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Efficient capture of nonlinear oscillations into resonance

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Abstract
The problem of efficient capture of nonlinear oscillations into resonance is discussed. The capture is guaranteed by passage through resonance when the system starts in equilibrium and the driving amplitude exceeds a threshold. The threshold problem is described by a universal nonlinear Schrödinger-type equation with a single parameter and cannot be analyzed by perturbation methods. A similar threshold phenomenon is a characteristic of two weakly coupled oscillators with a slow parameter if one of the oscillators starts in equilibrium, allowing efficient capture into resonance and subsequent adiabatic (autoresonant) control of strongly excited nonlinear oscillations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Autoresonance is a salient property of many nonlinear systems to stay in resonance with driving perturbations despite variation of system parameters. The phenomenon was first used in application to particle accelerators [1, 2] and, more recently, as a convenient method of controlling many other dynamical [3] and extended systems (see, for example, [4] and references therein). One way of implementing autoresonant manipulation of nonlinear systems is starting in resonance. This approach requires fine tuning of parameters which may be experimentally difficult, especially when the number of degrees of freedom of the driven system increases. Another approach is to capture the nonlinear system into resonance by slowly passing through the resonance. This resonant capture paradigm was triggered by problems in planetary dynamics [5]. The problem was analyzed in [6] showing that for *strongly* excited oscillations only a *small* subset of initial conditions ends up in resonance leading to the notion of probability of capture into resonance. The situation is different if the oscillator starts from (or near) its natural (zero amplitude) equilibrium. In this case, the phase locking in the driven system is guaranteed *prior* to arrival at the linear resonance [7] and the question becomes that of preserving the phase locking at later times. One finds that the phase locking in the system continues (the system remains in autoresonance) after the passage through resonance provided the driving amplitude exceeds a sharp threshold. At lower driving amplitudes the phase locking is lost. This threshold phenomenon was observed in experiments with magnetized electron clouds [8]. The effect was explained in terms of a quasi-particle in a tilted, washboard-type potential with the potential minima disappearing for driving amplitudes below the threshold, leading to the escape of the system from resonance. Although not fully rigorous, this theory gave a very good prediction for the threshold. In section 2 of this work we further discuss the threshold phenomenon for capture of nonlinear oscillations into resonance when the driven oscillators start in equilibrium. Later, in section 3 we consider a related threshold phenomenon and autoresonance in a system of two weakly coupled oscillators with a slow parameter, when one of the oscillators starts in equilibrium. We show that the autoresonance in this system is guaranteed provided the coupling between the oscillators is sufficiently strong, yielding efficient capture of strongly excited oscillations into resonance and their subsequent autoresonant control. Nevertheless, this control is unidirectional because one can only *decrease* the action of the resonantly captured strongly excited oscillator. In section 4 we suggest another method of efficient capture of large amplitude oscillations into resonance, which removes the aforementioned limitation. The approach is still based on coupling to another oscillator in equilibrium, but uses an oscillating coupling parameter having a slowly varying frequency. Finally, section 5 presents our conclusions.

2. The threshold phenomenon

Consider a driven nonlinear system governed by the Hamiltonian

$$H = H_0(p, q) + 2\varepsilon q \cos\varphi(t), \tag{1}$$

where q, p are canonical coordinate and momentum respectively, H_0 is the time-independent unperturbed Hamiltonian, the driving amplitude ε is small, while the driving frequency $\omega(t) = \mathrm{d}\varphi/\mathrm{d}t$ is a slow function of time. It is convenient to transform the problem to action-angle variables I, θ of H_0. Then equation (1) becomes

$$H_r = H_0(I) - \varepsilon a_1(I)\cos\Phi, \tag{2}$$

where we have expanded $q(I,\theta) = \sum a_n(I)\cos(n\theta)$ in Fourier series, used the standard isolated resonance approximation [9], i.e. left the fundamental ($n = 1$) resonance contribution by replacing $\sum a_n(I)\cos(n\theta)\cos\varphi$ by $\frac{1}{2}a_1\cos(\theta-\varphi)$ in the perturbation term in equation (2), and defined the phase mismatch $\Phi \equiv \theta - \varphi - \pi$. The corresponding evolution equations are

$$\begin{aligned} \dot{I} &= -\varepsilon a_1(I)\sin\Phi, \\ \dot{\Phi} &= \Omega(I) - \omega(t) - \varepsilon(\mathrm{d}a_1/\mathrm{d}I)\cos\Phi, \end{aligned} \tag{3}$$

where $\Omega = \mathrm{d}H_0/\mathrm{d}I$ is the unperturbed oscillations frequency. Equations (3) differ from those describing the usual nonlinear resonance by time variation of the driving frequency and characterize the autoresonance phenomenon, where, if initially in resonance, i.e. $\Omega(I(t_0)) \approx \omega(t_0)$ and $\Phi(t_0)$ near zero or π, depending on the sign of the Hessian $\mathrm{d}^2H_0/\mathrm{d}I^2$, the system automatically adjusts its action I so that $\Omega(I(t)) \approx \omega(t)$ despite the variation of the driving frequency. This also means a wide range of controllability of the driven nonlinear system by variation of parameter (the driving frequency). Starting in resonance is a precondition for successful autoresonance. Capture into resonance by passage through resonance is one way to fulfil this precondition. Here we consider the problem of accessibility of the autoresonance in the system, when the system is initially in equilibrium and the driving frequency passes the linear frequency ω_0 in the system (we set $\omega_0 = 1$ for simplicity). Thus, we focus on a weakly nonlinear version of equations (3), where $a_1 \approx \sqrt{2I}, \Omega \approx 1 + \beta I$ (assuming $\beta > 0$ for definiteness), and, locally, near the resonance, the driving frequency $\omega(t) \approx 1 + \alpha t, \alpha > 0$. Then equations (3) become

$$\begin{aligned} \dot{I} &= -\varepsilon\sqrt{2I}\sin\Phi, \\ \dot{\Phi} &= \beta I - \alpha t - (\varepsilon/\sqrt{2I})\cos\Phi. \end{aligned} \tag{4}$$

This problem is governed by the Hamiltonian

$$H_r'(I, \Phi, t) = \tfrac{1}{2}\beta I^2 - \alpha t I - \varepsilon\sqrt{2I}\cos\Phi \tag{5}$$

of the type studied in [6]. Another representation of the problem is obtained by defining $a = \sqrt{2I}$ as a new dynamical variable instead of I and rescaling, i.e., introducing dimensionless time $\tau = \alpha^{1/2}t$, new amplitude A via $A^2 = \frac{1}{2}\beta\alpha^{-1/2}a^2$, and the dimensionless driving amplitude $\mu = \frac{\varepsilon\beta^{1/2}}{\sqrt{2}\alpha^{3/4}}$. Then equations (4) become

$$\begin{aligned} \mathrm{d}A/\mathrm{d}\tau &= -\mu\sin\Phi, \\ \mathrm{d}\Phi/\mathrm{d}\tau &= A^2 - \tau - (\mu/A)\cos\Phi, \end{aligned} \tag{6}$$

or by defining a complex dependent variable, $\psi = A\exp(\mathrm{i}\Phi)$,

$$\mathrm{i}\mathrm{d}\psi/\mathrm{d}\tau + (|\psi|^2 - \tau)\psi = \mu. \tag{7}$$

This inhomogeneous nonlinear Schrödinger-type equation with a *single* parameter describes the passage through linear resonance in nonlinear dynamical systems. The form of this equation also shows the link to several weakly nonlinear driven *wave* problems with slow parameters [4].

We are interested in passage through the resonance and autoresonance in the system, i.e. in the asymptotic solution of equation (7) at large positive τ subject to the initial condition $\psi(\tau_0) = 0$ at some negative τ_0. For zero initial conditions, the initial phase mismatch Φ is ill defined. Furthermore, the second equation (6) has a singularity at $A = 0$. This singularity is due to the choice of representation of ψ in equation (7) via the amplitude and phase. One can avoid this problem by representing ψ as $\psi = x + \mathrm{i}y$, yielding nonsingular evolution equations for the real and imaginary parts x and y for initial $x(\tau_0) = y(\tau_0) = 0$. We used this approach in our numerics with the zero initial condition. Note that the zero initial condition corresponds to the equilibrium of the system, which is a preferred state in many physically relevant situations. Nevertheless, we have also found numerically that small deviations of initial conditions from zero did not lead to a significant change in the evolution described below (particularly in the threshold condition). However, detailed analysis and theory of capture into autoresonance with nonzero, but sufficiently small initial conditions remained outside the scope of the present work. Now, let us show that the zero initial condition on ψ yields phase locking in the system *prior* reaching the linear resonance at $\tau = 0$. Indeed, for $|\psi| \ll 1$ equation (7) can be linearized, and solved via Fresnel integrals, i.e.

$$\psi = -\mathrm{i}\mu\int_{\tau_0}^{\tau} \mathrm{e}^{\mathrm{i}(\tau'^2-\tau^2)/2}\,\mathrm{d}\tau' = -\mu[F(\tau) - F(\tau_0)\,\mathrm{e}^{-\mathrm{i}(\tau^2-\tau_0^2)/2}], \tag{8}$$

with $F(\tau) = f + \mathrm{i}g$ expressed in terms of the auxiliary Fresnel functions f and g [10]. Asymptotically, at sufficiently large τ (practically for $|\tau| > \tau_{\min} = 2$), $F \approx 1/\tau$ [10] and therefore, for $\tau, \tau_0 < -\tau_{\min}$, i.e. prior reaching the linear resonance, we obtain the solution

$$\psi \approx -\frac{\mu}{\tau}\left[1 - \frac{\tau}{\tau_0}\,\mathrm{e}^{-\mathrm{i}(\tau^2-\tau_0^2)/2}\right]. \tag{9}$$

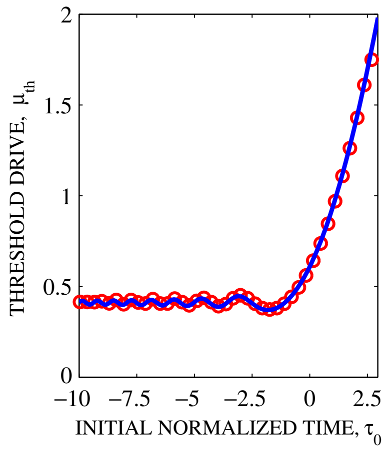

Figure 1. Threshold for autoresonance in the nonlinear Schrödinger equation (7) versus initial normalized time τ_0 (solid line). The circles show a similar threshold for the normalized coupling parameter in the two coupled oscillators system (18).

We observe that $\Phi = Arg(\psi)$ in this solution is bounded, meaning phase locking, with $\Phi \to 0$ at small τ/τ_0. Thus, the driven solution becomes phase locked immediately after the starting point (or somewhat later, if the analysis is extended to nonzero, but small initial conditions [4]). The evolution of the system beyond $-\tau_{\min}$ enters the nonlinear stage and the problem of its final asymptotic state at large positive τ depends on whether the phase locking in the system continues and we discuss this asymptotic limit next.

Finite amplitude asymptotic solutions of equation (7) at positive τ are the *constant* amplitude solution

$$\psi = A_0 \exp(-\mathrm{i}\tau^2/2) \tag{10}$$

and the phase locked ($\Phi = 0$) *growing* amplitude solution

$$\psi = \sqrt{\tau}. \tag{11}$$

The latter is the desired autoresonant limit. But how the system chooses between the two solutions for given initial conditions? The answer is simple; it is the value of the single parameter μ in the system which describes the bifurcation. One finds that for each τ_0 there exists a critical value μ_{th} separating the two types of solutions. Figure 1 shows the dependence of this critical μ on τ_0 found numerically. At large negative τ_0 and until $\tau_0 \approx -\tau_{\min}$, $\mu_{\mathrm{th}} \approx 0.41$ in average, with small oscillations around the average. Therefore, for $t_0 < -2\alpha^{-1/2}$, returning to our original parameters, we have

$$\varepsilon_{th} = 0.58\beta^{-1/2}\alpha^{3/4}. \tag{12}$$

For larger t_0 and even for $t_0 > 0$ a sharp threshold value $\varepsilon_{\mathrm{th}}$ for asymptotic transition to autoresonance still exists, but grows significantly for $t_0 > 0$. Note that in the vicinity of the threshold equation (7) has no small parameters and, thus, in principle, μ_{th} cannot be calculated via a perturbation theory. This is the consequence of the existence of two small parameters in our original problem, such that the rescaled driving amplitude μ involving the ratio of powers of these parameters happens to be of $O(1)$ near the threshold. Still, a heuristic approach [8] leads to the estimate of the threshold within a few percent accuracy. Nevertheless, presently, we have no explanation of small oscillations of μ_{th} around the average of 0.41 for finite initial $|\tau_0|$, leaving the challenge for future research.

In conclusion, for driving amplitudes exceeding the threshold and zero (or sufficiently small) initial condition on the initial amplitude of the driven oscillator, one guarantees a continuing phase locking (autoresonance) in the system allowing efficient control by slow variation of the driving frequency. The question remains regarding how to efficiently phase lock initially *strongly* excited oscillators by passage through resonance. The direct passage does not give a satisfactory answer to this question because the capture probability becomes small for strongly excited oscillations [6]. Nevertheless, one can efficiently capture such oscillations into resonance by coupling to another oscillator, which starts in equilibrium. We discuss this idea next.

3. Autoresonance of coupled oscillations

Consider a system of two oscillators governed by the Hamiltonian

$$H = H_{01}(p_1, q_1) + H_{02}(p_1, q_1, \lambda(t)) + 2\varepsilon q_1 q_2, \tag{13}$$

where H_{0i} describe two decoupled oscillators, one of which has a slow parameter $\lambda(t)$. The choice of the perturbation $2\varepsilon q_1 q_2$ is dictated by simplicity and describes a weak linear coupling. In studying resonant interaction of the oscillators, we again transform to the action-angle variables, I_i, θ_i, of the decoupled system, expand $q_1(I_1, \theta_1) = \sum a_n(I_1)\cos(n\theta_1)$ and $q_2(I_2, \theta_2, \lambda) = \sum b_m(I_2, \lambda)\cos(m\theta_2)$ in Fourier series, and use the isolated resonance approximation in the perturbation term, leaving a single, fundamental resonance ($m = n = 1$) contribution in the Hamiltonian. This yields the approximation

$$H_r = H_{01}(I_1) + H_{02}(I_2, \lambda) + \varepsilon a_1(I_1) b_1(I_2, \lambda)\cos\Phi, \tag{14}$$

where $\Phi = \theta_1 - \theta_2$. The corresponding evolution equations are

$$\begin{aligned} \dot{I}_1 &= +\varepsilon a_1(I_1) b_1(I_2, \lambda)\sin\Phi, \\ \dot{I}_2 &= -\varepsilon a_1(I_1) b_1(I_2, \lambda)\sin\Phi, \\ \dot{\Phi} &= \Omega_1(I_1) - \Omega_2(I_2, \lambda) + \varepsilon\left(b_1\frac{\mathrm{d}a_1}{\mathrm{d}I_1} - a_1\frac{\mathrm{d}b_1}{\mathrm{d}I_2}\right)\cos\Phi. \end{aligned} \tag{15}$$

The first two equations in (15) yield the conservation law $I_1 + I_2 = C = \text{const}$, which allows us to eliminate I_1 and reduce the problem to that of a one degree of freedom for the canonical variables I_2, Φ (we shall use notation I instead of I_2 in the following):

$$\begin{aligned} \dot{I} &= -\varepsilon V_1 \sin\Phi, \\ \dot{\Phi} &= \Omega_1|_{C-I} - \Omega_2|_I - \varepsilon(\mathrm{d}V_1/\mathrm{d}I)\cos\Phi, \end{aligned} \tag{16}$$

where $V_1(I, \lambda) = a_1|_{C-I} b_1|_I$ and the corresponding Hamiltonian

$$H_r' = -H_{01}|_{C-I} - H_{02}|_I - \varepsilon V_1(I, \lambda)\cos\Phi. \tag{17}$$

The similarity between equations (16) and (3) allows us to predict autoresonance in the system when starting in resonance, i.e. a continuing frequency matching $\Omega_1(C - I) \approx \Omega_2(I, \lambda)$ in the system despite variation of λ when this variation is sufficiently slow. As a consequence, one can dynamically control the state of the nonlinear system by varying a parameter, but needs a fine initial tuning of the system for choosing resonant initial conditions. However, if one starts far from resonance and oscillator 2 proceeds from equilibrium (zero amplitude), the autoresonance can be again guaranteed by slow passage through resonance. We demonstrate this effect in an example of coupled 'hard' nonlinear and linear oscillators described by

$$\begin{aligned} \ddot{q}_1 + 4q_1^3 &= 2\varepsilon q_2, \\ \ddot{q}_2 + \lambda^2(t) q_2 &= 2\varepsilon q_1, \end{aligned} \tag{18}$$

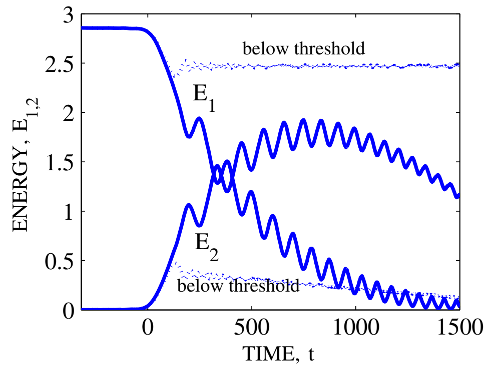

Figure 2. Autoresonant evolution of the energies $E_{1,2}$ of the two coupled oscillators in system (18). The evolution just below the threshold for capture into resonance is shown by dotted lines.

where the frequency $\lambda(t) = \lambda_0 - \alpha t$. Figure 2 shows the evolution of the energies $E_{1,2}$ of the two oscillators subject to initial conditions (at $t_0 = -320)q_{10} = 1.3$, $q_{20} = 0$, $\dot{q}_{10} = \dot{q}_{20} = 0$ and parameters $\lambda_0 = \Omega_1(I = 0) = 2.2$ (in this case the oscillators resonate at $t \approx 0$), $\varepsilon = 0.013$ and $\alpha = 0.001$. One can see a continuous autoresonant decrease of the energy of the nonlinear oscillator for $t > 0$ as the two oscillators stay in resonance for $t > 0$. The dotted line in the figure shows the evolution for the same parameters and initial conditions, but $\varepsilon = 0.0115$, which is just below the threshold value of $\varepsilon_{th} = 0.0119$ for resonant capture. The problem of the threshold in this case reduces to that described in section 2. Indeed, focusing on the case described by equations (18) for simplicity (in this case $H_{02} = \lambda(t)I$ and $b_1 = \sqrt{2I/\lambda(t)}$), a small I expansion in equation (17) yields

$$H_r'' \approx -\tfrac{1}{2}\beta I^2 + \alpha t I - \varepsilon a\sqrt{2I}\cos\Phi, \tag{19}$$

where $\beta = \partial^2 H_{01}/\partial I_1^2$ at $t = 0$ (i.e. at $I_1 \approx C$) and, to lowest order, $a = \lambda_0^{-1/2} a_1|_{I_1=C}$. This Hamiltonian differs from (5) only by the signs in the first two terms. Therefore, all conclusions of section 2 can be applied in the present case with the asymptotic autoresonant value of Φ at π instead of 0, while the autoresonant threshold when starting far from resonance in this case becomes [compare to equation (12)]

$$\varepsilon_{th} = 0.58a^{-1}\beta^{-1/2}\alpha^{3/4}. \tag{20}$$

We have also studied the problem of the threshold in equations (18) numerically for a set of initial times and show the results for $70\varepsilon_{th}$ in figure 1. One observes a full agreement with the results for μ_{th} in the externally driven oscillator case. But now we have a method for efficiently capturing a strongly excited oscillation into resonance. Nevertheless, the autoresonant control of the system by using this scheme comprises a unidirectional process, because the action of the initially excited oscillator can only be lowered after the capture into resonance, as follows from the conservation law $I_1 + I_2 = C$. In the next section, we present a modification of this scheme allowing us to both autoresonantly increase and decrease the action of the excited oscillator. But, prior to addressing this different autoresonant scheme, we discuss an additional phenomenon accompanying autoresonant evolution in figure 2, i.e. small oscillating modulations of the energy around the autoresonantly evolving average. Recall that due to the conservation law in the coupled oscillator problem, it reduces to that of autoresonance in a one degree of freedom system for the canonical variables I, Φ. In this case the smooth autoresonant evolution is always accompanied by slow oscillations having characteristic

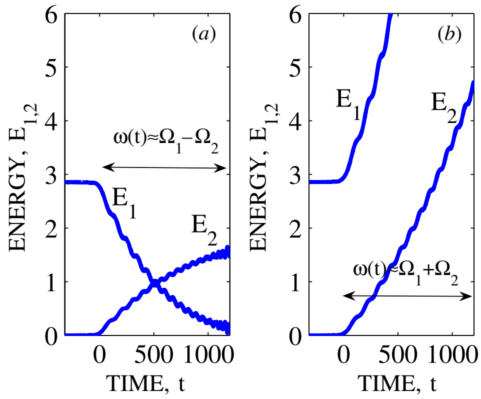

Figure 3. The energies $E_{1,2}$ of the coupled oscillators in system (18) with an oscillating coupling parameter. (a) Passage through $\Omega_1 - \Omega_2$ resonance; (b) passage through $\Omega_1 + \Omega_2$ resonance. The autoresonant decrease or increase of the energy of the initially excited oscillator 1 is obtained by choosing one of the resonances.

frequency of $O(\varepsilon^{1/2})$ [11]. These oscillations are expressions of stability of autoresonant evolution in driven one degree of freedom dynamical systems.

4. Capture into resonance by oscillating the coupling parameter

Consider a system of two coupled oscillators governed by the Hamiltonian

$$H = H_{01}(p_1, q_1) + H_{02}(p_2, q_2) + 2\varepsilon(t)q_1q_2, \tag{21}$$

where, in contrast to (13), the coupling parameter is a constant amplitude oscillation having a slowly varying frequency, $\varepsilon(t) = \varepsilon_0 \cos\int\omega(t)\,\mathrm{d}t$. As before, oscillator 2 is not excited initially. Now the chirping of $\omega(t)$ allows us to pass through either $\Omega_+ \equiv \Omega_1 + \Omega_2$ or $\Omega_- \equiv \Omega_1 - \Omega_2$ resonances, respectively, and to efficiently capture the initially excited oscillator 1 into resonance and subsequently control its state by variation of the driving frequency. As an illustration, we consider example (18) from the last section, but now oscillate the coupling parameter. Figure 3 shows the energies of the two oscillators when the driving frequency ω passes the two combination resonances Ω_+ (figure 3(*a*)) and Ω_- (figure 3(*b*)). We used $\varepsilon_0 = 0.025$ and the same initial conditions as in section 3, while the driving frequency was $\omega(t) = \omega_0 + \alpha t$ with $\omega_0 = 3.2, \alpha = 0.001$ and $\omega_0 = 1.2, \alpha = -0.001$ for the Ω_+ and Ω_- resonances respectively, while the frequency of the linear oscillator was kept constant $\lambda = \lambda_0 = 1$. One observes that the energy of the initially excited nonlinear oscillator in autoresonance can be both increased or decreased by passing through different resonances.

The theory of the process of capture and subsequent autoresonance in this system can be developed similarly to that in section 3. We again transform to the action-angle variables of the decoupled oscillators, yielding the isolated resonance Hamiltonian for the two resonances [compare to equation 14]

$$H_r^{\pm} = H_{01}(I_1) + H_{02}(I_2) + \frac{\varepsilon_0}{2}a_1(I_1)b_1(I_2, \lambda)\cos\Phi_{\pm}, \tag{22}$$

where $\Phi_{\pm} = \theta_1 \pm \theta_2 - \int \omega(t)\,\mathrm{d}t$. The corresponding evolution equations are

$$\begin{aligned} \dot{I}_1 &= +\frac{\varepsilon_0}{2} a_1(I_1) b_1(I_2, \lambda) \sin \Phi_{\pm}, \\ \dot{I}_2 &= \pm\frac{\varepsilon_0}{2} a_1(I_1) b_1(I_2, \lambda) \sin \Phi_{\pm}, \\ \dot{\Phi}_{\pm} &= \Omega_{\pm}(I_1, I_2) - \omega(t) + \frac{\varepsilon_0}{2}\left(b_1 \frac{\mathrm{d}a_1}{\mathrm{d}I_1} \pm a_1 \frac{\mathrm{d}b_1}{\mathrm{d}I_2}\right) \cos \Phi_{\pm}. \end{aligned} \tag{23}$$

These equations yield the conservation laws, $I_1 \mp I_2 = C_{\pm}$ for the $\Omega_{\pm}$ resonances respectively, allowing us to eliminate one of the actions (say I_1) in the problem and reduce it to that of one degree of freedom for the canonical pair I_2, Φ. Again, in the initial trapping stage, for small I_2, one can assume a constant value of a_1 at the resonance, while for the case of the linear oscillator 2, $b_1 = \sqrt{2I_2/\lambda_0}$. This leads to an effective Hamiltonian of form (19), yielding a similar scaling for the thresholds for capture into resonance as described above. Finally, as in the case illustrated in figure 2, one can see slow oscillating modulations of the energy in the advance autoresonant stage in figure 3. The origin of these oscillations was discussed at the end of the last section.

5. Conclusions

We have described the threshold phenomenon in the process of slow capture of driven nonlinear oscillations into resonance when starting in the equilibrium. The threshold problem is described via a generic nonlinear Schrödinger-type equation with a single $O(1)$ parameter and, thus, cannot be solved by using a perturbation theory. The problem of capture of initially large amplitude oscillations was approached by coupling to another oscillation starting in equilibrium. After the capture this coupled system yields unidirectional autoresonant *decrease* of the action of the initially excited oscillator. A different efficient capture into the resonance scheme, based on oscillating the coupling parameter in the coupled oscillators system and passage through combination resonances, allows us to fully control the system, i.e. to both autoresonantly *increase* or *decrease* the action of the initially excited oscillator.

Acknowledgments

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