A water bag model of driven phase space holes in non-neutral plasmas

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The formation and control of stable multiphase space hole structures and the associated Bernstein–Greene–Kruskal modes in trapped pure ion plasmas driven by an oscillating, chirped frequency perturbation are considered. The holes are formed by passing kinetic bounce resonances $\omega_d = n\pi u/L$ in the system, u and L are the longitudinal velocity of the plasma species and the length of the trap, and n is the multiplicity of the resonance (the number of the phase space holes). An adiabatic, quasi-one-dimensional water bag model of this excitation for an initially flat-top distribution of the ions in the trap is suggested, based on the isomorphism with a related problem in infinite quasineutral plasmas. A multiwater bag approach allows us to generalize the theory to other initial distributions. Numerical simulations yield a very good agreement with the theory until the coherent phase space structure is destroyed due to the resonance overlap when the decreasing driving frequency passes a critical value estimated within the water bag theory. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2969738]

I. INTRODUCTION

Excitation and control of large amplitude waves is one of the important goals in plasma physics. For example, carefully shaped electrostatic waves in plasmas can be used for charged particle acceleration[1] or pulse amplification via Raman scattering.[2] In many such applications a nonlinear fluid-type description of the waves in plasmas is sufficient. Nevertheless, when a wave resonates with a part of the velocity distribution of the plasma species, one must use the kinetic theory. An important class of such kinetic waves was discovered by Bernstein, Greene, and Kruskal (BGK), who predicted the existence of dissipationless nonlinear electrostatic modes in plasmas[3] with resonant particles playing an important role. Controlled generation of these waves is the main subject of this work. The BGK modes can be obtained as a final state of large amplitude electrostatic perturbations after relaxation via Landau dumping.[4] These modes were observed in trapped pure electron plasma experiments[5] and in a quasineutral plasma.[6] Phase space hole structures seen in the magnetosphere[7,8] and in the solar wind[9] have been also interpreted as BGK modes.[10,11] A relativistic BGK-type model was suggested in studying electron acceleration by supernova remnant electrostatic shocks waves.[12] Numerical simulations suggested that stable one-dimensional (1D) BGK modes can be also formed dynamically via a two-stream instability,[13] while 3D BGK structures were studied in Ref. 14.

A typical approach to generating BGK modes in aforementioned studies was a decay of a large electrostatic excitation to some BGK equilibrium. More recently, BGK modes in pure electron plasmas in a Penning–Malmberg trap were excited adiabatically[15] by driving the plasma by an external, oscillating, chirped frequency potential. The approach used the idea of autoresonance (AR) (see Ref. 16, and references therein), where the excited wave remained in resonance with the drive continuously despite the time variation of the driving frequency. In the case of driven BGK modes in the trap, this self-phase locking allowed efficient control of the wave amplitude by varying the external parameter. The theory of such autoresonant BGK modes was outlined in Ref. 17 using a kinetic approach, with an *ad hoc* assumption on the form of the electron distribution function in the resonant region. Autoresonant BGK modes can be also generated and controlled in infinite quasineutral plasmas via a persistent Cherenkov-type resonance.[18] An adiabatic, water bag-type theory of driven phase space holes (a particular case of BGK modes) in quasineutral plasmas was developed in Ref. 19. The water bag idea was introduced for studying coherent phase space structures in collisionless plasmas by Bertrand[20] and Berk.[21] The simplest version of the water bag theory describes a uniform (flat-top) distribution of particles in phase space confined between sharp boundaries (limiting trajectories). In this case, the knowledge of the dynamics of the limiting trajectories is sufficient for describing the evolution of the whole phase space structure, since the distribution function in the interior of the region bounded by the limiting trajectories remains constant. In more complex situation one views the phase space distribution function of the plasma particles as a superposition of elementary flat-top distributions, each described by the water bag model. A theory of excitation of BGK modes in infinite quasineutral plasmas via such a multiwater bag approach was developed in Ref. 19.

In this work we will apply the adiabatic water-bag approach to driven BGK modes in a single species plasma in Penning–Malmberg traps. This will remove limitations and *ad hoc* assumptions of the previous theory[17] and yield a description of the phenomenon with no adjustable parameters. In Sec. II we will illustrate formation of alternating-current-driven phase space holes in simulations in the case of a pure ion plasma trapped in a square well potential. In Sec. III we

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will show that the governing equations in this case are isomorphic to those for a driven, infinite quasineutral plasma with an additional symmetry condition preserved by the Vlasov–Poisson dynamics. Later in Sec. III we will use this isomorphism in formulating the water bag theory of driven phase space holes in trapped pure ion plasmas and compare the results of our simulations with the theory for initially flat-top or Maxwellian distributions. Finally, Sec. IV will present our conclusions.

II. TRAPPED PHASE SPACE HOLES IN SIMULATIONS

Consider a driven, pure ion plasma column of length L and radius R, trapped in a square well potential. We describe this system by the following Vlasov–Poisson system:

$$f_t + uf_x - (\varphi_x + \varphi_x^d)f_u = 0, \tag{1}$$

$$\varphi_{xx} - \kappa^2\varphi = \eta^2\left[1 - \int_{-\infty}^{\infty} f(u,x,t)du\right], \tag{2}$$

where various variables and parameters are dimensionless, i.e., the longitudinal velocity u is expressed in units of the characteristic (thermal) velocity v_{th} of the particles, the coordinate x is replaced by x/L, the time t by $v_{\mathrm{th}}t/L$, and the distribution function is normalized as $\int_0^1 dx\int_{-\infty}^{\infty} f(u,x,t)du = 1$. The self-potential $\varphi(x,t)$ and the external driving potentials $\varphi^d(x,t)$ in Eqs. (1) and (2) are expressed in units of mv_{th}^2/e, while $\eta = L/\lambda_D$, λ_D is the Debye length. We assume that the plasma is confined inside a grounded cylindrical wall and, for simplicity, adopt a quasi-one-dimensional theory, where the radial decay of the self-potential is modeled by the screening term $\kappa^2\varphi$ in the Poisson equation with parameter κ scaling as $\kappa \sim L/R \gg 1$ (small aspect ratio case). The square trapping potential well implies bouncing of the ions in the interval $x \in [0,1]$ and perfect reflection from the edges, i.e.,

$$f(u,0,t) = f(-u,0,t), \quad f(u,1,t) = f(-u,1,t), \tag{3}$$

guaranteeing conservation of the number of ions in the trap. Note that the governing Vlasov–Poisson system for a pure electron plasma in a square well potential is the same as above with φ, φ^d replaced by $-\varphi, -\varphi^d$. With this replacement, Eqs. (1) and (2) also describe the oscillatory part of the electron distribution function (and the associated self-potential) in a quasineutral plasma with frozen ion species (see Sec. III).

We will assume a flat-top initial phase space distribution of the ions in the trap in our simulations below, i.e., $f(u,x,t) = 1/2$ for $u \in [-1,1]$ and $f(u,x,t) = 0$ for $|u| > 1$. We will also assume that the plasma is driven by a small amplitude standing wave-type potential

$$\varphi^d = 2\varepsilon\cos(kx)\cos(\psi_d), \tag{4}$$

where the driving frequency, $\omega_d(t) = d\psi_d/dt$, is a slowly *decreasing* function of time, passing through the bounce resonances $\omega_d = ku$ with different plasma particles, where $k = n\pi$, n is an integer. We illustrate formation of phase-space holes and the associated growing amplitude BGK mode as obtained in simulations in this system in Figs. 1 and 2. A standard pseudospectral method[22] was used in the simulations.

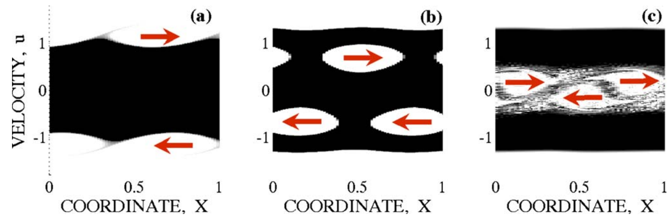

FIG. 1. (Color online) The formation of autoresonant phase space holes in a trapped pure-ions plasma at different driving phase velocities: (a) A surface wave in the phase space fluid at $u_d = 1.16$, (b) fully developed phase locked holes at $u_d = 0.7$, and (c) destruction of the holes at $u_d = 0.21$ ($u_d < u_d^{\mathrm{cr}}$). The arrows show the direction of motion of the coherent structures.

For avoiding numerical difficulties characteristic of Vlasov codes for distributions having large phase-space gradients, we have introduced an artificial high frequency filters at grid scales in x and u. We used parameters $\kappa = 10$, $\eta = 5$, and $n = 3$ (this yields formation of three holes in phase space) in our numerical example. The amplitude of the external driving potential was $\varepsilon = 0.01$, the initial driving wave phase velocity $u_d(0) = \omega_d(0)/k = 1.7$ was outside the initial ion distribution and the driving frequency decreased at uniform chirp rate of $\alpha = 0.06$. The characteristic physical parameters of the trapped plasma in this example could model the case of $L = 10$ cm, $R = 1$ cm, and $\lambda_D = 2$ cm.

Figure 1 shows snapshots of the phase space distributions at three different times corresponding to different stages of evolution of the phase space distribution in the driven system. The initial excitation stage [Fig. 1(a)], characterized by a wave on the surface of the phase space fluid, continues as long as the driving phase velocity $u_d(t) = \omega_d(t)/k$ is outside the bulk of the distribution. After passage of u_d through the boundary of the phase space fluid, one observes formation of trapped phase space holes [Fig. 1(b)]. The holes bounce in the trap resulting in two counterpropagating (the arrows in the figure show the direction of propagation) hole systems in the phase space. The bounce fre-

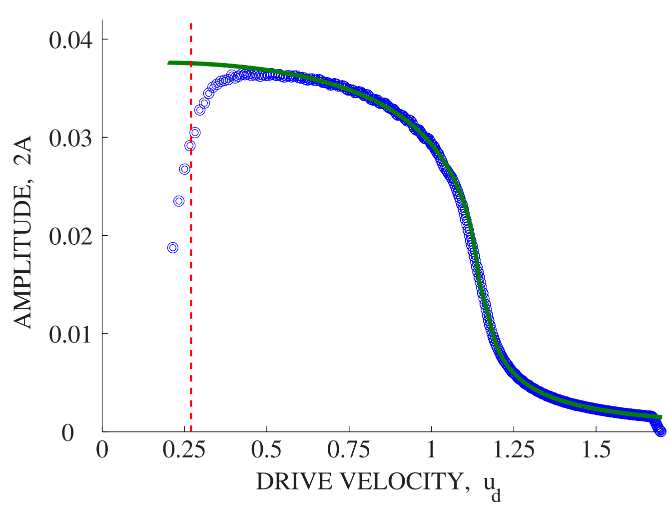

FIG. 2. (Color online) The amplitude $2A$ of the self-potential of the autoresonant BGK mode in the example in Fig. 1 vs the driving phase velocity $u_d(t)$ as given by the water bag theory (solid line) and simulation (circles). The dashed line indicates the critical driving velocity u_d^{cr} at which the coherent hole structure is destroyed due to the resonance overlap (see Sec. III).

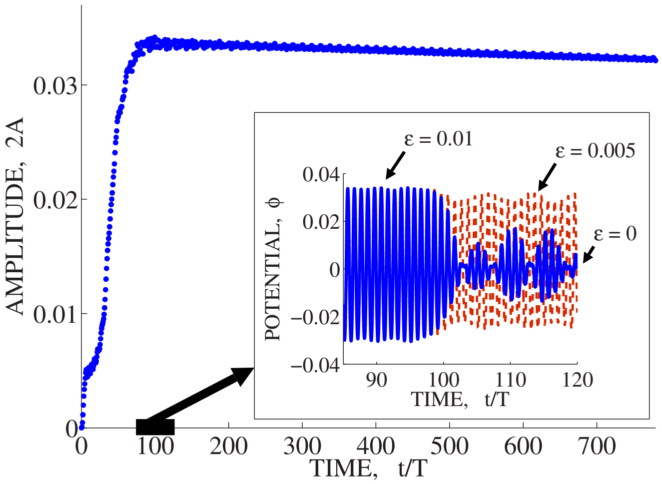

FIG. 3. (Color online) Stability of driven trapped BGK modes. The amplitude $2A$ of the self-potential in simulations (dots) vs normalized time for driving frequency fixed at $t/T>100$. The inset shows the evolution of the self-potential $\varphi(t)$ in a small time window around $t/T=100$ in two cases: The driving amplitude is reduced by half to $\epsilon=0.005$ (dashed line) or to zero (solid line) at $t/T=100$. The driven structure is destroyed in the latter case.

quency of the holes is locked to that of the chirped frequency driving perturbation, a characteristic of autoresonance in the driven system. As the driving frequency decreases, the distance between the counterpropagating structures in velocity space decreases until the holes overlap at certain times [Fig. 1(c)], leading to a rapid phase space mixing and destruction of the coherent structures. The amplitude of the wave potential (a BGK mode) associated with the phase space holes is shown in Fig. 2 versus $u_d(t)$. The solid line in the figure shows the prediction of our water bag theory (see Sec. III), while the open circles are the results of the simulations, showing a very good agreement. One also observes a continuing growth of the wave amplitude with the decrease of $u_d(t)$, until the mode is rapidly destroyed due to the overlap of the counterpropagating hole structures in phase space [see Fig. 1(c)]. The vertical dashed line in Fig. 2 indicates the critical driving phase velocity of $u_d^{cr}=0.27$, when the overlap and strong interaction of the driven holes in phase space begins (see our theory below).

We complete this section by discussing the stability of the phase space holes. It is known[21] that free phase space holes in an infinite quasineutral plasma interact, yielding a decay of the associated BGK modes. We have observed the same phenomenon in our simulations in trapped, pure ion plasmas. Nevertheless, we have found that *driven* phase space holes are stable as illustrated in Fig. 3. Figure 3 shows the results of a simulation with the same parameters as in Fig. 1 and the initial driving phase velocity, $u_d(0)=1.3$ (again outside the bulk of the distribution). The phase velocity in the simulation slowly decreased in time reaching the value of $u_d^f=0.8$ (with fully developed phase space holes at this stage) and remained fixed at this value at later times. The period of the driving perturbation in this final quasisteady state was $T=2\pi/\omega_d^f=2.618$. One can see in the figure that the self-field amplitude decreased by 5% only after more than 600 periods of oscillation, indicating the stability of the driven phase space structure. The observed slow decay is probably due to the above-mentioned artificial viscosity, introduced in simulations for numerical stability. We further studied the stability of the excited phase space structures by reducing the driving field after reaching the quasisteady state. The inset in Fig. 3 shows the actual self-potential $\varphi(t)$ in a relatively short time window in two such simulations. In the first run (the solid line) we reduced the driving amplitude ε by half starting at $t/T=100$. One can see that the self-potential remained nearly the same but with small modulations introduced by a sudden change in the driving amplitude. In the second simulation in the inset in Fig. 3 (the dotted line) the driving potential was switched off at $t/T=100$, resulting in the destruction of the phase space structure and a rapid (few oscillation periods) decay of the self-potential. Thus, a sufficiently strong external drive can overcome the interaction between the phase space holes and stabilize the BGK mode at longer times.

III. THE WATER BAG MODEL FOR TRAPPED PURE ION PLASMAS

Our theory of the driven, trapped phase-space holes in pure ion plasmas uses the isomorphism allowing application of the results of the water bag description of driven BGK modes in quasineutral plasmas[19] in the *trapped* single species system. Consider an infinite, quasineutral plasma, where the ions are stationary, while the electron distribution function and the oscillatory part of the self-potential are described by Eqs. (1) and (2) (with a change of sign of the potentials as mentioned above). Assume that the driving potential φ^d, as well as $f(u,x,t)$ and $\varphi(x,t)$ are spatially periodic, i.e., $\varphi^d(x-1,t)=\varphi^d(x+1,t)$ and, similarly, for f and φ. Assume also that f and φ in this system satisfy the following symmetry conditions:

$$f(u,x,t) = f(-u,-x,t), \tag{5}$$

$$\varphi(x,t) = \varphi(-x,t). \tag{6}$$

Then, one can show that these f and φ in the interval $x\in[0,1]$ are also solutions for a single species system (taking into account the sign of the charge) trapped in a square well potential. Indeed, the main difference between the two systems is in using the periodic boundary conditions in the unbounded plasma case instead of the reflecting boundary conditions for a trapped, single species plasma. However, the combination of the symmetry and periodicity conditions compensate for this difference. Indeed, by substitution of $x=0$ and $x=1$ in the symmetry conditions (5) we obtain $f(u,0,t)=f(-u,0,t)$, and $f(u,1,t)=f(-u,-1,t)=f(-u,1,t)$, i.e., the desired boundary conditions. Similarly, the symmetry condition (6) on φ applied at x=1, yields $\varphi(1,t)=\varphi(-1,t)$, which is consistent with the periodicity condition. Finally, for applications of this isomorphism argument, one must show that solutions satisfying the above-mentioned periodicity and symmetry condition exist in the unbounded quasineutral plasma case. Indeed, such solutions can be constructed by choosing the initial conditions $[f(u,x,0)$ and $\varphi(x,0)]$ and the external driving potential $\varphi^d(x,t)$ all satisfying the symmetry and the periodicity conditions. Then, since the Vlasov–Poisson system preserves the reflection symmetry, the solutions $f(u,x,t)$ and $\varphi(x,t)$ at later times will also satisfy the additional symmetry conditions and, therefore, evaluated in the interval $x \in [0,1]$, will also solve our original trapped, non-neutral plasma problem.

At this stage, we take advantage of the above-mentioned isomorphism and apply the water bag model for infinite quasineutral plasmas (see the Appendix) to our trapped pure ion plasma case. We recall that the pure ion plasma in a square potential well is described by the Vlasov–Poisson systems (1) and (2) with a perfect reflection boundary conditions (3) and is driven by a chirped frequency standing wave equation (4). Since $f(u,x,t)$ and $\varphi^d(x,t)$ in this case satisfy conditions (5) and (6), we can use the isomorphism to solve the trapped plasma problem in the interval $x \in [0,1]$ by constructing the Vlasov–Poisson solution in the interval $x \in [-1,1]$ for the corresponding infinite, quasineutral plasma with periodic boundary conditions. The goal is achieved by writing the driving potential in the infinite plasma as a sum of two traveling waves $\varphi^d = \varphi^d_+ + \varphi^d_+$, where $\varphi^d_\pm = \varepsilon \cos(kx \pm \psi_d)$. The Poisson equation (2) is linear, hence, one can write the self-potential as a sum of two potentials, $\varphi = \varphi_+ + \varphi_-$, each associated with the driving waves φ^d_+ and φ^d_-, respectively. The interaction between these two waves in the Poisson equation is indirect via the density $n(x,t) = \int f du$ described by the (nonlinear) Vlasov equation (1). However, since each of the driven counterpropagating hole structures is phase-locked (see below) to a different driving component $\varphi^d_\pm$, as long as the corresponding phase velocities $u_d = \pm \omega_d / k$ are well separated, this interaction is *nonresonant*. As a result, if the driving velocity $u_d = \omega_d / k$ is sufficiently large ($u_d > u_d^{\rm cr}$, see below), we can write the total self-potential as a sum of the two *independent* contributions, $\varphi = \varphi_+ + \varphi_-$, each induced by its driving potential $\varphi^d_\pm$, Eq. (A1).

The formation and subsequent evolution of phase space holes by a single chirped frequency traveling wave in a quasineutral plasma was discussed in Ref. 19 by using an adiabatic water bag model. Here we adopt the same approach and briefly describe it in the Appendix for completeness. The approach uses the perfect phase-locking assumption and the adiabatic invariance of the action integrals associated with limiting trajectories bounding the flat-top distribution in phase space for calculating the resonant Fourier component amplitude A of the driven self-potential in the problem [see Eqs. (A3), (A7), and (A8)]. In our case, we simultaneously apply two driving waves, each resulting in the same self-potential amplitudes A of the corresponding noninteracting BGK structures $\varphi_\pm$, because the two structures differ in the direction of propagation only, having no effect on the Fourier amplitude A, Eq. (A3). Then, the amplitude of the self-potential in the corresponding trapped pure ion plasma is simply $|\varphi|_{\max} = |\varphi_+ + \varphi_-|_{\max} = 2A$. The solid line in Fig. 2 shows the results of the water bag theory for $|\varphi|_{\max}$. One can see that the theory is in a very good agreement with simulations until the interaction between the counterpropagating hole structures becomes important, as the driving frequency decreases and reaches some critical value $u_d^{\rm cr}$ indicated by the vertical dashed line in the figure. We estimate this critical velocity by calculating the half-width Δu of the phase space holes in the velocity space and setting $u_d^{\rm cr} = \Delta u$. The overlap of the resonances associated with the two driving waves at this driving velocity leads to the violation of the single resonance approximation in the theory. The overlap leads to a complex interaction of the holes in phase space, resulting in a rapid phase space mixing and subsequent decay of the self-potential. Within the single resonance approximation, the boundary u_0 (limiting trajectory) of a single hole in phase space is given by Eq. (A6) and its maximum half-width is $\Delta u = \sqrt{2(H_0 + A)}$, where H_0 is the energy of the limiting trajectory. Our water bag model yields $\Delta u = 0.27$ (dashed line in Fig. 2) in the example in Fig. 1, which is in good agreement with the simulation results.

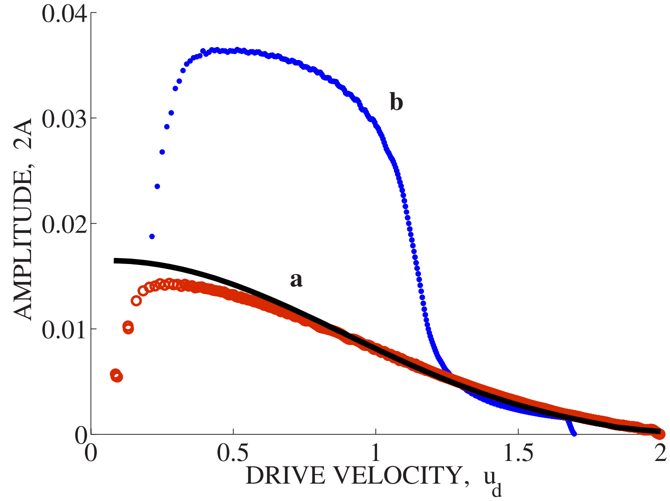

FIG. 4. (Color online) The amplitude $2A$ of the trapped autoresonant BGK mode vs the driving phase velocity u_d for (a) the Maxwellian distribution in simulations (open circles) and via the multiwater bag theory (solid line); (b) the simulation results for the flat-top initial distribution (from Fig. 2).

We complete our discussion of driven, trapped BGK modes in pure ion plasmas by comparing the theory and simulations for the case of initially Maxwellian distribution $f(u,x,t=0) = (2\pi)^{-1/2} \exp(-u^2/2)$ of the ions in the potential well. The theory in this case uses the multiwater bag approach,[19] which views the initial distribution as a collection of many small height flat-top layers in phase-space. The dynamics of the phase space structures in each of these elementary flat-top distributions is the same as described above, but with the coupling via the self-potential governed by the Poisson equation. The results of these calculations for the amplitude of the self-potential are shown in Fig. 4 by the solid line for the same parameters as in Figs. 1 and 2, but with $\varepsilon = 0.0036$, while the simulation results are shown by open circles. In the same figure, for comparison, we show the results of simulations from Fig. 2 for the flat-top distribution (dotted line). One can observe a slower growth of the amplitude in the Maxwellian distribution case, but a similar decay of the field at smaller driving phase velocity due to the resonance overlap.

IV. CONCLUSIONS

(a) We have studied the formation and control of phase space holes of a pure ion plasma confined in a square well potential. The goal was achieved by applying an oscillating, external driving potential with a slowly down-chirped frequency resonating in the tail of the ion distribution initially. The number of the particles captured into resonance remains nearly constant during the excitation process, yielding an almost empty phase locked resonant buckets in phase space. The associate self-field does not exhibit Landau damping and, thus can be viewed as a BGK mode. When the chirped driving frequency becomes small, the excited phase space structure is destroyed due to the resonance overlap in the problem.

(b) We have used the isomorphism between trapped non-neutral and infinite quasineutral plasmas under appropriate symmetry conditions. This isomorphism allowed us to apply a water bag theory developed for a related problem in quasineutral plasmas to the trapped BGK modes. The Vlasov–Poisson system in this model is replaced by a set of algebraic equations for the limiting trajectories in phase space. We have applied this approach in calculating the self-potential associated with the phase space holes for initially flat-top and Maxwellian initial distribution (via a multiwater bag theory), yielding a very good agreement with the results of the numerical simulations.

(c) The simulations have shown robust stability of the trapped, driven BGK modes, until the chirped driving frequency reached some critical value beyond which the hole structures were destroyed due to the resonance overlap in the associated dynamics. The water bag theory allowed us to estimate this critical frequency.

(d) We have found that the phase-locked, driven phase space structures with more than one hole were stable at longer times (for over 600 oscillations) with a sufficiently strong driving. Nevertheless, when the drive was switched off, the excited structure was rapidly destroyed due to the interaction between neighboring holes.

(e) A theory describing the stability of the trapped, driven BGK modes must remove the perfect phase locking assumption of our water bag theory. It also seems interesting to extend the theory to trapping potentials different from a square well.

ACKNOWLEDGMENTS

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APPENDIX: THE DRIVEN WATER BAG MODEL

In this appendix, we present a brief description of the water bag model introduced in Ref. 19 for a driven, unbounded quasineutral plasma with an initially flat-top velocity distribution. The model allows us to replace the Vlasov–Poisson system of integro-differential equations by a set of algebraic equations for the amplitude of the self-potential and the limiting trajectories bounding the electron distribution in phase space. Consider infinite, quasineutral, driven Vlasov–Poisson systems (1) and (2), where f is the distribution function of the electrons, while the ions are stationary. In contrast to the 1D theory of Ref. 19, we use a quasi-1D plasma model by adding a radial screening term in the Poisson equation. We will see that this addition leads to similar results with a rescaled wave vector in the problem. We assume that the initial distribution is flat-top of height $1/2$ and confined between $u=\pm 1$ in the velocity space. For generating phase space holes in this system we apply a traveling wave-type external potential,

$$\varphi^d(x,t) = -\varepsilon \cos\left[kx - \int \omega_d(t)dt\right], \tag{A1}$$

where $k=n\pi$. The driving frequency is chirped, $\omega_d(t)=\omega_0-\alpha t$, and we assume that initially, at $t=0$, the driving wave does not Cherenkov resonate with the distribution, i.e., $\omega_0/k>1$. Later, as the driving frequency decreases, the phase velocity of the driving potential enters the bulk of the distribution and passes successive Cherenkov resonances with different plasma electrons in phase space. It was shown in Ref. 19 that this process results in the formation of electron phase space holes locked to the chirped frequency drive, so that the holes drift in the velocity space at the rate of variation of the driving phase velocity. The self-potential associated with these holes is a BGK mode. The amplitude of the mode, as well as the position of the holes in the phase-space are fully controlled by the external driving frequency.

At all stages of formation of the electron holes in the water bag model one views the phase space distribution as confined between slowly moving curves (*limiting trajectories*) in phase space, representing the evolving sharp edges of the distribution. The incompressibility of phase space fluid ensures that the distribution function remains constant between these limiting trajectories during the evolution, so the dynamics of the limiting trajectories only is necessary for calculating the self-potential in the problem. The theory of this evolution involves two levels of description. First, one considers a stationary, traveling driven state, when the driving frequency is constant. Later, using the adiabatic theory, one follows the dynamics of the limiting trajectories in the case of slowly varying parameters (ω_d and/or ε). This yields a complete description of formation and evolution of the driven, chirped BGK modes in quasineutral plasmas. The two levels of the theory are briefly described below.

1. Stationary driven BGK modes

In the case of constant driving parameters, the dynamics of an electron on a limiting trajectory is given by

$$\ddot{x} = -\varepsilon k \sin(kx - \omega_d t) - \varphi_x. \tag{A2}$$

We make two assumptions in the theory. The first is the *ideal phase locking* in the driven system, i.e., that the self-potential has the same phase as the drive. In addition, we assume the simple harmonicity of the self-potential,

$$\varphi = -A\cos(kx - \omega_d t), \quad \text{(A3)}$$

i.e., neglect the higher harmonics in representing φ. Then, the total potential in the system is

$$\varphi^{\text{total}} = -A'\cos(kx - \omega_d t), \quad \text{(A4)}$$

where $A' = A + \varepsilon$ and the limiting trajectory is governed by the Hamiltonian $H = \frac{1}{2}u^2 - A'\cos(kx - \omega_d t)$. Next, it is convenient to use a different canonical set, i.e., u and $x' = x - u_d t$ (coordinate in the frame moving with the phase velocity $u_d = \omega_d/k$ of the wave) instead of u and x. The corresponding Hamiltonian is

$$\tilde{H}(u,x') = \frac{1}{2}(u - u_d)^2 - A'\cos(kx'), \quad \text{(A5)}$$

which can be used to write the corresponding trajectory in phase space

$$u = u_d \pm \sqrt{2[\tilde{H} + A'\cos(kx')]}. \quad \text{(A6)}$$

By choosing different energies $\tilde{H} = \tilde{H}_i$ one can specify different limiting trajectories $u_i(x')$, while the whole distribution function is confined between these trajectories. Two different sets of limiting trajectories are relevant in our case. One set involves two trajectories, $u_{1,2}$, with $\tilde{H}_{1,2} > A'$ and negative signs in Eq. (A6). These two passing trajectories correspond to the case when the wave phase velocity is outside the bulk of the distribution, $u_d > 1$. The Poisson equation, in this case within the water bag model yields

$$A(k^2 + \kappa^2) = \eta^2(F_2 - F_1), \quad \text{(A7)}$$

where $F_{1,2} = \frac{1}{2}\int_{-1}^{1} u_{1,2}\cos(kx')dx'$. This equation allows us to find (by iterations, for example) the self-field amplitude in the problem. In the case when the phase space hole is inside the electron distribution, one chooses three trajectories, where the two outer trajectories $u_{1,2}$ are passing, i.e., $\tilde{H}_{1,2} > A'$, but have different signs at the square root in Eq. (A6), while the internal trajectory u_0 is bound, $\tilde{H}_0 < A'$, and describes the boundary of the hole. The Poisson equation (A7) in this case gives

$$A(k^2 + \kappa^2) = \eta^2(F_1 + F_2 + F_0), \quad \text{(A8)}$$

where $F_0 = \oint u_0 \cos(kx')dx'$. The process of solution of the equations for A by iteration simplifies by observing that various terms in Eqs. (A7) and (A8) can be expressed via elliptic integrals,

$$F_{1,2} = \frac{2}{3\pi}\sqrt{2A'(g_{1,2} - 1)}[(g_{1,2} + 1)K(\mu_{1,2}) - g_{1,2}E(\mu_{1,2})] \quad \text{(A9)}$$

and, similarly,

$$F_0 = \frac{4}{3\pi}\sqrt{A'}[2g_0E(\mu_0) + (1 - g_0)K(\mu_0)], \quad \text{(A10)}$$

where $g_i = \tilde{H}_i/A'$, $\mu_{1,2} = 2/g_{1,2} - 1$ and $\mu_0 = 1 + g_0/2$, and K and E are the complete elliptic integrals of the first and second kind, respectively.

2. Adiabatic driven phase space holes

The transition from the stationary driven water bag equilibria described above to the case of slowly varying ε and ω_d can be accomplished by observing that dynamically, we still have a Hamiltonian picture of the limiting trajectories, where now the Hamiltonian $\tilde{H}(u,x';\varepsilon,u_d)$ depends on slow parameters. Thus, by the adiabatic theory, each limiting trajectory possesses an adiabatic invariant, i.e., $J_{1,2}(\tilde{H}_{1,2},\varepsilon,u_d) = \frac{1}{4}\int_{-1}^{1} u_{1,2}dx'$ and $J_0(\tilde{H}_0,\varepsilon,u_d) = \frac{1}{4}\oint u_0 dx'$ remain constant despite variation of parameters. This yields an additional equation for each limiting trajectory, which, in combination with the Poisson equations (A7) or (A8), comprise a complete set for finding A and $H_{1,2}$ (or A and $H_{0,1,2}$ for the holes inside the distribution) as functions of $u_d(t)$. The detailed discussion of passage through the distribution boundary in phase space and formation of the holes can be found in Ref. 19 and is based on the assumption of a small variation of the self-potential during the passage through the boundary. This analysis allows us to connect the two stages of evolution (with and without the holes) in the adiabatic description.

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