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[6]The cooperative molecular amplitudes a_i can be related to a more familiar formalism in the case of a collection of *two-level* systems. In this case the collective (dimensionless) angular momentum operators l_1, l_2, l_3 are given by $l_1 = \frac{1}{2}(a_1 a_2^\dagger + a_1^\dagger a_2)$, $l_2 = -\frac{1}{2}i(a_1 a_2^\dagger - a_1^\dagger a_2)$, $l_3 = \frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1)$. Cooperation corresponds to the invariance of l^2. The present model is subject, of course, to the same limitations as the model for a collection of two-level systems in which l^2 is invariant. The latter model and its limitations are discussed by I. R. Senitzky, Phys. Rev. A 6, 1171 (1972), and by P. D. Drummond and H. J. Carmichael, Opt. Commun. 27, 160 (1978).
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[9]The similarity of Fig. 1 to a phase diagram is obvious. If $|x_2|$ is considered (formally) to be the "order parameter", then the transition $S \leftrightarrow Z$ corresponds to a second-order phase transition, while $S \leftrightarrow P$ and $Z \leftrightarrow P$ correspond to first-order phase transitions. Nonequilibrium phase transitions involving cooperative atomic behavior of two-level systems are discussed by D. F. Walls, P. D. Drummond, S. S. Hassan, and H. J. Carmichael, Prog. Theor. Phys. (Japan). Suppl. 64, 307 (1978), who also give additional references.
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[11]The apparent—seemingly puzzling—stability of P and S in the limit $\omega \to 0$ is due to the fact that relaxation in the (1,3) transition has been ignored. The present model is applicable only when the effect of this relaxation is small compared to that of the pump. Otherwise, sufficiently small ω leads to a fourth stable state, to be described in detail elsewhere, which is irrelevant to the effects presently considered. With a more complex model (beyond the scope of the present discussion), it can be shown that this state is a weakly excited steady state of two-level systems involving only levels 1 and 3.
[12]The idealizations and approximations used may limit the time during which the present theory is valid.

Free-Electron Laser with a Strong Axial Magnetic Field

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A small-signal theory is given for gain in a free-electron laser comprising a cold relativistic electron beam in a helical periodic transverse, and a strong uniform axial, magnetic field. Exact finite-amplitude, steady-state helical orbits are included. If perturbed, these orbits oscillate about equilibrium, so that substantial gain enhancement can occur if the electromagnetic perturbations resonate with these oscillations. This gain enhancement need not be at the cost of frequency upshift.

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Intensive activity is underway to exploit the gain properties of a relativistic electron beam undulating in a periodic transverse magnetic field. Such free-electron laser (FEL) configurations have provided oscillation at 3.4 (Ref. 1) and 400 μm,[2] and amplification at 10.6 μm.[3] Theory has advanced apace,[4] and elaborate schemes have been proposed for obtaining high FEL efficiency.[5] A factor which limits the practical application of this interaction at wavelengths shorter than perhaps a few microns is the rapid decrease in small-signal gain G_0 as the electron energy increases. This is shown explicitly in the well-known result[6] for G_0 in the single-particle limit (i.e., when collective effects are negligible)

$$G_0 = (\omega_p \xi / k_0 c)^2 (k_0 L / 2\gamma)^3 F'(\theta). \tag{1}$$

Here ω_p and γ are the beam plasma frequency $Ne^2/m\epsilon_0$ and normalized energy W/mc^2, k_0 and ξ are the helical transverse magnetic field wave number $2\pi/l$ and normalized strength $eB_\perp/mck_0$, L is the interaction length, and $F'(\theta) = (d/d\theta)(\sin\theta/\theta)^2$ is the line-shape factor, with $\theta = [k_0 v_{30} - \omega(1 - v_{30}/c)](L/2c)$, where v_{30} is the unperturbed electron axial velocity. The peak gain occurs at $\theta = 1.3$, where $F'(\theta) = 0.54$. For example, with γ

$=10$, $l=1.05$ cm, $\omega_p=5\times10^7$ sec^{-1}, $\xi=1$ ($B_\perp=10.2$ kG), and $L=130$ cm, the peak gain is $G_{0p}=0.002\,47$ at a wavelength of 105 μm. For $\gamma=100$, $l=10.5$ cm, $\omega_p=2\times10^9$ sec^{-1}, $\xi=1$ ($B_\perp=1.02$ kG), and $L=260$ cm, the peak gain is $G_{0p}=0.003\,16$ at a wavelength of 10.5 μm. These gain values may be large enough to sustain oscillations if highly reflecting mirrors are judiciously added but the strong helical fields required (particularly the 10.2-kG case) may be beyond the capability of present superconducting coil technology.[7]

A suggestion has appeared for enhancing the small-signal gain above values given by Eq. (1) (or for achieving comparable gains with smaller $B_\perp$) by employing a strong axial magnetic field so as to exploit resonance between the cyclotron frequency and the undulatory frequency.[8] The present Letter presents a single-particle derivation for the small-signal gain of a FEL in a uniform axial magnetic field B_0. We shall demonstrate that careful adjustment of the system parameters will allow enhancement of the FEL small-signal gain by an order of magnitude or more (for the above examples) *without increasing the undulatory velocity*. This result goes beyond that predicted by Sprangle and Granatstein[8] who have suggested that the only effect of the axial magnetic field would be to add a multiplicative factor $(1-\Omega/k_0c\gamma)^{-2}$ to Eq. (1), due to the aforementioned resonance giving an enhanced undulatory velocity $v_\perp$, where $\Omega=eB_0/m$. This result is in fact predicted by our analysis as a limiting case. Of course, any mechanism which increases the undulatory velocity $v_\perp$ would increase the gain, but this would also reduce the relativistic frequency upshift, since

$$\omega\simeq k_0c(1-v_{30}/c)^{-1}=2\gamma^2k_0c(1+\gamma^2v_\perp{}^2/c^2)^{-1}.$$

If, for example, $\gamma v_\perp/c=1$ without the axial magnetic field, then a given gain enhancement η achieved through this resonance alone would result in a reduction in frequency upshift by a factor $(1+\eta)/2$. The process we describe in this Letter will be shown to permit significant gain enhancement without undue sacrifice in frequency upshift. The gain enhancement originates when the electromagnetic perturbations resonate with the natural frequency of oscillation of electrons on finite amplitude equilibrium helical orbits. Prior workers have not considered this effect.

A full derivation of our result will be presented elsewhere.[9] Exact unperturbed relativistic orbits are considered in the customary FEL model magnetic field

$$\vec{B}(\vec{r})=B_0\hat{e}_z+B_\perp(\hat{e}_x\cos k_0z+\hat{e}_y\sin k_0z). \tag{2}$$

These orbits, which have been the subject of recent study,[10] can possess more than one steady state, depending upon γ, B_0, $B_\perp$, and k_0. These steady states are characterized by the normalized velocity components (i.e., $u_i=v_i/c$)

$$u_{10}=0,\quad u_{20}=k_0\xi u_{30}/(k_0u_{30}\gamma-\Omega/c),$$
$$u_{30}=(1-u_{20}{}^2-\gamma^{-2})^{1/2}, \tag{3}$$

where the basis vectors $\hat{e}_1(z)=-\hat{e}_x\sin k_0z+\hat{e}_y\times\cos k_0z$, $\hat{e}_2(z)=-\hat{e}_x\cos k_0z-\hat{e}_y\sin k_0z$, and $\hat{e}_3(z)=\hat{e}_z$ have been introduced to track the symmetry of the transverse magnetic field. Figure 1 shows u_{30} vs Ω/c for $k_0=6.0$ cm^{-1}, $\xi=1.0$, and $\gamma=10$. For $\Omega>\Omega_{\rm cr}\equiv k_0c[(\gamma^2-1)^{1/3}-\xi^{2/3}]^{3/2}$, it is seen that only one branch exists (branch C). But for $\Omega<\Omega_{\rm cr}$ two additional branches (A and B) are allowed: Branch B has been shown to be unstable, in that the orbits exhibit nonhelical, highly anharmonic motions, while branches A and C have orderly helical orbits. Stability is insured if $\mu^2\equiv a^2-bd>0$, where $a=k_0cu_{30}\xi/\gamma u_{20}$, $b=\Omega u_{20}/\gamma u_{30}$, and $d=k_0c\xi/\gamma$. The quantity μ is the natural resonance frequency in response to small perturbations of the orbit: We shall show that strong resonance response of the electrons to electromagnetic perturbation can lead to enhanced FEL gain for small μ, i.e., for Ω close to $\Omega_{\rm cr}$.

The derivation of FEL gain proceeds by solving the single-particle equations of motion, subject to weak electromagnetic perturbing fields $\vec{E}=\hat{e}_xE_0\cos(kz-\omega t)$ and $\vec{B}=\hat{e}_y(kc/\omega)E_0\cos(kz-\omega t)$,

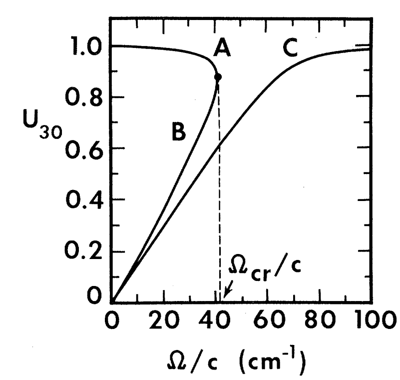

FIG. 1. Steady-state normalized axial velocity u_{30} as a function of normalized axial magnetic field Ω/c. For this example $k_0=6.0$ cm^{-1}, $\xi=1.0$, and $\gamma=10$. Gain enhancement discussed in this work is for orbits on either branch A or branch C.

about the equilibrium orbits on either branch A or C as discussed above. These equations are

$$\dot{u}_1 = (k_0 c u_3 - \Omega/\gamma)u_2 - (k_0 c\xi/\gamma)u_3 - (\dot{\gamma}/\gamma)u_1 + (eE_1/mc\gamma)(kcu_3/\omega - 1), \tag{4}$$

$$\dot{u}_2 = -(k_0 c u_3) - \Omega/\gamma)u_1 - (\dot{\gamma}/\gamma)u_2 + (eE_2/mc\gamma)(kcu_3/\omega - 1), \tag{5}$$

$$\dot{u}_3 = (k_0 c\xi/\gamma)u_1 + (kc/\omega - u_3)(\dot{\gamma}/\gamma), \tag{6}$$

where $\dot{\gamma} = -(e/mc)(u_1E_1 + u_2E_2)$ and

$$2(E_2 + iE_1) = -E_0 \exp\{i[(k_0 + k)u_3ct - \omega t + \alpha]\}$$

with α the random initial electron phase. When time variations and electromagnetic fields are absent, Eqs. (4)–(6) lead to the exact steady states given by Eq. (3). To linearize Eqs. (4)–(6), we introduce the velocity perturbations $w_i = u_i - u_{i0} \ll u_{i0}$ and retain only the lowest-order quantities. This results in $\ddot{w}_1 + \mu^2 w_1 = AE_0\cos(\beta t + \alpha)$, or

$$w_1 = \frac{AE_0}{\mu^2 - \beta^2}[\cos(\beta t + \alpha) - \cos\mu t\cos\alpha + (\beta/\mu)\sin\mu t\sin\alpha] + \mu^{-1}\dot{w}_1(0)\sin\mu t, \tag{7}$$

where

$$A = (a + \beta)(1 - u_{30}) + bu_{20}, \quad \beta = c(k + k_0)u_{30} - \omega, \quad \omega \simeq kc, \quad \dot{w}_1(0) = (eE_0/2\gamma mc)(1 - u_{30})\sin\alpha,$$

and $w_1(0) = 0$. The other components follow from

$$\dot{w}_2 = -aw_1 + (eE_0/2mc\gamma)(1 - u_{30} - u_{20}^2)\cos(\beta t + \alpha), \quad w_2(0) = 0; \tag{8}$$

and

$$\dot{w}_3 = dw_1 + (eE_0/2mc\gamma)u_{20}(1 - u_{30})\cos(\beta t + \alpha), \quad w_3(0) = 0. \tag{9}$$

Equation (7) for w_1 exhibits the aforementioned natural resonance at frequency μ, while the electromagnetic perturbation drives the transverse motion at frequency β. Gain enhancement can be expected when μ is close to β.

The energy gain for an electron is calculated from $(mc/e)d\gamma/dt \simeq -w_1E_{10} - w_2E_{20} - u_{20}E_{21}$. The first-order variation in electric field E_{21} originates from small phase variations as u_3 changes. Thus this becomes

$$(mc/e)d\gamma/dt = -w_1E_{10} - w_2E_{20} - \tfrac{1}{2}E_0(k + k_0)cu_{20}\sin(\beta t + \alpha)\int_0^t dt' w_3(t'). \tag{10}$$

The third term in Eq. (10) is much larger than the other two on account of the factor $k + k_0$. The dominant single-particle energy transfer in the FEL (even with an axial magnetic field) is seen to be by work ecu_2E_2 done along the transverse undulatory motion, enhanced by the strong variation in E_2 as its phase varies through w_3. The energy variation [Eq. (10)] is averaged over random phase α to give $\langle d\gamma/dt\rangle$, which in turn leads to the gain through $G = 2(\epsilon_0 E_0^2)^{-1}Nmc^2\int_0^T dt\langle d\gamma/dt\rangle$, where N is the beam electron density and $T = L/c$ is the total interaction time for the electrons in a system of length L.

The final result is

$$G = \frac{\omega_p^2 k_0 c}{16\gamma}u_{20}^2T^3\left\{\left[1 + \frac{a}{\mu_2}\left(a + \beta + \frac{u_{20}b}{1 - u_{30}}\right)\right]\left[F'(\theta) - \frac{F(\theta + \varphi) - F(\theta - \varphi)}{2\varphi}\right] + \frac{F(\theta + \varphi) - F(\theta - \varphi)}{2\varphi} - \frac{a}{\mu^2 T}\left[P'(\theta) - \frac{P(\theta + \varphi) - P(\theta - \varphi)}{2\varphi}\right]\right\}, \tag{11}$$

where $\theta = \beta T/2$, $\varphi = \mu T/2$, $F(x) = (\sin x/x)^2$, and $P(x) = xF(x)/2$; and where we have approximated $(k + k_0)(1 - u_{30}) \simeq k_0$. We shall examine Eq. (11) in several limits.

For $\mu \gg \beta$, only the terms involving $F'(\theta)$ and $P'(\theta)$ in Eq. (11) are significant, and on branch A the latter of these is smaller than the former by at least a factor 2φ. Thus to a good approximation we may write

$$G(\mu \gg \beta) = Z(\omega_p^2/16\gamma)k_0cu_{20}^2T^3F'(\theta), \tag{12}$$

where $Z = 2 + \mu^{-2}[a\beta + bd(1 - u_{30})^{-1}]$. In the case where the axial magnetic field is absent, $\Omega = 0$, $\mu = a = k_0cu_{30} \gg \beta$, and $u_{20} = \xi/\gamma$. Thus, $Z \simeq 2$ and Eq. (12) goes over to Eq. (1). When $\Omega \neq 0$ and μ

$\gg\beta$, gain enhancement can be achieved as claimed by the prior workers,[8] due to resonant enhancement of u_{20}, but not without sacrificing frequency upshift, as discussed above.

However a more attractive possibility exists when μ is small, and approaches β. Here one can approximate $Z \simeq \mu^{-2}bd(1-u_{30})^{-1} \gg 1$; this results from resonance between the electromagnetic perturbation which gives oscillatory motion to the electron at a frequency β, close to its natural oscillation frequency μ. Gain enhancement due to large Z is seen to be possible without simultaneously increasing u_{20}, so that the desirable frequency upshift property of the FEL need not be sacrificed.

We define a gain enhancement factor $\eta = G/G_0$ to compare two free-electron lasers, identical except that one has a strong axial magnetic field, while the second does not. In the first laser, the transverse magnetic field $B_\perp$ is reduced so that u_{20} is the same for both lasers. (This assures that both enjoy the same frequency upshift.) Then

$$\eta = Z\{1 - [F(\theta+\varphi) - F(\theta-\varphi)]/2\varphi F'(\theta)\}. \qquad (13)$$

We have evaluated Eq. (13) for two examples with the parameters cited in the first paragraph of this Letter, holding $|\theta| = 1.3$ where $|F'(\theta)|$ has its maximum value. The results are shown in Fig. 2 for the $\gamma = 10$ example. In Fig. 2(a) we plot the gain enhancement factor η as a function of the transverse magnetic field normalized strength ξ for the FEL with the axial guide magnetic field. The solid curves are for steady-state orbits on branch C; the dashed curves for branch A. On branch A, gain occurs for $\theta > 0$, while on branch C gain occurs for $\theta < 0$. Two transverse magnetic fields for the FEL without axial field corresponding to $\xi_0 = 1$ and 0.5 are shown. Figure 2(b) shows the required values of axial guide field. One sees a gain enhancement of 31 (on branch C) at $\xi = 5 \times 10^{-3}$ for an axial guide field of 102 kG. The transverse magnetic field required is reduced to 51 G, and the gain is increased to 0.0766 at $\lambda = 105$ μm. Higher gain is predicted on branch A. For the $\gamma = 100$ example at $\lambda = 10.5$ μm, we find a gain enhancement of 16 (on branch A) at $\xi = 3 \times 10^{-2}$ for an axial guide field of 99.5 kG. The transverse magnetic field required is reduced to 30.6 G, and the gain is increased to 0.0506.

Of course when the predicted single-pass gain is large (say > 0.1) this theory must be modified. Furthermore, finite electron momentum spread (neglected here) will mitigate against gain, as for a FEL without a guide field. These effects

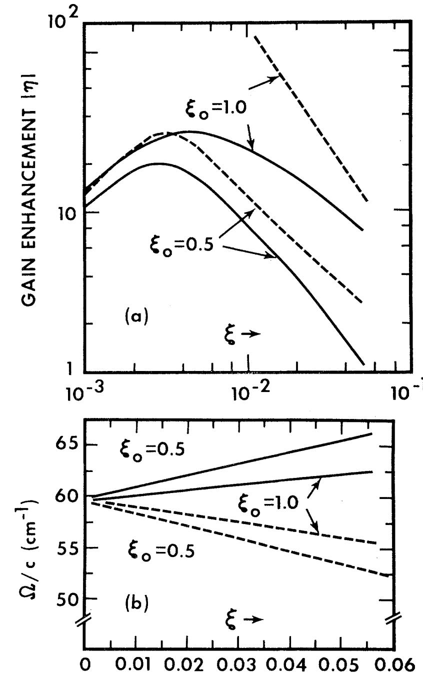

FIG. 2. (a) Gain enhancement $|\eta|$ and (b) corresponding normalized axial magnetic field Ω/c, vs transverse magnetic field parameter ξ. The values $\xi_0 = 0.5$ and 1.0 are for the FEL without axial field, and provide the same u_{20} as do the indicated (smaller) values of ξ for the FEL with the indicated axial field strength. Example is for $\gamma = 10$, $k_0 = 6.0$ cm^{-1}, and $L = 130$ cm. Solid curves, orbits on branch C; dashed curves, orbits on branch A. For high enhancement values, such as on the $\xi_0 = 1.0$ branch A example, the numerical precision required to compute accurate results suggests that the phenomenon is very sensitive to the system parameters.

deserve careful study. However, to the extent that these effects are negligible, our theory shows that provision of a strong uniform axial magnetic field can allow significant small-signal gain enhancement, and significant reduction in the required transverse magnetic field strength in a FEL, without undue compromise in operating frequency below that given by the idealized upshift value $2\gamma^2 k_0 c$.

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