{\vec{k}}]\cdot\omega\underline{\epsilon}^{T}-\tfrac{1}{2}\vec{a}_0'^{H}\cdot[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\,. \tag{56}$$

Now it can be easily shown that if the amplitude $\vec{a}_0'$ is expressed in terms of the base vectors $\vec{v}_i:\vec{a}_0'=\alpha_1'\vec{v}_1+\alpha_2'\vec{v}_2+\alpha_3'\vec{v}_3$, then $\alpha_2'=\alpha_3'=0$ and therefore $\vec{a}_0'=\alpha_1'\vec{v}_1$. Then, on multiplying (56) by $\vec{u}_1$ from the right, one gets an analog to Eq. (49):

$$\vec{v}_1^{H}\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\vec{u}_1\frac{d\alpha_1'^{*}}{dt}=\beta'\alpha_1'^{*}\,, \tag{57}$$

where

$$\beta'=\left\{\nu\vec{v}_1^{H}\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}-\frac{\partial\vec{v}_1^{H}}{\partial t}\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}-\tfrac{1}{2}\vec{v}_1^{H}\cdot\left[\frac{\partial}{\partial t}\left(\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\right)\right]+[(\nabla\vec{v}_1^{H})^T\cdot\nabla_{\vec{k}}]\cdot\omega\underline{\epsilon}^{T}+\tfrac{1}{2}\vec{v}_1^{H}\cdot[\vec{\nabla}\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\right\}\cdot\vec{u}_1\,. \tag{58}$$

Finally, on multiplying Eq. (49) by $\alpha_1'^{*}$, adding it to Eq. (57) multiplied by α_1, and using (48), one has

$$\frac{\partial G}{\partial t}+\vec{\nabla}\cdot(\omega_{\vec{k}}G)=2\nu G\,, \tag{59}$$

where

$$G=\alpha_1'^{*}\alpha_1\vec{v}_1^{H}\cdot\frac{\partial\omega\underline{\epsilon}}{\partial\omega}\cdot\vec{u}_1\simeq\alpha_1'^{*}\alpha_1\frac{\partial\omega\epsilon_1}{\partial\omega}\,. \tag{60}$$

In the stationary case ($\partial G/\partial t=0$), one can integrate (59) in the volume element $\delta V=\delta S\,|\omega_{\vec{k}}|dt$ of the infinitsimal flux tube of cross section δS containing the ray. Then on using Gauss' theorem one gets

$$\frac{dI}{dt}=2\nu I\,, \tag{61}$$

where $I=\delta S\,|\omega_{\vec{k}}|G$. Equations (59) and (61) represent the analog of the energy- and flux-conservation equations in a plasma with Hermitian dielectric tensor, where $\alpha'=\alpha$ and the object $G/8\pi$ is identified as the sum of the electromagnetic and reactive kinetic energy densities averaged over a period $2\pi/\omega$.[2] In the non-Hermitian case, G is in general complex and is not related to the energy density in a simple way.

One interesting feature of a system in which $\underline{\epsilon}\neq\underline{\epsilon}^{H}$ is that the absorption of the electromagnetic energy may not be due only to the presence of the imaginary correction $i\nu$ to the frequency. To illustrate this, assume that we have a stationary case and start a ray in a Hermitian region of the plasma (in a vacuum, for example). Then the energy density flux along the ray at the starting point will be equal to $F_0=I_0/8\pi$. Suppose now that the ray passes through a non-Hermitian region and then arrives at a point where the plasma dielectric tensor is again Hermitian. One can also integrate (61) and get at the final point $I=I_0\exp(2\int_0^t\nu dt')$, where the time integration is along the ray. The energy density flux at the final point, however, will be

$$F=(\alpha^{*}/\alpha'^{*})(I/8\pi)=(\alpha^{*}/\alpha'^{*})(I_0/8\pi)\exp\left(2\int_0^t\nu dt'\right).$$

Thus if the value of α' at the final point differs from α, one gets a change in the flux due to the factor α^{*}/α'^{*}, even if $\nu=0$.

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[1]For a collection of the most recent papers on the application of the geometric-optics approximation in a variety of problems see IEEE Tr. Plasma Science PS-8, (1980).

[2]I. B. Bernstein, Phys. Fluids 18, 320 (1975).
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[4]D. B. Batchelor, Report No. TM-630, Oak Ridge National Laboratory, 1978 (unpublished).
[5]B. Noble, *Applied Linear Algebra* (Prentice-Hall, Englewood Cliffs, New Jersey, 1969), pp. 335–338.