Free-Electron Laser with a Strong Axial Magnetic Field

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A small-signal theory is given for gain in a free-electron laser comprising a cold relativistic electron beam in a helical periodic transverse, and a strong uniform axial, magnetic field. Exact finite-amplitude, steady-state helical orbits are included. If perturbed, these orbits oscillate about equilibrium, so that substantial gain enhancement can occur if the electromagnetic perturbations resonate with these oscillations. This gain enhancement need not be at the cost of frequency upshift.

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Intensive activity is underway to exploit the gain properties of a relativistic electron beam undulating in a periodic transverse magnetic field. Such free-electron laser (FEL) configurations have provided oscillation at 3.4 (Ref. 1) and 400 \mu m, and amplification at 10.6 \mu m. Theory has advanced since, and elaborate schemes have been proposed for obtaining high FEL efficiency. A factor which limits the practical application of this interaction at wavelengths shorter than perhaps a few microns is the rapid decrease in small-signal gain \( G_0 \) as the electron energy increases. This is shown explicitly in the well-known result for \( G_0 \) in the single-particle limit (i.e., when collective effects are negligible)

\[
G_0 = (\omega_p \xi / h c)^2 (k_e L / 2\gamma) F'(\theta).
\]

Here \( \omega_p \) and \( \gamma \) are the beam plasma frequency \( Ne^2 / m e \) and normalized energy \( W / mc^2 \), \( k_e \) and \( \xi \) are the helical transverse magnetic field wave number \( 2\pi / L \) and normalized strength \( e B_z / m c^2 \), \( L \) is the interaction length, and \( F'(\theta) = (d/d\theta) (\sin \theta / \theta)^2 \) is the line-shape factor, with \( \theta = (\gamma_{30}^2 - \omega (1 - v_{30}^2 / c^2)) (L / 2c) \), where \( v_{30} \) is the unperturbed electron axial velocity. The peak gain occurs at \( \theta = 1.3 \), where \( F'(\theta) = 0.54 \). For example, with \( \gamma \)
= 10, \( I = 1.05 \) cm, \( \omega_p = 5 \times 10^7 \) sec\(^{-1}\), \( \xi = 1 \) \((B_\perp = 10.2\) kG\), and \( L = 130 \) cm, the peak gain is \( G_{\text{peak}} = 0.00247 \) at a wavelength of 105 \( \mu \)m. For \( \gamma = 100 \), \( I = 10.5 \) cm, \( \omega_p = 2 \times 10^8 \) sec\(^{-1}\), \( \xi = 1 \) \((B_\perp = 1.02\) kG\), and \( L = 260 \) cm, the peak gain is \( G_{\text{peak}} = 0.00316 \) at a wavelength of 105 \( \mu \)m. These gain values may be large enough to sustain oscillations if highly reflecting mirrors are judiciously added but the strong helical fields required (particularly the 10.2-kG case) may be beyond the capability of present superconducting coil technology.

A suggestion has appeared for enhancing the small-signal gain above values given by Eq. (1) (or for achieving comparable gains with smaller \( B_\perp \)) by employing a strong axial magnetic field so as to exploit resonance between the cyclotron frequency and the undulatory frequency.\(^6\) The present Letter presents a single-particle derivation for the small-signal gain of a FEL in a uniform axial magnetic field \( B_\parallel \). We shall demonstrate that careful adjustment of the system parameters will allow enhancement of the FEL small-signal gain by an order of magnitude or more (for the above examples) without increasing the undulatory velocity. This result goes beyond that predicted by Sprangle and Granatstein\(^8\) who have suggested that the only effect of the axial magnetic field would be to add a multiplicative factor \((1 - \xi \Omega/\Gamma)^k\) to Eq. (1), due to the aforementioned resonance giving an enhanced undulatory velocity \( v_\perp \), where \( \Omega = \sqrt{B_\parallel/\mu} \). This result is in fact predicted by our analysis as a limiting case. Of course, any mechanism which increases the undulatory velocity \( v_\perp \) would increase the gain, but this would also reduce the relativistic frequency upshift, since

\[
\omega = \sqrt{\gamma + \left( 1 - \frac{\Omega}{\sqrt{1 + \gamma^2}} \right)^2 - \frac{\Omega^2}{\gamma^2}}.
\]

If, for example, \( \gamma v_\perp = \gamma = 1 \) without the axial magnetic field, then a given gain \( \eta \) achieved through this resonance alone would result in a reduction in frequency upshift by a factor \((1 + \eta)/2\). The process we describe in this Letter will be shown to permit significant gain enhancement without undue sacrifice in frequency upshift. The gain enhancement originates when the electromagnetic perturbations resonate with the natural frequency of oscillation of electrons on finite amplitude equilibrium helical orbits. Prior workers have not considered this effect.

A full derivation of our result will be presented elsewhere.\(^9\) Exact unperturbed relativistic orbits are considered in the customary FEL model magnetic field

\[
\vec{B}(z) = B_{\parallel} \hat{\kappa}_x + B_{\perp} \left( \hat{\kappa}_x \cos \kappa_z + \hat{\kappa}_y \sin \kappa_z \right).
\]

These orbits, which have been the subject of recent study,\(^10\) can possess more than one steady state, depending upon \( \gamma \), \( B_{\parallel} \), \( B_{\perp} \), and \( k_0 \). These steady states are characterized by the normalized velocity components (i.e., \( u_{i} = \gamma v_{i}/c \))

\[
u_{\parallel 0} = 0, \quad u_{30} = \frac{k_0 \omega_{30}}{(k_0 \gamma - \Omega/c)},
\]

\[
u_{30} = \left(1 - \nu_{30}^2 - \gamma^2 \right)^{1/2},
\]

where the basis vectors \( \hat{\kappa}_x (-\hat{\kappa}_y \sin \kappa_z + \hat{\kappa}_x \cos \kappa_z) \), \( \hat{\kappa}_y \sin \kappa_z - \hat{\kappa}_x \cos \kappa_z \), and \( \hat{\kappa}_z \) have been introduced to track the symmetry of the transverse magnetic field. Figure 1 shows \( u_{30} \) vs \( \Omega/c \) for \( k_0 = 6.8 \) cm\(^{-1}\), \( \xi = 1.0 \), and \( \gamma = 10 \). For \( \Omega > \Omega_{cr} \equiv k_0 \gamma \left( 1 - \frac{1}{2} - v_\perp^2 \right)^{1/2} \), it is seen that only one branch exists (branch C). But for \( \Omega < \Omega_{cr} \) two additional branches (A and B) are allowed: Branch B has been shown to be unstable, in that the orbits exhibit nonhelical, highly anharmonic motions, while branches A and C have orderly helical orbits. Stability is insured if \( \mu^2 \equiv a^2 - bd > 0 \), where \( a = k_0 \gamma \omega_{30}/\gamma m \), \( b = \Omega_{30}/\gamma \), and \( d = k_0 \xi \gamma /\gamma \). The quantity \( \mu \) is the natural resonance frequency in response to small perturbations of the orbit: We shall show that strong resonance response of the electrons to electromagnetic perturbation can lead to enhanced FEL gain for small \( \mu \), i.e., for \( \Omega \) close to \( \Omega_{cr} \).

The derivation of FEL gain proceeds by solving the single-particle equations of motion, subject to weak electromagnetic perturbing fields \( E = \hat{\kappa}_x E_0 \cos(kz - \omega t) \) and \( \vec{B} = \hat{\kappa}_y (k_0/c) E_0 \cos(kz - \omega t) \).
about the equilibrium orbits on either branch A or C as discussed above. These equations are
\[ \dot{u}_1 = (k_c u_{30} - \Omega / \gamma) u_2 - (k_c \xi / \gamma) u_3 - (\xi / \gamma) u_1 + (\varepsilon E_1 / mc \gamma)(k_c u_{30} / \omega - 1), \]
\[ \dot{u}_2 = (k_c u_{30} - \Omega / \gamma) u_1 - (\xi / \gamma) u_2 + (\varepsilon E_2 / mc \gamma)(k_c u_{30} / \omega - 1), \]
\[ \dot{u}_3 = (k_c \xi / \gamma) u_1 + (k_c / \omega - u_3)(\xi / \gamma), \]  
where \( \gamma = (\varepsilon / mc)(u_{10} + u_{20} / 2) \) and
\[ 2(E_2 + E_1) = -E_0 \exp\left[ i(k_0 + k)u_{30}ct - \omega t + \alpha \right]. \]

with \( \alpha \) the random initial electron phase. When time variations and electromagnetic fields are absent, Eqs. (4)-(6) lead to the exact steady states given by Eq. (3). To linearize Eqs. (4)-(6), we introduce the velocity perturbations \( \dot{w}_1 = u_1 - u_{10}, \dot{w}_2 = u_{10}, \) and retain only the lowest-order quantities. This results in
\[ \dot{w}_1 + \mu^2 w_1 = A E_0 \cos(\beta t + \alpha), \]
where
\[ A = (\alpha + \beta)(1 - u_{30}) + b u_{20}, \beta = c(k + k_0)u_{30} - \omega, \omega = k_c, \dot{w}_1(0) = (\varepsilon E_0 / 2mc)(1 - u_{30}) \sin \alpha, \]
and \( w_1(0) = 0 \). The other components follow from
\[ \dot{w}_2 = -aw_1 + (\varepsilon E_0 / 2mc \gamma)(1 - u_{30} - u_{20} / 2) \cos(\beta t + \alpha), \]
\[ w_2(0) = 0, \]
and
\[ \dot{w}_3 = dw_1 + (\varepsilon E_0 / 2mc \gamma)u_{20}(1 - u_{30}) \cos(\beta t + \alpha), \]
\[ w_3(0) = 0. \]

Equation (7) for \( w_1 \) exhibits the aforementioned natural resonance at frequency \( \mu \),(9,594),(988,992)

The energy gain for an electron is calculated from \( mc \dot{\gamma} / dt = -w_1 E_{10} - w_2 E_{20} - u_{20} E_{21}. \) The first-order variation in electric field \( E_{21} \) originates from small phase variations as \( u_3 \) changes. Thus this becomes
\[ \frac{mc}{e} \dot{\gamma} / dt = -w_1 E_{10} - w_2 E_{20} - \frac{1}{2}E_0(k + k_0)u_{20} \sin(\beta t + \alpha) \int_0^T dt' w_3(t'). \]

The third term in Eq. (10) is much larger than the other two on account of the factor \( k + k_0 \). The dominant single-particle energy transfer in the FEL (even with an axial magnetic field) is seen to be by work \( e c u_{20} E_2 \) done along the transverse undulatory motion, enhanced by the strong variation in \( E_2 \) as its phase varies through \( u_3 \). The energy variation [Eq. (10)] is averaged over random phase \( \alpha \) to give \( \langle \dot{\gamma} / dt \rangle \), which in turn leads to the gain through \( G = 2(\varepsilon E_0)^2 \) \( \int_0^T dt \langle \dot{\gamma} / dt \rangle \), where \( N \) is the beam electron density and \( T = L / c \) is the total interaction time for the electrons in a system of length \( L \).

The final result is
\[ G = \frac{\omega^2 k_o c u_{20}}{16 \gamma} \int_0^T \left[ 1 + \frac{a + \beta + u_{20} \beta}{1 - u_{30}} \right] \left[ F'(\theta) - F(\theta + \varphi) - F(\theta - \varphi) \right] \frac{\theta T}{2 \varphi} - a \frac{\mu^2}{\gamma^2} \left[ P'(\theta) - P(\theta + \varphi) - P(\theta - \varphi) \right] \frac{\theta T}{2 \varphi}, \]
where \( \theta = \beta T / 2, \varphi = \mu T / 2, \theta = (\sin \theta / \theta)^2, \) and \( P(\theta) = x^2 F'(\theta) / 2 \). We have approximated \( (k + k_0)(1 - u_{30}) \approx k_0 \). We shall examine Eq. (11) in several limits.

For \( \mu \gg \beta, \) only the terms involving \( F'(\theta) \) and \( P'(\theta) \) in Eq. (11) are significant, and on branch A the latter of these is smaller than the former by at least a factor \( 2 \varphi \). Thus to a good approxi-
$\beta$, gain enhancement can be achieved as claimed by the prior workers, due to resonant enhancement of $u_{20}$ but not without sacrificing frequency upshift, as discussed above.

However a more attractive possibility exists when $\mu$ is small, and approaches $\beta$. Here one can approximate $Z = \mu^2 \beta d (1 - u_{20})^{-1} \gg 1$; this results from resonance between the electromagnetic perturbation which gives oscillatory motion to the electron at a frequency $\beta$, close to its natural oscillation frequency $\mu$. Gain enhancement due to large $Z$ is seen to be possible without simultaneously increasing $u_{20}$, so that the desirable frequency upshift property of the FEL need not be sacrificed.

We define a gain enhancement factor $\eta = G / G_0$ to compare two free-electron lasers, identical except that one has a strong axial magnetic field, while the second does not. In the first laser, the transverse magnetic field $B_z$ is reduced so that $u_{20}$ is the same for both lasers. (This assures that both enjoy the same frequency upshift.) Then

$$\eta = \left| 1 - \left[ F(\theta + \phi) - F(\theta - \phi) \right] / 2 \phi F'(\theta) \right|.$$  \hspace{1cm} (13)

We have evaluated Eq. (13) for two examples with the parameters cited in the first paragraph of this Letter, holding $|\theta| = 1.3$ where $|F'(\theta)|$ has its maximum value. The results are shown in Fig. 2 for the $\gamma = 10$ example. In Fig. 2(a) we plot the gain enhancement factor $\eta$ as a function of the transverse magnetic field normalized strength $\xi$ for the FEL with the axial guide magnetic field. The solid curves are for steady-state orbits on branch C; the dashed curves for branch A. On branch A, gain occurs for $\theta > 0$, while on branch C gain occurs for $\theta < 0$. Two transverse magnetic fields for the FEL without axial field corresponding to $\xi_0 = 1$ and 0.5 are shown. Figure 2(b) shows the required values of axial guide field. One sees a gain enhancement of 31 (on branch C) at $\xi = 5 \times 10^{-3}$ for an axial guide field of 102 kG. The transverse magnetic field required is reduced to 51 G, and the gain is increased to 0.0766 at $\lambda = 105 \mu m$. Higher gain is predicted on branch A. For the $\gamma = 100$ example at $\lambda = 10.5 \mu m$, we find a gain enhancement of 16 (on branch A) at $\xi = 3 \times 10^{-2}$ for an axial guide field of 99.5 kG. The transverse magnetic field required is reduced to 38 G, and the gain is increased to 0.0506.

Of course when the predicted single-pass gain is large (say $> 0.1$) this theory must be modified. Furthermore, finite electron momentum spread (neglected here) will mitigate against gain, as for a FEL without a guide field. These effects deserve careful study. However, to the extent that these effects are negligible, our theory shows that provision of a strong uniform axial magnetic field can allow significant small-signal gain enhancement, and significant reduction in the required transverse magnetic field strength in a FEL, without undue compromise in operating frequency below that given by the idealized upshift value $2\gamma^2 B_0$.

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