Autoresonance of coupled nonlinear waves

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Abstract. Resonant three-wave interactions (R3WIs) and their dynamical counterpart, three-oscillator interactions (R3OIs) play a fundamental role in many fields of physics. Consequently, controlling R3WI/R3OIs is an important goal of both basic and applied physics research. We have developed new control schemes based on a recent approach of wave autoresonance. This approach is based on the intrinsic property of many nonlinear waves and oscillations to stay in resonance (phase-lock) even when parameters of the system vary in time and/or space. We review autoresonance in several new coupled wave systems including externally driven R3OI systems and multidimensional R3WIs. Particularly, we have focused on autoresonant stimulated Raman scattering in nonuniform plasmas. This research comprises an important step toward understanding of adiabatic synchronization of nonlinear waves in space-time varying media with a potential of many new applications in plasma physics and related fields, such as fluid dynamics, nonlinear optics, and acoustics.

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I. INTRODUCTION

Resonant three-wave interactions (R3WIs) and their dynamical counterpart, three-oscillator interactions (R3OIs) play a fundamental role in physics because they represent lowest order (in terms of wave amplitudes) nonlinear effects in systems approximately described by a linear superposition of discrete waves and oscillations [1]. For example, an incoming laser beam in a plasma can decay via R3WIs into another electromagnetic wave and an ion-acoustic or electrostatic plasma waves [2]. In particular, stimulated Raman scattering (SRS) is of great interest as a deleterious reflection mechanism in inertial confinement fusion [3]-[7], or, potentially, as a beneficial mechanism for optical pulse compression in plasma-based Raman amplifiers [8]-[11]. R3WIs are also characteristic of many other plasma related phenomena [12]-[14] and other fields of nonlinear physics, such as nonlinear optics [15], hydrodynamics [16], and acoustics [17]. Consequently, controlling R3WI/R3OIs is an important goal of both basic and applied physics research. Varying the three-wave resonance condition by time and/or space inhomogeneity of the background medium is one approach to affecting R3WIs [18, 19]. An alternative is to influence the resonance condition by nonlinearity of the medium [20]. By combining the space/time variation and the nonlinearity of the background one may use autoresonance for controlling R3WIs [21]-[23]. This more recent approach uses intrinsic property of many nonlinear waves and oscillations to stay in resonance (phase-lock) even when parameters of the system vary in time and/or space [24]. In the case of three coupled waves or oscillations the phase-locking is achieved as the nonlinear frequency/wave vector shifts self-adjust (via variation of the waves' amplitudes) to compensate the linear dispersion shifts due to space-time inhomogeneity of the background. The effect of autoresonance was used for excitation and control of many systems in a variety of fields of interest (see e.g. Refs. 22-36 in [25]).

Nonlinear waves in inhomogeneous, time dependent medium are usually described by systems of partial differential equations. For instance, in the case of a plasma medium in the fluid approximation, there are numerous coupled variables such as the electric and magnetic vector-fields, the densities of plasma components, their fluid velocity vector fields, etc. The problem seems intractable because of nonlinearities and a typical space/time dependence of the medium. Nevertheless, in the case of slow space/time nonuniformity and weak nonlinearity, one can use a multidimensional WKB formalism for multicomponent waves in reducing the complexity. The difficulty caused by the multicomponent structure of the waves can be resolved by applying congruent reduction [26] of a largest possible number of wave components from the problem leading to the lowest order system of differential equations. Resonant three-wave interactions involve three modes whose frequencies and wave vectors are related by the matching conditions $\omega_1 + \omega_2 \approx \omega_3$ and $\mathbf{k}_1 + \mathbf{k}_2 \approx \mathbf{k}_3$. In these cases, using the congruent reduction, the original slow system of $3N$ (N components for each wave) coupled differential equations may be reduced to a set of $3N-3$ algebraic equations plus the following 3 differential equations [25]:

$$L_1 A_1 = -p_1 \varepsilon A_2 A_3 e^{-i\Psi} \quad (1)$$

$$L_2 A_2 = p_2 \varepsilon A_1 A_3^* e^{+i\Psi} \quad (2)$$

$$L_3 A_3 = p_3 \varepsilon A_1 A_2^* e^{+i\Psi}, \quad (3)$$

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where the operators L_j are defined as $L_j = \partial/\partial t + \mathbf{V}_j \cdot \nabla + \Gamma_j + iC_j \left|A_j\right|^2$, and we use slow complex amplitudes A_j, group velocities $\mathbf{V}_j$, coupling coefficient ε, and weakly nonlinear frequency shifts assumed to be of form $\delta\omega_j = C_j \left|A_j\right|^2$. The wave energy signs above are $p_j = \pm 1$, while Γ_j include the linear damping coefficients and the divergence effect of the geometric optics rays associated with the waves. $\Psi = \psi_1 - \psi_2 - \psi_3$ is the phase mismatch between the three coupled waves, where ψ_j is the eikonal phase of the j-th wave.

We used this set of reduced equations in several applications, described in the rest of this review. In Sec. II a system of externally driven three coupled oscillators is discussed. The process of stimulated backward Raman scattering in an inhomogeneous plasma is examined in Sec. III. Finally, the general scenario of resonant three wave interactions in a nonuniform $3D$ and time-dependent medium is presented in Sec. IV.

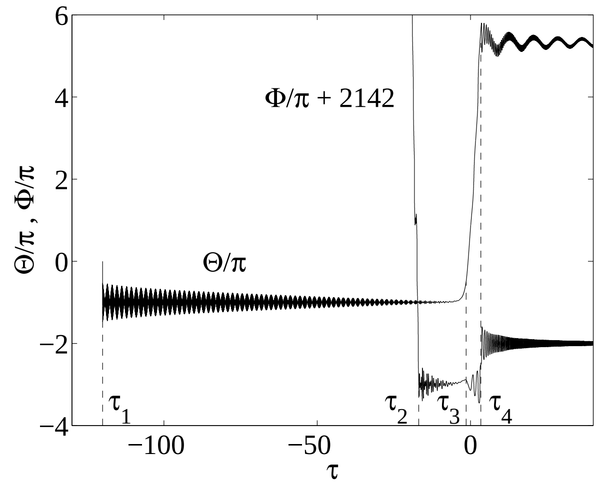

FIGURE 1. The evolution of phase mismatches Θ and Φ ($\lambda = 2$, $\mu = 0.9$).

II. DRIVEN THREE OSCILLATOR INTERACTIONS

Multidimensional autoresonant three-wave interactions involve matching of frequencies ω_j and wave vectors $\mathbf{k}_j$, i.e. adjustment of *four* parameters of the interacting waves. As a first step in understanding this complicated phenomenon, we have considered a nonlinear system of three externally driven coupled oscillators [27], where only *one* matching condition must be satisfied $\omega_1 + \omega_2 = \omega_3$:

$$\ddot{Z}_j + \omega_j^2 Z_j = -\varepsilon Z_k Z_3, \quad j,k = 1,2, \;\; j \neq k, \tag{4}$$

$$\ddot{Z}_3 + \omega_3^2 Z_3 = -\varepsilon Z_1 Z_2 - F_d. \tag{5}$$

Here we neglected the nonlinear frequency shifts, but added a constant amplitude, chirped frequency driving perturbation $F_d = \eta \cos\psi_d$, $\psi_d = \int \omega_d(t)dt$ in the equation for the third oscillator. We have focused our analysis on a case in which the drive frequency varies linearly in time $\omega_d = \omega_3 + \alpha t$ and passes the linear frequency of the third wave at $t = 0$. We have developed an analytical model of this interaction when the driving frequency is a slow function of time and compared its predictions with the results of numerical simulations [27]. We assumed that the amplitudes are slow on the scale of the largest period of linear oscillations (adiabatic approximation) and made a weak nonlinearity assumption. By introducing the rescaled amplitudes $B_j = A_j\sqrt{\omega_j}$, time $\tau = t\sqrt{\alpha}$, coupling parameter $\lambda = \varepsilon/\left(4\sqrt{\omega_2\omega_3\alpha}\right)$ and driving amplitude $\mu = \eta/\left(2\sqrt{\omega_3\alpha}\right)$ and defining the phase mismatches between the interacting oscillators, $\Phi \equiv \psi_3 - \psi_1 - \psi_2$, and between the drive and the driven third oscillator, $\Theta \equiv \psi_d - \psi_3$, the following results were obtained:

(a) We have demonstrated efficient control of resonant three-oscillator interactions (R3OI) by using an external chirped frequency drive. In analyzing this effect we have divided the evolution of the driven R3OI system into several stages characterized by different resonance types in each stage. Fig. 1 shows the time evolution of the phase mismatches Φ and Θ. The starting times of the successive interaction stages is denoted by $\tau_1 ... \tau_4$. Analytic expressions describing the time dependence of the slow amplitudes of the oscillators were derived.

(b) A quasi-steady state of the system, where the slow amplitudes of the interacting oscillators increase linearly with time was found such that

$$B_{1s} = B_{2s} = \frac{\tau}{\sqrt{2}\lambda}, \tag{6}$$

$$B_{3s} = \frac{\tau}{2\lambda}. \tag{7}$$

The quasi-steady state stage discontinues at a certain time τ_e given by

$$\tau_e \approx \frac{\frac{2}{3}\varepsilon\eta\sqrt{\omega_1\omega_2} - 8\omega_3\alpha}{\alpha^{3/2}\left(8\omega_1\omega_2 + 3\omega_3^2\right)}. \tag{8}$$

Beyond this time, the energy of the driven system saturates. By increasing the driving amplitude η, one extends the duration of the autoresonant state, allowing higher energy excitations.

(c) We have shown that for weak excitations, such that the nonlinear frequency shifts due to interaction between the oscillators are small, the quasi-steady state is characterized by *double phase-locking*, i.e. the phase

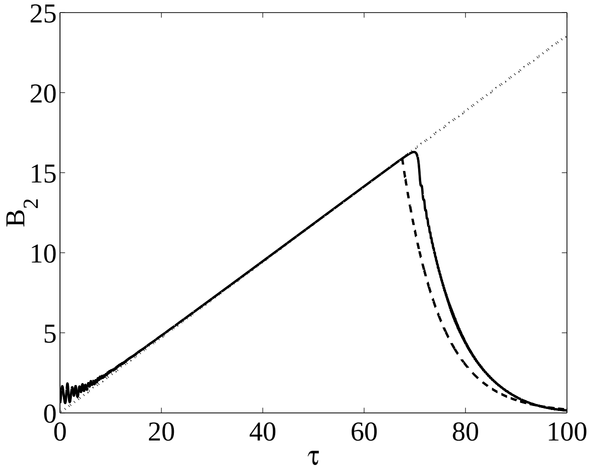

FIGURE 2. Autoresonance in a dissipative system. The numerical solution of the adiabatic set of equations (solid line) is compared to that given by Eq. (9) (dashed line). The dotted line represents the quasi-steady state solution (6). The parameters are $\lambda = 3$, $\mu = 5$ and $\nu = 0.4$.

mismatches Φ and Θ are locked at $\Phi \approx 0$ *mod* 2π and $\Theta \approx$ *const* (demonstrated in Fig. 1 for $\tau > \tau_4$). We have shown that this double phase-locking is linearly stable.

(d) We have studied the threshold μ_{th} on the amplitude of the drive for successful excitation of the autoresonant state. We have shown analytically that the product $\lambda\mu_{th}$ must exceed 1.5 for having the asymptotic autoresonant quasi-steady state. This threshold condition may be expressed using the original parameters as $\varepsilon\eta_{th}/\left(\alpha\omega_3\sqrt{\omega_1\omega_2}\right) > 12$.

(e) The addition of a dissipation term of form $\left(-\beta\dot{Z}_3\right)$ on the RHS of Eq. (5) results in the escape of the system from autoresonance at time $\tau_d = (2\lambda\mu - 3)/\nu$, where $\nu = \beta/\left(2\sqrt{\alpha}\right)$. For $\tau > \tau_d$ the amplitudes decay exponentially with time:

$$B_j(\tau) = B_{js}(\tau_d)\, e^{-\frac{\nu}{3}(\tau-\tau_d)}, \tag{9}$$

where B_{js} are given by (6) and (7). The effect of adding dissipation to the system is illustrated in Fig. 2.

III. STIMULATED RAMAN SCATTERING IN NONUNIFORM PLASMAS

The goal of this work was formation of large amplitude, spatially autoresonant plasma waves in a one dimensional (1D) nonuniform plasma via stimulated Raman scattering (SRS) [28]. The transition to spatial autoresonance during pulsed application of the pump/seed laser waves and the role of the autoresonant threshold phenomena in this transitional 1D process was investigated. In the past, this threshold effect was studied in the context of autoresonance in *externally* driven dynamical and nonlinear wave problems only. These studies assumed that the driving perturbation was unaffected by the excited nonlinear wave or oscillation. We proposed to add self-consistency to the problem, by considering the autoresonance of *mutually* interacting nonlinear waves.

Our starting point was a system of envelope equations describing the SRS process in a stationary, one-dimensional (along z), weakly-nonuniform underdense plasma [5], which represents a special case of Eqs. (1-3):

$$L_a a = -\varepsilon \tfrac{\omega_g}{\omega_a}\, b g\, e^{-i\Psi}, \tag{10a}$$

$$L_b b = \varepsilon \tfrac{\omega_g}{\omega_b}\, a g^* e^{i\Psi}, \tag{10b}$$

$$L_g g + i\beta\,|g|^2\, g = \varepsilon\, a b^* e^{i\Psi}. \tag{10c}$$

Here, the complex envelopes a and b describe the electromagnetic waves (pumps) and g describes the plasma Langmuir wave (seed). The associated dimensionless r.m.s. electric fields are

$$\begin{aligned} E_z &= \frac{mc}{e}\frac{\omega_p(z)}{\sqrt{2}} g e^{i\psi_g} + c.c., \\ \boldsymbol{E}_\perp &= \frac{mc}{e}\left(\tfrac{\omega_a}{\sqrt{2}} a e^{i\psi_a} + \tfrac{\omega_b}{\sqrt{2}} b e^{i\psi_b}\right)\hat{\boldsymbol{e}}_\perp + c.c., \end{aligned} \tag{11}$$

where c is the vacuum speed of light, m is the electron mass, and $\hat{\boldsymbol{e}}_\perp$ is the common transverse polarization of the laser fields (linear or circular), $\psi_\ell = \int dz\, k_\ell(z) - \omega_\ell t$, $(\ell = a, b, g)$, are the eikonal phases of the waves, and $\Psi = \psi_a - \psi_b - \psi_g$. The frequencies ω_ℓ of the waves are assumed to be constant and satisfy the three-wave resonance condition $\omega_a - \omega_b = \omega_g$, while the wave vectors are slowly-varying functions of longitudinal position z, so as to satisfy the local dispersion relations $\omega_{a,b}^2 = \omega_p^2(z) + c^2 k_{a,b}^2(z)$ and $\omega_g^2 = \omega_p^2(z) + 3v_{th}^2 k_g^2(z)$, where $v_{th} = \sqrt{K_B T_0/m}$ is the electron thermal velocity, assumed uniform, and $\omega_p = \omega_p(z)$ is the local linear plasma frequency, which is assumed to satisfy $\omega_p'(z) \ll \omega_p(0)k_{a,b}$. The differential operators $L_\ell = \frac{\partial}{\partial t} + v_\ell\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial v_\ell}{\partial z}$ are those of slowly-varying linear geometric optics [29], with $v_{a,b} = c^2 k_{a,b}/\omega_{a,b}$ and $v_g = 3v_{th}^2 k_g/\omega_g$ are the group velocities of the corresponding waves. The three waves are quadratically coupled via the right-hand sides of (10), while the additional term $i\beta\,|g|^2$ represents a nonlinear frequency shift of the plasma wave. The coupling coefficient ε and the nonlinear frequency shift coefficient β are given by:

$$\varepsilon = \frac{c k_g \omega_p}{2\sqrt{2}\omega_g}, \quad \beta = \frac{15}{2}\frac{\omega_g^2}{\omega_p^2}\frac{v_{th}^2 c^2}{v_{ph}^4}\omega_g - \frac{3}{8}\frac{\omega_g^2}{\omega_p^2}\omega_g. \tag{12}$$

We are interested in analyzing the passage through *spatial* resonance in the plasma and, consequently, assume that in the vicinity of the resonance at $z=0$, $k_a-k_b-k_g\approx\alpha z$, where α parameterizes the spatial non-uniformity. We solve the system (10) between two fixed plasma boundaries, at $z=z_{\rm L}<0$ and $z=z_{\rm R}>0$, and assume that the electromagnetic field $a=a(z,t)$ is switched on at $z_{\rm L}$ at time $t=0$, propagates in the positive z direction ($v_a>0$), and remains at a prescribed constant amplitude at the left boundary $z_{\rm L}$ for all $t\geq 0$: i.e., $a(z_{\rm L},t)=\Theta(t)a_0$, where $\Theta(t)$ is the Heaviside step function. We suppose the electromagnetic field $b=b(z,t)$ is also switched on at $t=0$, and either [for the case of Raman Forward Scattering (RFS)] $v_b>0$, and $b(z_{\rm L},t)=\Theta(t)b_0$, or else [for the case Raman Backward Scattering (RBS)] $v_b<0$, and $b(z_{\rm R},t)=\Theta(t)b_0$. We seek the solution of this initial/boundary problem in the space-time domain $z_{\rm L}\leq z\leq z_{\rm R}$, $t\geq 0$.

At this stage, we rewrite the coupled equations (10) in a more convenient dimensionless form. We introduce the dimensionless longitudinal coordinate $\xi\equiv\sqrt{|\alpha|}z$ scaled to the plasma gradient, and the dimensionless time $\tau\equiv v_g(z=0)\sqrt{|\alpha|}t$. Furthermore, we define the scaled action amplitudes of the waves,

$$A(\xi,\tau)=\frac{\sqrt{v_a}}{\sqrt{v_a(z_{\rm L})}}\frac{a(z,t)}{a_0}, \tag{13a}$$

$$B(\xi,\tau)=\frac{\sqrt{|v_b|}}{\sqrt{v_a(z_{\rm L})}}\frac{\sqrt{\omega_b}}{\sqrt{\omega_a}}\frac{b(z,t)}{a_0}, \tag{13b}$$

$$G(\xi,\tau)=\frac{\sqrt{v_g}}{\sqrt{v_a(z_{\rm L})}}\frac{\sqrt{\omega_g}}{\sqrt{\omega_a}}\frac{g(z,t)}{a_0}e^{-i\Psi}, \tag{13c}$$

which simplify the operators L_j and the coupling terms. The set of governing equations is transformed to the following one:

$$\frac{v_g}{v_a}\frac{\partial A}{\partial\tau}+\frac{\partial A}{\partial\xi}=-\tilde{\varepsilon}BG, \tag{14a}$$

$$\frac{v_g}{|v_b|}\frac{\partial B}{\partial\tau}+\sigma\frac{\partial B}{\partial\xi}=\quad\tilde{\varepsilon}AG^*, \tag{14b}$$

$$\frac{\partial G}{\partial\tau}+\frac{\partial G}{\partial\xi}-i\sigma\left(\tilde{\beta}\,|G|^2-\xi\right)G=\quad\tilde{\varepsilon}AB^*, \tag{14c}$$

where $\sigma=\mathrm{sign}(v_b)$, and $\tilde{\varepsilon}$ and $\tilde{\beta}$ are dimensionless parameters describing respectively the wave coupling and plasma nonlinearity:

$$\tilde{\varepsilon}=\frac{a_0\sqrt{\omega_g}}{\sqrt{|\alpha|\,\omega_b v_g v_b}}\,\varepsilon,\quad \tilde{\beta}=\frac{a_0^2\omega_a v_a}{\sqrt{|\alpha|\omega_g v_g^2}}\;|\beta|\,.$$

In this notation, our boundary conditions become $A(\xi_{\rm L},\tau)=\Theta(\tau)$ and either $B(\xi_{\rm L},\tau)=B_0\Theta(\tau)$ (RFS) or $B(\xi_{\rm R},\tau)=B_0\Theta(\tau)$ (RBS). For simplicity we will assume that all parameters in Eqs. (14a-14c) are constant in the following discussion.

As a first step we examine the case of a stationary system where the time derivatives in (14) are omitted. The resulting set of equations is

$$\frac{\partial A}{\partial\xi}=-\tilde{\varepsilon}BG, \tag{15a}$$

$$\sigma\frac{\partial B}{\partial\xi}=\quad\tilde{\varepsilon}AG^*, \tag{15b}$$

$$\frac{\partial G}{\partial\xi}-i\sigma\left(\tilde{\beta}\,|G|^2-\xi\right)G=\quad\tilde{\varepsilon}AB^*, \tag{15c}$$

and A, B, and G depend only on ξ. This set of equations was studied in Ref. [21] for FS ($\sigma=+1$), where it was demonstrated, under certain criteria, to yield spatially-autoresonant solutions. These solutions are obtained when the nonlinearity of the plasma wave approximately cancels the plasma gradient in (15c), so that

$$|G|^2\approx\xi/\tilde{\beta}. \tag{16}$$

A well known characteristic of autoresonant solutions is phase-locking, i.e.,

$$\Phi\equiv\arg(A)-\arg(B)-\arg(G)\approx\text{constant.}$$

We present such RBS solution for the dimensionless wave intensities in Fig. 3a, for which we solve the stationary equations (15) for the parameters $\tilde{\beta}=48.2$, $\tilde{\varepsilon}=0.25$, while the boundary conditions are $A(\xi_{\rm L})=1$, $B(\xi_{\rm R})=0.8$, $G(\xi_{\rm L})=0$, with $\xi_{\rm L}=-10$ and $\xi_{\rm R}=+60$.

In Fig. 3a, we observe that beyond the linear resonance at $\xi=0$, the plasma wave grows according to Eq. (16). The three waves are continuously phase-locked, i.e., $\Phi\approx$

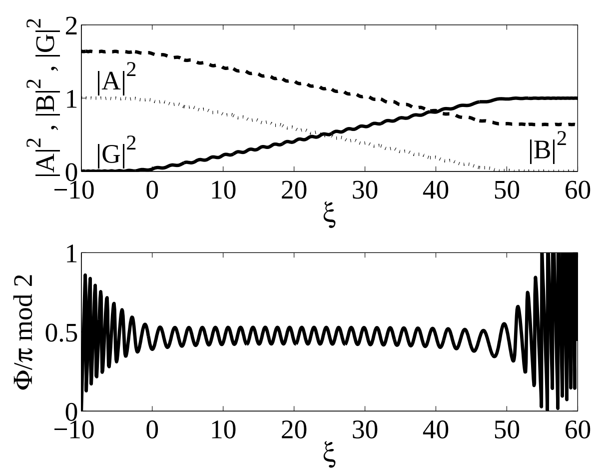

FIGURE 3. Spatially autoresonant evolution. (a) Normalized wave intensities $|A|^2,|B|^2$ and $|G|^2$ versus ξ. (b) Phase mismatch Φ versus ξ. From [28].

$\pi/2$, as shown in Fig. 3b. Furthermore $|A|^2$ and $|B|^2$ are also approximately linear functions of ξ due to the stationary Manley-Rowe conditions:

$$|A|^2 + |G|^2 = \text{constant}, \tag{17a}$$

$$|B|^2 - \sigma |G|^2 = \text{constant}. \tag{17b}$$

After essentially all the pump action has been transferred to the plasma wave, so that $|A|^2 \approx 0$ and $|G|^2 \approx 1$, autoresonance is lost. This happens near the point $\xi \approx \tilde{\beta}$.

We now turn to numerical solution of the fully space-time dependent, initial/boundary three-wave problem (14) and find naturally-arising, quasi-stationary solutions that have many of the same essential features as the autoresonant solutions in the time-independent case studied above. We show the resulting wave intensities as functions of ξ for three different scaled times $\tau = 10, 20, 30$ in Fig. 4, using the same parameters and boundary conditions as in Fig. 3, but with the pump and seed waves switched on suddenly at $\tau = 0$. Where it is excited, the plasma wave $|G|^2$ is again a nearly linear function of ξ, except in this case it has a steep front moving with the group velocity. The location of the front ξ_f is approximately $\xi_f(\tau) \approx \tau$, and therefore, prior to the depletion of the pump at $\xi = \tilde{\beta}$, the plasma wave can be approximated by

$$|G(\xi,\tau)|^2 \approx \begin{cases} \xi/\tilde{\beta} & \text{if} \quad 0 < \xi < \xi_f(\tau) \\ 0 & \text{otherwise} \end{cases}. \tag{18}$$

At the same time, because of the near stationarity of the solution behind the front, the amplitudes $|A|^2$ and $|B|^2$ of the pumps waves approximately satisfy the algebraic Manley-Rowe conditions of the time-independent problem (17), and, thus, also evolve approximately linearly in ξ.

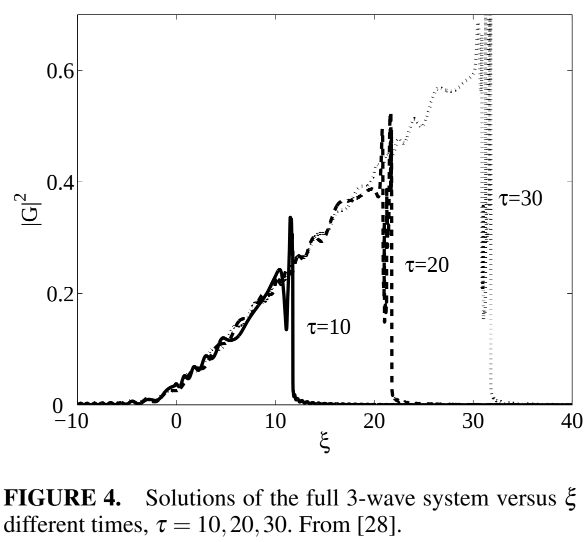

FIGURE 4. Solutions of the full 3-wave system versus ξ at different times, $\tau = 10, 20, 30$. From [28].

IV. MULTIDIMENSIONAL THREE WAVE INTERACTIONS

The theory of autoresonance of three-wave interactions was recently generalized to more than one space or time variation of the background medium [25]. This research extended the ideas of a previous work on autoresonant two-wave interactions, the process known as multidimensional mode conversion [30]. A general theory for a 3D, time dependent medium was developed based on the reduced equations (1-3). We considered a 4-dimensional space/time wave evolution through a 3-dimensional resonance "surface", where the matching conditions on combinations of linear frequencies and wave vectors are satisfied exactly. Asymptotic solutions describing the dynamics of the three waves far away from the resonance "surface" were obtained. Numerical simulations in two dimensional cases were performed for testing our theory, e.g. a steady, spatially autoresonant state in a 2D system or autoresonance in a time-dependent, 1D medium.

Note that the time t and the space variables x_j in our system (1-3) appear on equal footing. Therefore, for simplicity, we shall drop the time derivatives in the following and consider a 3D, time independent problem.

It is convenient at this stage to rescale all dependent and independent variables and parameters, i.e. to introduce: $\mathbf{u}_j = \mathbf{V}_j/V_j$, $\mathbf{R} = |K|^{1/2}\mathbf{r}$, $K = \kappa\mathbf{q}\cdot\mathbf{u}_3$, $\xi = \mathbf{q}\cdot\mathbf{R}$, $B_{1,2} = V_{1,2}^{1/2}A_{1,2}$, $B_3 = V_3^{1/2}A_3\exp[-i\Psi] = V_3^{1/2}A_3\exp\left[-i\kappa(\mathbf{q}\cdot\mathbf{r})^2/2\right]$, $\gamma_j = \left(V_j|K|^{1/2}\right)^{-1}\Gamma_j$, $c_j = \left(V_j^2|K|^{1/2}\right)^{-1}C_j$, $\eta = |V_1V_2V_3K|^{-1/2}\varepsilon$. Then our system becomes

$$\left(\mathbf{u}_1\cdot\nabla_R + \gamma_1 + ic_1|B_1|^2\right)B_1 = -p_1\eta B_2B_3 \tag{19}$$

$$\left(\mathbf{u}_2\cdot\nabla_R + \gamma_2 + ic_2|B_2|^2\right)B_2 = p_2\eta B_1B_3^* \tag{20}$$

$$\left(\mathbf{u}_3\cdot\nabla_R + \gamma_3 - i\sigma\left[|c_3||B_3|^2 - \xi\right]\right)B_3 = p_3\eta B_1B_2^*. \tag{21}$$

where $\nabla_{R_i} \equiv \partial/\partial\mathbf{R}_i$ and $\sigma = -\operatorname{sign}(c_3) = -\operatorname{sign}(\kappa)$. Here κ is a nonzero eigenvalue computed from the corresponding linear problem and $\mathbf{q}$ is a unit vector perpendicular to the resonance surface. The system (19-21) yields the following Manley-Rowe relations

$$P_\alpha\left(\mathbf{u}_\alpha\cdot\nabla_R|B_\alpha|^2 + 2\gamma_\alpha|B_\alpha|^2\right) + P_3\left(\mathbf{u}_3\cdot\nabla_R|B_3|^2 + 2\gamma_3|B_3|^2\right) = 0 \tag{22}$$

for $\alpha = 1,2$, where $P_1 = p_1$, $P_2 = -p_2$ and $P_3 = p_3$. In order to simplify the analysis, we choose the coordinate system such that the R_1 and R_2 axis are in the plane $\xi = 0$, so $R_3 \equiv \xi$.

We focus on a nonlinear boundary value problem in a given volume of the medium containing the linear resonance surface as defined above. We assume that pump waves 1 and 2 are launched externally from plane surfaces $S_{1,2}$ located sufficiently far away from the resonance surface, so the three waves are decoupled on $S_{1,2}$ because of the rapidly varying phase mismatch Ψ. In contrast to waves 1 and 2, the seed wave 3 is assumed to be excited internally (in the vicinity of the resonance surface) via the resonant three-wave interaction process. In the pre-resonant region ($\xi < 0$) the solution for the pump waves ($\alpha = 1,2$) is

$$|B_\alpha| = |B_{\alpha 0}|\, e^{-\gamma_\alpha s_\alpha}, \tag{23}$$

where s_α is the distance from point $\mathbf{R}$ to the boundary surface S_α along the line parallel to $\mathbf{u}_\alpha$ (recall that $|\mathbf{u}_\alpha| = 1$) and $|B_{\alpha 0}|$ is the value of B_α on the boundary S_α. By substituting (23) into (21) and neglecting the nonlinear frequency shift term in this equation, the asymptotic solution of B_3 vanishing at $|\xi| \to \infty$, is

$$|B_3| \approx \eta \frac{|B_{10}B_{20}|}{|\gamma_3 + i\sigma\xi|} e^{-(\gamma_1 s_1 + \gamma_2 s_2)}. \tag{24}$$

It is possible to get autoresonant solutions in the region ($\xi > 0$) when the system parameters satisfy some known threshold criterion [25]. Being in a supercritical case, the solution of the equations in the particular case $c_1 = c_2 = 0$ is as follows:

$$|B_3|^2 \approx \xi / |c_3|. \tag{25}$$

Analytical closed-form solutions for $|B_{1,2}|^2$ were derived as well, but since they are quite complicated they are not presented here. For details see Ref. [25]. Note that Eq. (25) is an analogous of the one-dimensional result Eq. (16), i.e. the autoresonant solution in a multi-dimensional system is essentially one-dimensional along the direction perpendicular to the resonance surface.

Next we present a numerical example illustrating our theory. We consider a 2D problem described by

$$\left(\frac{\partial}{\partial X} + \gamma_1\right) B_1 = -\eta B_2 B_3, \tag{26}$$

$$\left(\frac{\partial}{\partial X} + \gamma_2\right) B_2 = \eta B_1 B_3^*, \tag{27}$$

$$\left(\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right] + \gamma_3 - i\left[|c_3|\,|B_3|^2 - q_X X - q_Y Y\right]\right) B_3 = \eta B_1 B_2^*. \tag{28}$$

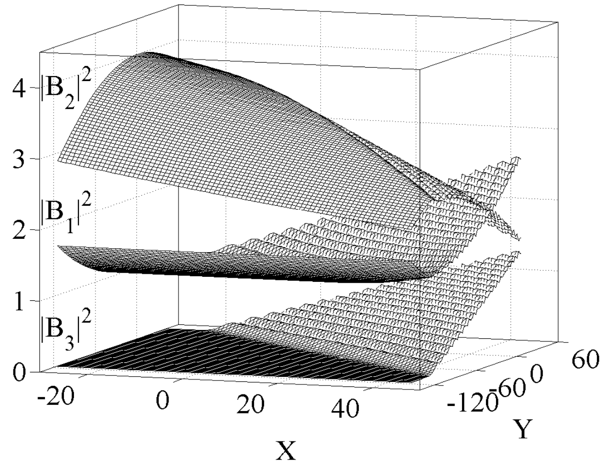

FIGURE 5. Autoresonant evolution of wave intensities $|B_j|^2$ - numerical solution.

With a replacement of the coordinate Y by time, this set of equations is characteristic of laser driven plasma waves in *space-time varying plasma* (see Sec. III for autoresonant $1D$ stimulated Raman scattering (SRS) in *time-independent plasma*).

We have solved this problem numerically in the rectangle $X_{\min} \le X \le X_{\max}$ and $Y_{\min} \le Y \le Y_{\max}$. The following set of parameters was used in our example: $q_X = 0.89$, $q_Y = 0.45$, $c_3 = -45$, $\gamma_1 = 0$, $\gamma_2 = \gamma_3 = 0.002$, $\eta = 0.0337$, and the domain of integration was between $X_{\min} = -30$, $X_{\max} = 45$, and $Y_{\min} - 115$, $Y_{\max} = 45$.

Our boundary conditions on the amplitudes of the pump waves are nonuniform, i.e. set $|B_{10}(Y)| = 1 + 0.3\,(Y/Y_{\min})^2$, $|B_{20}(Y)| = 2 - 0.3\,(Y/Y_{\min})^2$. Note that the direction of propagation of B_3 is inclined at 45^o in

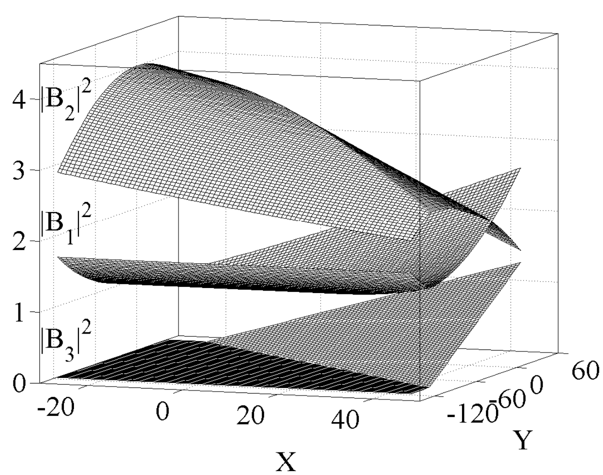

FIGURE 6. Smooth analytical solution for the autoresonant wave intensities $|B_j|^2$ for parameters of Fig. 5.

(X,Y)-plane and, therefore, the seed wave amplitude $|B_3|$ in the simulations is assumed to be given by (24) on both $X = X_{\min}$ and $Y = Y_{\min}$.

In this example one finds that our system is supercritical within the whole pre-resonant integration domain, therefore leading to autoresonant evolution. The results of our simulations in this case are shown in Fig. 5. The monotonic decay of $|B_2|^2$ in the preresonant stage is demonstrated in the Figure. One observes emergence of nearly flat autoresonant seed wave beyond the resonance line in the Figure with the autoresonant pump waves following a smooth, but more complex pattern. The autoresonant solutions are accompanied by small autoresonant oscillations. Finally, for comparison, Fig. 6 shows the smooth analytic solution in this case, which is in a very good agreement with the averaged numerical solution in Fig. 5.

V. CONCLUSIONS

In this review we have described several recent interrelated studies of autoresonance in coupled waves systems. Approximate analytic, closed-form solutions for the evolution of interacting waves were presented. The threshold criterion for autoresonance and the effect of dissipation in the system were discussed.

We have assumed that the nonuniformity in time or space is locally linear in space-time in all investigated systems. Nevertheless, we have found that the functional dependence of the excited autoresonant solutions for the wave amplitudes space time is a characteristic of the system in question. In the driven three-oscillator system, the frequency of the drive was chirped linearly in time, resulting in a linear autoresonant amplitude growth of each oscillator. On the other hand, in the coupled three-wave systems in a nonuniform and/or time dependent medium, the autoresonant evolution is such that the square of the excited wave amplitude is linear in space-time.

The developed theories are generic and may be applied in various fields of physics such as plasma physics, nonlinear optics, acoustics etc. The process of Raman scattering in a nonuniform plasma was discussed in detail, as an example of a possible application.

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