Driven chirped vorticity holes

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The formation and control of \textit{m}-fold symmetric vorticity hole structures in a two-dimensional vortex patch with a line vortex core is studied within an adiabatic contour dynamics theory. The holes are formed by subjecting an initially circular vortex patch to an \textit{m}-fold symmetric, oscillating, chirped frequency straining flow. The theory uses adiabatic invariants associated with the boundaries of the patch and describes all stages of evolution in the driven system, i.e., the emergence of the \textit{m}-fold symmetric \textit{V}-state, resonant passage through the boundary of the \textit{V}-state, formation of vorticity holes, and autoresonant dynamics of the driven holes inside the vortex structure. The results of the theory are in a good agreement with the fast multipole-type simulations. In contrast to free (unstrained) \textit{m}-fold symmetric vorticity hole structures, where only \textit{m}=1 case is stable, resonantly driven phase-locked \textit{m} \gg 1 vorticity holes can be stabilized by the external strain. More complex, stable \textit{m}-fold symmetric vorticity structures with local minima in vorticity distributions can be formed from initially axisymmetric distributions by external, chirped frequency strains. © 2008 American Institute of Physics. [DOI: 10.1063/1.2964361]

I. INTRODUCTION

Formation of coherent vortex structures with controlled properties is an important problem of modern physics. The knowledge of realizable routes to emergence of such structures may be useful in many physical applications. The goal of this paper is to study formation and control of two-dimensional (2D) \textit{m}-fold symmetric vortex states involving vorticity holes in ideal fluids. The existence of stable \textit{m}-fold symmetric coherent vortex configurations by itself is a non-trivial problem of fluid dynamics and is closely related to that of axisymmetrization of vortices. In this context, Melander et al.\textsuperscript{1} studied spatially smooth elliptical vortices. It is well known\textsuperscript{2} that a uniform elliptical (Kirchhoff) vortex patch is a steady state solution of the Euler equation. This solution is linearly stable if the aspect ratio of the ellipse is less than 3 and remains stable in the presence of a strain.\textsuperscript{3} Thus, one could expect that spatially smooth elliptical vortices would have similar properties. However, it was found in simulations\textsuperscript{1} that the dynamics of these vortices is different and they may tend to relax toward axisymmetry via generation of filaments and vorticity holes. Later, similar vorticity holes were studied in experiments on trapped magnetized electron clouds.\textsuperscript{4} The azimuthal dynamics of such clouds is isomorphic to 2D vortex dynamics in ideal fluids with the electron density playing the role of vorticity and the electric potential that of the stream function in fluid dynamics. Different radial electron density distributions of the cloud could be prepared in experiments in Ref. 4. Threefold symmetric density holes in the distribution were unstable, relaxing first to \textit{m}=2 state, which later decayed into \textit{m}=1 (single hole) state. The instability was explained by a beat-wave resonance between the azimuthal modes. The numerical simulations of analogous vortex structures in Ref. 5 led to similar conclusions. Additional results on the axisymmetrization problem were obtained in Ref. 6, wherein the dynamics of two types of vortices are compared both having constant vorticity lines of the form of ellipses. One of these vortices had the same structure as in Ref. 1, while the second had a finite size boundary. The authors concluded that the smooth distribution tends to axisymmetry through the filamentation process as suggested in Ref. 1, but the vortex on the finite support retained its initial elliptical symmetry.

In addition to the axisymmetrization problem, there was another line of research of a family of \textit{m}-fold symmetric coherent vortex structures on a finite support, which started with the discovery of the \textit{V}-states by Deem and Zabusky.\textsuperscript{7} These states were generalizations of Kirchhoff elliptic vortices to \textit{m}-fold symmetry. Later, similar spatial structures have been obtain for 2D irrotational liquid drops\textsuperscript{8} and electron systems.\textsuperscript{9} The most complete theory of stability of the \textit{V}-states can be found in Ref. 10, studying a single parameter (\textit{p}) family of vortices, which could be transformed from a compact, top-hat-type vorticity distribution (for \textit{p}=0) to a Gaussian at \textit{p} \rightarrow \infty. The top-hat vortices allowed isolated, stable \textit{m}-fold symmetric linear and nonlinear Kelvin modes.\textsuperscript{11} It was shown in Ref. 10 that when approaching the Gaussian vorticity distribution, the Kelvin modes become weakly damped, but the damping is algebraic and not exponential, so the mode is transformed into a Kelvin quasimode. One of the important results was the existence of the threshold on the initial perturbation amplitude such that below the threshold the quasimodes remained damped, leading to a complete axisymmetrization of the perturbed vortex, while above the threshold, the damping process was arrested and the final vortex state retained its initial \textit{m}-fold symmetry. To

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some extent, this situation was similar to the emergence of dissipationless Bernstein–Greene–Kruskal (BGK) modes in plasmas.12 The theory suggested an explanation of the results obtained in Refs. 1 and 6, where some initially smooth elliptical vorticity distributions retain their elliptic symmetry during their evolution, whereas others lost their ellipticity. In addition to V-states, Crowdy13,14 found a number multipolar vortex structure equilibria by complex analysis.

The formation of all m-fold symmetric vortex structures discussed above required either very special initial conditions or poorly controllable strong excitation-relaxation approach. Recently, in a series of papers, a different adiabatic autoresonance idea had been applied15–18 to the formation of m-fold symmetric vortices and did not require nontrivial initial conditions or intermediate relaxation stage. The autoresonance is a salient property of many nonlinear systems (vortices, for example) to automatically self-adjust their parameters for staying in resonance with chirped frequency external perturbations (an oscillating strain in vortex applications). Generally, in autoresonance the nonlinear system (oscillators, waves, vortices, etc.) remains phase-locked with an external drive, despite the slow variation of the driving frequency, allowing efficient control of the system by variation of external parameters. Furthermore, the driving amplitude can be small for autoresonance to occur when the frequency chirp is sufficiently slow, i.e., the controllability can be achieved by using small perturbations. These ideas were tested in experiments on excitation and control of the m = 1 diocotron mode in magnetized, pure electron plasmas15 (the analog of a line vortex inside a cylindrical boundary). Later, a similar approach was used in controlling ellipticity of Kirchhoff vortices16 and formation of the V-states.17,18

In the present work, we discuss the autoresonant formation and control of m-fold symmetric vorticity holes. The problem is similar to that for electron phase-space holes (a class of BGK modes) in alternate-current- (ac)-driven plasmas. This BGK paradigm was studied recently via a mixed fluid-kinetic approach,19 followed by a contour dynamics theory for driven phase-space holes and more complex BGK structures.20 Here we will apply similar ideas to the formation and control of vorticity holes in 2D ideal fluids. Our presentation will be as follows. We will illustrate our ideas in numerical simulations in Sec. II. In Sec. III we will develop a theory of driven contour dynamics equilibria with fixed driving parameters for a flat-top initial vorticity distribution with a line vortex core. Later, in Sec. IV, we will use adiabatic invariants in our driven problem and generalize the theory to describe all stages of formation and evolution of chirped frequency, driven vorticity holes inside a uniform vortex patch. In addition, we will present simulations of adiabatic formation of m-fold symmetric vortices of a more general nonmonotonic shape by starting from smooth axisymmetric vorticity distributions and discuss a multicontour dynamics extension of the theory for this case. Finally, our conclusions will be presented in Sec. V.

II. AUTORESONANT VORTICITY HOLES IN SIMULATIONS

The driven 2D vorticity dynamics in ideal fluids is governed by a system of the Euler and Poisson equations for vorticity \( \omega(x,y,t) \) and the stream function \( \psi(x,y,t) \)

\[
\begin{align*}
\frac{\partial \omega}{\partial t} + v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} &= 0, \\
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= -\omega,
\end{align*}
\]

where the fluid velocity field is \( v = -e_\times \nabla (\psi + \psi_{ext}) \) and \( \psi_{ext}(x,y,t) \) represents an external straining flow. There are several methods to solve this set of equations numerically. The simplest is based on the fast Fourier transform algorithm,15 but is applicable with periodic boundary conditions only. In addition, large gradient characteristics of frequent filamentation of the vorticity field require addition of artificial viscosity terms in the Euler equation for stability of the numerical algorithm. Another numerical approach, based on contour dynamics, was suggested in Ref. 21. This method describes the evolution of constant vorticity contour lines and reduces the problem to one-dimensional (1D) integral equations instead of partial differential equations. The approach encounters difficulties in dealing with the filamentation, and special regularization procedures were suggested to treat this problem.22 A newer numerical approach is based on the fast multipole method (FMM) suggested by Greengard and Rokhlin.23 Roughly speaking, the method allows fast evaluation of pairwise interactions between N particles [term “fast” denotes the need of only O(N) numerical operations]. The interaction has to be of a potential type with the potential being a solution of the Poisson equation. The implementation of the method is simple in 2D cases (the potential has log r-type solutions) and is more sophisticated for higher dimensionality. The electrons and ions in plasmas are natural objects for FMM applications. In 2D vortex dynamics, the fluid is viewed as a collection of fluid elements (point vortices) and the equations of motion have to be written in Lagrangian coordinates. First such application was reported in Ref. 6, where the approach was compared to other methods. One of the deficiencies of the FMM method is that the local density of vortices is not controlled and, therefore, there exist fluctuations (numerical noise) due to the discreetness of the point vortices. Consequently, a sufficiently large number of fluid elements must be used in simulations to minimize this problem.

In the present work, we use the FMM algorithm suggested in Ref. 6 in application to our driven vorticity problem. We replace Eq. (1) with the dynamical equations for N point vortices and the Poisson equation.
\[
\frac{dx_i}{dt} = v_{x_i}, \quad v_{x_i} = \frac{\partial \psi}{\partial y}, \quad v_{y_i} = -\frac{\partial \psi}{\partial x}, \quad (2)
\]

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\sum_{i=1}^{N} \omega_i \delta(r - r_i).
\]

\(\omega_i=\text{const}\) being the vorticity of the elements and \(r_i=(x_i, y_i)\), \(v_i=(u_i, v_i)\), \(i=1, \ldots, N\) their position and velocity vectors. The FMM calculates the stream function from the last equation in Eq. (2). Since the equation is linear, the stream function at point \(r\) equals the sum of the stream functions due to each element, i.e. (in 2D case) \(\psi=1/2 \pi \sum_{i=1}^{N} \omega_i \ln|\mathbf{r} - \mathbf{r}_i|\).

Thus, it seems that one needs to perform \(O(N^2)\) operations to calculate the velocities \(v_i\) of \(N\) elements, but the FMM algorithm reduces the computing cost to \(O(N)\) operations only and allows to perform simulations with a large enough number of elements using a usual personal computer (PC). Note that, formally, the logarithmic divergence in Eq. (2) (and \(1/r\) divergence of velocity) would require very small time steps in following short range collisions. We used a finite size time steps in our simulations and, therefore, do not calculate these short range collisions accurately. However, long range interactions become increasingly more important for sufficiently large number of line vortices. Thus, the inaccuracy in following short range interactions translates into an additional numerical noise in the system. Below we present the results of our simulations, illustrating autoresonant formation and control of vortice holes.

Our simulations describe evolution of initially circular vortex patch of constant vorticity \(\omega_0\) at \(r < r_0\). Such a vortex performs a solid body rotation with a uniform angular frequency \(\Omega = \omega_0r/2\) of all fluid elements.\(^2\) To introduce a radial dependence of this rotation frequency (this dependence is necessary for autoresonance in the system, see below), we add a line vortex of strength \(\omega_0\) at the center of our vortex patch. Then, initially, \(\Omega(r) = \omega_0r/2 + \omega_a/(2\pi r^2)\). We drive this system by an oscillating \(m\)-fold symmetric (in polar angle \(\varphi\)) straining flow given by

\[
\psi_{\text{ext}} = e(t)r^m \cos(m \varphi) \cos \left( m \int \Omega_D(t) dt \right), \quad (3)
\]

where the driving frequency \(\Omega_D\) and amplitude \(e\) vary slowly in time on the scale of the oscillation period. In our illustrations we will show two examples with \(m=2\) and \(3\), set the initial driving frequencies \(\Omega_D\) below the frequency of the fluid elements at the boundary \([\Omega_D(t=0) < \omega_0/2 + \omega_a/(2\pi r_0^2)]\), and slowly increase the frequency in time until it reaches some value \(\Omega_F\) at \(t=T\). The driving amplitude \(e(t)\) is ramped up from zero to some constant \(e_0\) first and later is further increased, until reaching a value \(e_F\) at \(t=T\). Both, \(\Omega_D\) and \(e\) remained constant for \(t > T\). The results of the simulations are shown in Fig. 1 for \(m=2\) and Fig. 2 for \(m=3\) at two different evolution times. We used parameters \(\Omega_D(t=0) = 1.97, \Omega_F = 5.93, \omega_0 = 4\pi, \omega_a = 0.75\) in combination with \(e_0 = 0.0027, e_F = 0.53, T = 330\) in \(m=2\) case, and \(e_0 = 0.0018, e_F = 0.295, T = 325\) in \(m=3\) case. The detailed time variations of the driving parameters in these simulations are shown in Fig. 3. The number of the fluid elements was \(N=40000\) and we tested the accuracy of the method by varying \(N\) and the time step in solving the evolution equations. The typical simulation time (covering \(0 < t < 1000\)) was about 100 h when using Intel Fortran Compiler 9.1 on an Intel Core 2 Duo, 2.4 GHz PC. The dots in Figs. 1(a), 1(c), 2(a), and 2(c) are the snapshots of the actual vortex elements in our simulations at two different times during the evolution, while the solid lines in Figs. 1(b), 1(d), 2(b), and 2(d) show the vortex boundaries at the same times as predicted by our adiabatic contour dynamics theory developed in Sec. IV. The observed driven vortex evolution was as follows. In the initial evolution stage (not shown in the figure), the driving flow captured the vortex into resonance, transforming its initial circular shape into an \(m\)-fold symmetric autoresonant \(V\)-state.\(^17\) Later, during the passage of the driving frequency through the resonance with the rotating fluid elements at the vortex boundary, one observed emergence of filaments as seen in Figs. 1(a) and 2(a) and the evolution entered a new stage.

**FIG. 1.** (Color online) The emergence of driven vorticity holes from a circular vortex patch with a line vortex core by a twofold symmetric external strain. (a) and (c) show snapshots of numerical simulations at times \(t = 77\) and \(400\), respectively, while (b) and (d) represent vortex boundaries obtained from the adiabatic contour dynamics theory at the same times.
The filaments encircled a zero vorticity region and merged again with the main vortex, creating vorticity holes. At this time, the boundary of the vortex divided into the external and internal parts. As the increase in the driving frequency continued, the holes inside the vortex moved toward the vortex core [see Figs. 1(c) and 2(c)]. By analyzing the data, we found that the whole vortex structure (its external boundary and the holes) were continuously phase-locked with the drive, i.e., their rotation frequencies follow the time varying driving frequency $\Omega_D(t)$. This can be seen in Figs. 1 and 2 showing that the rotating vortex structures in simulations have nearly the same instantaneous angular orientations as the corresponding theoretical boundaries having rotation angle $\int \Omega_D dt$ associated with the driving flow. The phase-locking is the main signature of the adiabaticity in the system and allows to efficiently control (move) the position of the holes by varying the driving frequency. One can see in the figures that in addition to the rotation angle in the simulations, the general vortex shapes at different times are in a very good agreement with those predicted by the theory. Before describing this adiabatic theory, the next section addresses the problem of existence of uniformly rotating, driven vorticity hole equilibria.

III. STATIONARY PHASE-LOCKED VORTICITY HOLE EQUILIBRIA

Here, we set the driving amplitude $\epsilon$ and frequency $\Omega_D$ constant and use a rotating straining flow having $\psi_{ext} = 1/2r^m \cos[m(\varphi - \varphi_D)]$, where $\varphi_D = \Omega_D t$ [compare to Eq. (3)]. The Poisson equation for the stream function $\psi$ associated with the vorticity field in polar coordinates is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} = -\omega(r, \varphi, t).$$

We seek rotating $m$-fold symmetric vortex patch solutions synchronized with the drive, i.e., assume $\psi = \psi[r, m(\varphi - \varphi_D)]$, and

$$\omega = \omega[r, m(\varphi - \varphi_D)] = \left\{ \begin{array}{ll} \tilde{\omega} = \text{const}, & (r, \varphi) \in S, \\ 0, & (r, \varphi) \notin S, \end{array} \right.$$ (5)

where $S$ is a rotating, constant shape 2D region bounded by limiting trajectories (as in our simulations in Figs. 1 and 2, for example). This leads to a contour dynamics-type problem, where the vorticity dynamics is fully governed by the evolution of the limiting trajectories bounding the rest of the vorticity field. The contour dynamics idea was used in many physical applications and we adopt a similar approach.

We expand $\psi$ and $\omega$ in Fourier series with time independent coefficients and neglect all terms in these expansions, but with $k=0$ and $m$, i.e., approximate

$$\psi = \psi_0(r) + \psi_m(r) \exp[im(\varphi - \varphi_D)] + \text{c.c.},$$

$$\omega = \omega_0(r) + \omega_m(r) \exp[im(\varphi - \varphi_D)] + \text{c.c.}$$

This approximation does not mean linearization and allows to formulate a tractable theory of the driven vorticity hole equilibria (see below), yielding a good agreement with simulations. The nonlinearity in this formalism enters through the dynamics of the limiting trajectories in the combined simplified internal [see Eq. (6)] and full external flows. The coefficients $\psi_{0,m}$ and $\omega_{0,m}$ in Eq. (6) are related via the Poisson equation

$$\frac{d^2 \psi_0}{dr^2} + \frac{1}{r} \frac{d \psi_0}{dr} = -\omega_0(r),$$

$$\frac{d^2 \psi_m}{dr^2} + \frac{1}{r} \frac{d \psi_m}{dr} - \frac{m^2}{r^2} \psi_m = -\omega_m(r),$$

where $\omega_0 = (2\pi)^{-1} \int_0^{2\pi} \omega(r, \varphi) \cos(k \varphi) d\varphi$, $k=0,m$ can be calculated within the contour dynamics model if one knows the form of the limiting trajectories (the internal and external boundaries of the rotating vortex patch). We discuss these trajectories next.

The motion of the fluid elements in the vortex patch is given by

$$\dot{r} = -\frac{1}{r} \frac{\partial}{\partial \varphi}(\psi + \psi_{ext}) = -m \tilde{\mu} \sin[m(\varphi - \varphi_D)],$$

$$\dot{\varphi} = -\frac{1}{r} \frac{\partial}{\partial r}(\psi + \psi_{ext}) = -\frac{1}{r} \frac{d \psi_0}{dr} - \frac{1}{r} \frac{d \tilde{\mu}}{dr} \cos[m(\varphi - \varphi_D)],$$

where $\tilde{\mu} = 2^m \psi_m + 1/2r^m$. By transforming to the action-angle variables via $I=\pi r^2$ and $\Phi = \varphi - \varphi_D$ and changing $-2\dot{\Phi} \to \psi(I)$ for convenience, we write the last equations in a Hamiltonian form

$$\dot{I} = -\frac{\partial H}{\partial \Phi} = m \mu \sin(m \Phi),$$

$$\dot{\Phi} = \frac{\partial H}{\partial I} = \frac{d \psi_0}{dI} - \Omega_D + \frac{d \mu}{dI} \cos(m \Phi),$$

where $\mu = 2^m \psi_m(I) - \epsilon I^{m/2}$ and the one-degree-of-freedom Hamiltonian

$$H(I, \Phi) = \psi_0(I) - \Omega_D I + \mu \cos(m \Phi)$$

is the integral of motion. Given a set of parameters $H, \Omega_D, \epsilon$, and functions $\psi_0(I)$ and $\psi_m(I)$, one can find a closed trajectory $I(\Phi)$ or $\Phi(I)$ from the last equation, which can serve as a limiting trajectory in the problem and define a singly connected, driven contour dynamics equilibrium. By choosing two such sets of parameters we can find two enclosed trajectories and, by confining a constant vorticity field between the
two trajectories, we form a rotating vorticity hole equilib-
rium of the type seen in our simulations. Thus, the problem
reduces to the self-consistent calculation of $\psi_{0,m}(I)$. This
can be accomplished numerically by using the following itera-
tive procedure.

(i) As a first guess, we choose the unperturbed circular vortex
patch distribution ($\omega=\omega_0$ at $r<r_a$ and zero other-
wise) with a line vortex core of strength $\omega_0$ at its center. The
stream function of such a vortex is

$$
\psi_0(I) = \begin{cases} 
\frac{\omega_0}{2\pi} \ln(I) + \frac{\omega_0}{2} I, & I < r_a^2, \\
\frac{\omega_0}{2\pi} \ln(I) + \frac{\omega_0}{2} r_a^2 \left( \ln \frac{I}{r_a^2} + 1 \right), & I > r_a^2,
\end{cases}
$$

and both $\psi_m(I)$ and $\omega_m(I)$ vanish.

(ii) For a given set $(H, \Omega_D, \varepsilon)$ and using the trial func-
tions $\psi_{0,m}(I)$ from (i), we define new boundaries $\Phi(I)$ of
the vortex from Eq. (10). Note that $H$ must be less than
some limiting value $H_I$ and the actions on this bound-
ary are confined to an interval $I_1 < I < I_2$. Further-
more, for some values of $H$ ($H_I < H \leq H_2$), Eq. (10)
can have two solutions, and thus, we have two bound-
aries, internal end external, instead of one. For the
limiting value of $H=H_2$ these boundaries cover $0
\leq \Phi \leq 2\pi$, whereas for $H<H_2$ they exist only in $m$
angular regions: $\Phi_0 + 2\pi k/m \leq \Phi < 2\pi m/\Phi_0
+ 2\pi k/m$, where $k=0, \ldots, m-1$ and the limiting
angle $0 < \Phi_0 < \pi/m$ is uniquely defined by the value
of $H$. Therefore, for having a vortex patch with a
single (external) boundary we need to specify only one
value of $H=H_2$. The vortex with the external
boundary and the holes needs specification of two val-
ues, $H=H_i,h$. These values (together with $\varepsilon$ and $\Omega_D$)
completely define the shape of the external boundary
and the holes. Note that solutions with the holes exist
only if $\omega_0 > 0$. If $\omega_0 = 0$ then $\psi_0$ increases mono-
tonically with $I$ and Eq. (10) has only a single solution
covering the whole range $0 < \Phi \leq 2\pi$.

(iii) Next we calculate $\omega_{0,m}$. For the patch of constant
vorticity $\omega_0$ with a single external limiting trajectory we have

$$
\omega_0(I) = \begin{cases} 
\omega_0, & I < I_1, \\
\frac{\omega_0}{\pi} \Phi_0(I), & I_1 < I < I_2, \\
0, & I > I_2,
\end{cases}
$$

where $0 < \Phi_0(I) < \pi/m$ and $\Phi_0(I)$ (describing the ex-
ternal boundary of the vortex) decreases monotonically
from $\Phi_0(I_1) = \pi/m$ to $\Phi_0(I_2) = 0$. Similarly, one obtains

Note that $\omega_m$ reaches its maximum, $\omega_m(\pi)$, at the value
of $I$ such that $\Phi_m(I) = \pi/(2m)$. Therefore the value
of the maximum does not depend on $H$, $\varepsilon$, and $\Omega_D$, and
is entirely defined by $\omega_0$. In contrast, the characteristic
width of $\omega_m$ between $I_{1,2}$ depends on the parameters
$H$, $\varepsilon$, and $\Omega_D$. In the case with the holes, the cal-
ulation of vorticity harmonics is similar to that described
above. The only difference is that one needs to take
into account two types of boundaries, i.e., the external
boundary of the vortex and the boundaries of the
holes. The vorticity in the contour dynamics model
vanishes outside the external boundary $\Phi_m(I)$ and in-
side the holes, while $\omega=\omega_0$ in the rest of the vortex.
The detailed calculation in the case with the holes
shows that in the region $I_{1,2} < I < I_{2,2}$ ($I_{1,2}$ and $I_{2,2}$
being the minimum and the maximum of the actions defin-
ing the boundaries of the holes) $\omega_m$ is a doubly peaked
function of $I$ and each of its maxima equals $\omega_0/\pi$. In contrast, in the region of the external bound-
ary (in the presence of the holes) $\omega_m$ has a form of an
inverted peak with a single minimum equal to $-\omega_0/\pi$.

(iv) After finding $\omega_{0,m}$, we compute the new trial functions
$\psi_{0,m}$ by solving the system of ordinary differential
equations (ODEs)

$$
2I^2 \frac{d^2 \psi_0}{dI^2} + 2 \frac{d \psi_0}{dI} - \omega_0(I) = 0,
$$

$$
4I^2 \frac{d^2 \psi_m}{dI^2} + 4 \frac{d \psi_m}{dI} - \frac{m^2}{I} \psi_m - 2 \omega_m(I) = 0
$$

obtained from Eq. (7) (note that we changed our not-
ations, replacing $-2\psi_0(r^2)$ with $\psi_0(I)$ as mentioned
above). The first ODE in Eq. (14) comprises an initial
value problem with $\psi_0$, $d\psi_0/dI$ at $I=0$ as those in
the unperturbed initial vortex. The problem can be solved
numerically by a standard algorithm (the fourth order
Runge–Kutta method in our case). The solution of the
second ODE in Eq. (14) has the form $\psi_m = C_1 I^{m/2}$ at
$I \rightarrow 0$ and $\psi_m = C_2 I^{m/2}$ at $I \rightarrow \infty$. Thus, we have
a mixed boundary value problem that can be solved nu-
merically by a standard shooting method.

Finally, the steps (ii)-(iv) are repeated until the process
yields the desired accuracy. We found that the convergence
of the solution in all our examples was very fast and, typi-
cally, after seven iterations the accuracy was similar to that
of representation of real numbers on our PC, while the char-
acteristic time of the calculations was about 10 s.

Figure 4 shows examples of snapshots of uniformly ro-
tating, phase-locked, driven vorticity hole structures obtained
by using the iterative approach described above for $m=2$ and
$3$. We used parameters $\varepsilon=0.1, \Omega_D=4, \omega_0=0.75, \omega_0=4 \pi,$
and $H_b=3.8,-4.0, H_h=-3.24,-3.23$ for $m=2$ and $3$, re-
spectively ($H_b$ and $H_h$ being the energies of the external vortex

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boundary and the holes, respectively). We checked our theory via FMM simulations by using the theoretical vorticity distributions as the initial data in the simulations. Figures 5 and 6 show the comparison between \( \omega_0 \) and \( \omega_m \) in theory and simulations at the initial times, \( t=0 \) and \( t=300 \) (after nearly 200 periods of vortex rotation). These figures illustrate a good agreement between the theory and simulations and the stability of theoretically predicted vorticity hole structures. For example, the theoretical values of the extrema of \( \omega_m \) are \( \pm \omega_{0}/\pi \). We show these values as obtained in our simulations versus time in Fig. 7 in the two examples in Fig. 4. The simulation results agree well with the theory, but include also higher frequency modulations, which are probably due to the neglect of higher harmonics in Eq. (6). In addition, there exists a slow decay of the extrema as the result of the observed smoothing of the boundaries of the vorticity holes at larger times, probably due to the discreteness and consequent numerical noise in the adopted FMM scheme as described above.

Finally, we observe that \( \mu = 2^* \psi_m(l) - e l^{m/2} \) in the Hamiltonian (10) plays a role of the driving amplitude in the problem and includes the contribution of the self-field \( \psi_m(l) \). Therefore solutions with \( e=0 \) supported by the presence of the self-field alone can be found by the iterative procedure described above. Nevertheless, we found that free (unstrained) rotating vorticity hole equilibria are unstable for all \( m \), but \( m=1 \). An example of such instability in \( m=2 \) case is shown in Fig. 8, where a two-hole structure generated via our theory quickly relaxes in simulations to a single hole configuration. The evolution of a vortex with three holes is more complex and includes several competing transitions, \( 3 \rightarrow 2, 3 \rightarrow 1, \) and \( 2 \rightarrow 1 \). These results are in agreement with the previous studies of different free (unstrained) \( m \)-fold symmetric vortex equilibria, such as in Ref. 27 (pure electron plasmas between two concentric electrodes), 5 (a vortex patch without a line core), and 4 (pure electron plasma experiments). Our simulations show that the addition of the driving flow can stabilize rotating \( m \geq 2 \) symmetric vorticity hole equilibria described above. However, most importantly, in addition to stabilization, as illustrated in Sec. II, oscillating chirped frequency straining flows can be used in forming and controlling these vorticity hole equilibria by starting from trivial initial conditions. In the next section, we develop the adiabatic contour dynamics theory of this vortex formation process.

IV. ADIABATIC DRIVEN CONTOUR DYNAMICS MODEL

The contour dynamics theory of the last section required prescription of three or four parameters [(\( e, \Omega_D, H_b \)) or \( (e, \Omega_D, H_b, H_h) \)] depending on the vortex being without or with the holes] for a complete description of a class of driven, \( m \)-fold symmetric, phase-locked vortex structures. In the chirped frequency drive case we will assume a continuing phase-locking in the driven system, so one can think of the vortex as slowly passing a sequence of the rotating equilibria described above. However, in contrast to above, the energies of \( H_b, H_h \) of the limiting trajectories must be determined self-consistently during the evolution taking into account the slow variation of the parameters. Inclusion of this variation comprises our next goal in the theory. Focusing again on a vortex patch with a line vortex core, we assume that the driving amplitude \( e \), frequency \( \Omega_D \), and the Hamiltonian (10) of the limiting trajectories of the previous section are slow functions of time (see Fig. 3 for variation of \( e \) and \( \Omega_D \) in our simulations). We start from a circular vortex patch...
equilibrium with $\varepsilon=0$ and slowly increase $\varepsilon$ (and $\Omega_D$) in time. The vortex boundary transforms because of the action of the external flow. Nevertheless, if the parameters $\varepsilon$ and $\Omega_D$ change slowly enough, the action integral

$$J(\varepsilon, \Omega_D, H) = \frac{1}{2\pi} \int_0^{2\pi} I(H, \Phi) d\Phi$$

is an adiabatic invariant. Here $I(H, \Phi)$ is the solution of Eq. (10) (i.e., a trajectory of a vortex element) with the values of $(\varepsilon, \Omega_D, H)$ fixed at some $t$ and the slowness above means a small relative change during one period along the trajectory. The action integral $J(\varepsilon, \Omega_D, H)$ is conserved, despite of time variation of $(\varepsilon, \Omega_D, H)$ for any closed initial trajectory $I(\Phi)$, but in the contour dynamics theory one needs to know the boundary of the vortex only. Initially, at $\varepsilon=0$, the limiting trajectory associated with the boundary is a circle of radius $r_0$ and, therefore, $J_0 = I = r_0^2$, so, under adiabatic conditions,

$$J(\varepsilon, \Omega_D, H) = J_0$$

despite the variation of parameters. Note that the conservation of this action integral means also conservation of the total vorticity enclosed by the limiting trajectory

$$\int_0^\infty dI \int_0^{2\pi} d\Phi \omega I(\Phi) = \omega_a \int_0^\infty dI = 2\pi \omega_a J_0.$$  

Of course, the conservation of the total vorticity is guaranteed by the incompressibility of the vorticity field, but the adiabatic theorem also means that the limiting trajectory deforms so that at any time $t$ of evolution, the trajectory can be found from Eq. (10) by using fixed values of $(\varepsilon, \Omega_D, H)$ at this time. We have a closed system of equations describing the deformation of the boundary of the adiabatically driven vortex, i.e., one integral and one algebraic equations,

$$J_0 = \frac{1}{2\pi} \int_0^{2\pi} I(H(t), \Phi) d\Phi,$$

$$H(I, \Phi, t) = \psi_0(I) - \Omega_D(t) I + \mu \cos(m \Phi)$$

for the Hamiltonian $H(t)$ and the boundary $I(\Phi, t)$, two algebraic equations for $\omega_0(I, t), \omega_a(I, t)$ [see Eqs. (12) and (13)] and two ordinary differential equations [Eq. (14)] for $\Psi_0(I, t), \Psi_m(I, t)$. As in the stationary case, one can use iterations (see Sec. III) in solving the problem numerically and use the stream function and vorticity found at time $t$ as the initial approximation for finding the vortex state at time $t + \Delta t$. Such iterative procedure is computationally fast and, for example, the calculation of the vortex shapes for $0 < t < 500$ with the time step $\Delta t=0.5$ (1000 frames) took about 15 min on our PC.

The adiabatic process of development of a driven singly connected $V$-state continues as described above until, at some $t=t_{res}$, the external limiting trajectory given by the solution of Eqs. (18) and (19) seizes to exist. Let us discuss this stage in more detail using one of the examples of the theory illustrated in Fig. 2. We rewrite Eq. (19) as

$$\cos(m \Phi) = e^{-\frac{1}{4}[H - \psi_0(I) + \Omega_D I]} = G(I)$$

and show function $G(I)$ at two successive times approaching $t_{res}$ in this example in Fig. 9(a). The thick solid lines in the figure show the corresponding trajectories on the external boundary, while the thick dashed line corresponds to $t=t_{res}$. One can see that $G(I)$ has a single maximum, which decreases with time and reaches 1 at $t_{res}$. The maximum passes below 1 for $t > t_{res}$, indicating the disappearance of a closed trajectory covering the entire region $0 \leq \Phi \leq 2\pi$. Nevertheless, there appears a new internal trajectory describing the boundaries of newly formed vorticity holes. The vortex filaments seen in simulations in Figs. 1(a) and 2(a) show the formation of this new boundary enclosing a zero vorticity region, i.e., the formation of vorticity holes. We can see also that, to $O(\varepsilon)$, $t_{res}$ corresponds to the time of passage of the resonance between the driving frequency and the frequency associated with the external limiting trajectory. Indeed, for finding $I=I_m$ corresponding to the maximum of $G(I)$, we differentiate in Eq. (20) to obtain $\Omega_D(t_m)=\Omega_D+O(\varepsilon)$, where $\Omega_D = \partial \Phi_0 / \partial I$ is the frequency associated with the azimuthally averaged component of the vorticity. After the formation of the holes their boundaries encircle a region having the area of the holes $S_h$ and $J_2=J_0/(2\pi)$ becomes a new adiabatic invariant in the problem. The continuing adiabatic deformation of this boundary is described by Eqs. (18) and (19) with $J_0$ replaced with $J_2$ and integration in Eq. (18) is taken along the boundary of the hole. At the time of formation of the holes the old external trajectory is lost. Nevertheless, the
The angle averaged vorticity vs radius at times (a) $t=66$, (b) $t=92$, and (c) $t=381$ for chirped frequency drive. The dashed lines are predictions of the adiabatic contour dynamics model, while the solid lines represent the simulations.

The presence of the holes increases the area inside a possible new external limiting trajectory by that of the holes. Therefore, this new external boundary is again described by Eqs. (18) and (19) with $J_1=J_0+J_2$ replacing $J_0$. The solution covering the entire $0 \leq \Phi \leq 2\pi$ angular region is again allowed for this new external limiting trajectory and the new boundaries of the vortex evolve adiabatically as illustrated in Figs. 9(b) and 9(c). This completes our discussion of the transition through the boundary, formation of the driven holes, and the self-consistent slow evolution of the new external and internal boundaries of the vortex. The calculations based on this theory were compared with numerical simulations in Figs. 1 and 2. We observed a very good agreement of the predicted overall form of the rotating vortex with the theoretical predictions, and confirmed the continuing phase-locking (autoresonance) in the adiabatically varying system, as the rotation angle of the vortex followed the phase of the driving flow continuously, despite the variation of the driving frequency. More detailed comparison of the theory and simulations is shown in Figs. 10 and 11 [where $\omega_m$ versus $r$ at (a) $t < t_{res}$, (b) $t=t_{res}$, (c) $t > t_{res}$, and 12 (the extrema of $\omega_m$ versus time)]. One can see that the theoretical predictions are in a very good agreement with the results of the numerical simulations. This seems to be surprising in view of a severe wave number truncation in the Fourier expansion for the stream function and vorticity field [see Eq. (6)]. A possible explanation of this result is the smoothness of the driven vortex and vorticity holes boundaries as functions of $\Phi$. In studying a related problem of driven phase space holes in plasmas, the effect of inclusion of higher harmonics in the theory was studied via the perturbation analysis, justifying the neglect of these harmonics as long as the angular size of the holes was sufficiently large. In principle, a similar calculation can be performed in the case of vorticity holes, but such a development is outside the scope of the present work.

At this stage, we discuss the conditions for validity of our theory. One of the conditions is the adiabaticity of the parameters, i.e., a small relative change during one period of underlying oscillations. However, the adiabaticity is lost at $t_{res}$ as one passes the separatrix, where the frequency of the trajectory approaches zero. Nevertheless, if following the formation of the new boundaries, the latter moves away from the separatrix, the adiabaticity condition may be satisfied again at $t > t_{res}$. The simulations in our examples above illustrate this continuing process with restored adiabaticity and continuing phase-locking in the driven vorticity hole system. However, what guarantees the desired departure of the limiting trajectories from the separatrix at $t > t_{res}$? This question reduces to understanding the dynamics of the boundaries of the vorticity holes and we discuss this issue next.

We represent the trajectory defining the boundary of the holes by writing $I = I_0(t) + \Delta(t)$, where $I_0(t)$ is slow average defined via

![FIG. 11. (Color online) The $m=3$ Fourier component of vorticity vs radius at times (a) $t=66$, (b) $t=92$, and (c) $t=381$ for chirped frequency drive. The dashed lines are predictions of the adiabatic contour dynamics model, while the solid lines represent the simulations.](image1)

![FIG. 12. (Color online) The extrema of $\omega_m$ vs time in simulations for $m=2$ (a) and $m=3$ (b). The dashed lines are theoretical predictions of the adiabatic contour dynamics model.](image2)
while $\Delta$ is oscillating. Then, assuming that the amplitude of these oscillations is small enough, we expand $H$ [see Eq. (10)] to second order in $\Delta$

$$H_1 = \frac{1}{2} \beta \Delta^2 + \mu_0 \cos(m\Phi),$$

(22)

where $\beta = d^2 \psi_0/dt_0^2$ and $\mu_1 = \psi_0(I_0) - \epsilon I_0^{m/2}$ [we have neglected $\Delta$ in the term with $\cos(m\Phi)$ assuming that $\epsilon$ is sufficiently small]. This pendulum Hamiltonian has a simple phase-space portrait, with closed and passing trajectories depending on the value of $H_1$, while the separatrix between the two types of motion corresponds to $H_1 = |\mu_0|$. The area in $(I, \Phi)$-space enclosed by the separatrix is given by

$$S = 16|\mu_0|/\beta^{1/2}.$$  

The area $S_h = 2\pi J_2$ of the holes in our driven vorticity distributions above is equal to the value of $S$ at $t=t_0$ and remains constant at later times. Thus, the condition for departure of the boundaries from the separatrix is the departure of the ratio

$$\rho = S/S_h$$

from unity after the formation of the holes. For an estimate one can use the initial stream function $\psi_0$ [see Eq. (11)] yielding

$$S \approx 16 \left( \frac{2\pi}{\omega_c} \epsilon I_0^{m/2} \right)^{1/2}.$$  

(24)

One can see that for a constant driving amplitude $\epsilon$, $S$ decreases in time (because of the decrease of $I_0$ as the hole moves toward the vortex core). Therefore, for removing the hole from the separatrix, the driving amplitude $\epsilon$ would increase at least as $\epsilon \sim I_0^{2-m/2} \sim I_0^{-4-m}$ ($r_0 = H_1^{1/2}$ is the radial position of the holes). This condition was satisfied in the numerical examples shown in Figs. 1 and 2.

The second assumption of our theory was a continuous ideal phase-locking in the driven system, i.e., the smallness of the average $\Phi$ over one oscillation along the boundary of the holes. One can show that the validity of this assumption depends on the smallness of the driving frequency chirp rate $\alpha = d\Omega_D/dt$. Indeed, we can use the Hamiltonian (22) to write the equations of the trajectory at the boundary

$$\dot{\Delta} = m \mu_0 \sin(m\Phi),$$

(25)

$$\dot{\Phi} = \beta \Delta - \Omega_D(t).$$

(26)

For constant $\Omega_D$ these equations have a characteristic form describing nonlinear resonance in many dynamical system and we analyze the problem with slowly varying $\Omega_D$ similarly. We differentiate Eq. (26) with respect to $t$ and substitute $\Delta$ from Eq. (25) to obtain

$$\dot{\Phi} = m \beta \mu_0 \sin(m\Phi) - \alpha.$$  

(27)

This equation describes a pendulum with slowly varying parameters under the action of a small torque $\alpha$. Viewing $\alpha$ locally as a constant, we obtain a condition for having closed trajectories

$$\alpha < |m \beta \mu_0|.$$  

(28)

We can calculate the average $\langle \sin \Phi \rangle = \alpha |m \beta \mu_0|^{-1}$. Therefore, our assumption of $\langle \Phi \rangle = 0$ reduces to having a strong inequality in Eq. (28). If, in addition, one neglects $\psi_m$ in Eq. (28) and uses the initial stream function (11) for an estimate, the condition for the ideal phase-locking in the system becomes

$$\frac{2 \pi \alpha}{m \omega_c \epsilon I_0^{2-m/2}} \ll 1.$$  

(29)

Thus, one can increase $\alpha$ and still satisfy Eq. (29) if the driving amplitude $\epsilon$ increases, as long as $\epsilon = \epsilon(t)$ grows faster than $\alpha$. The largest value of the left hand side in Eq. (29) in our simulations was 0.15. The last inequality shows that for a given driving frequency chirp rate, one must have a finite value ($\omega_c$) of the central line vortex for formation of driven chirped vorticity holes in the system and, if $\omega_c$ decreases, one must use smaller and smaller chirp rates.

Finally, we observe that the effective driving amplitude in Eq. (22) is $\mu_0 = \psi_0(I_0) - \epsilon I_0^{m/2}$ and, therefore, $\psi_m(I_0)$ plays a role of an additional self-driving field, while the ratio of the two driving factors measures the importance of the self-field effects in the problem. We have found that the maximum ratio $\psi_m(I_0)/(\epsilon I_0^{m/2})$ in our simulations in Figs. 1 and 2 was about 0.6, meaning a significant contribution of the self-field component. Nevertheless, Figs. 10–12 show that our adiabatic contour dynamics theory yields a good agreement with simulations in this case.

Until now, we have assumed a flat-top initial vorticity distribution with a vortex line core. What happens if we start from a continuous axisymmetric initial distribution of vorticity? In this case, our theory can be generalized via the multicontour dynamics approach. A similar generalization was suggested for driven BGK structures in plasmas. The idea is to view the smooth initial distribution as a superposition of $n$ concentric circular vortex patches of thickness $\Delta \omega_c$. Each of these elementary patches can be considered as driven by the combination of the external, chirped frequency drive and the total self-flow. The latter couples to all elementary patches via the Poisson equation, with each patch contributing as a part of a linear superposition in the source term. Thus, formally, the formation of vorticity holes in each of the elementary vortex patches can be described by the same theory outlined above, and the whole problem reduces to the solution of a system of algebraic and ordinary differential equations. Implementation of this idea in the theory goes beyond the scope of the present work. Nevertheless, we illustrate the formation of a multicontour dynamics hole equilibrium in the following numerical simulations. We have assumed a stepwise constant radial vorticity distribution composed of five superimposed “elementary” vorticity patches shown in Fig. 13(a). This distribution models a Lorentzian vorticity profile indicated by a thin line in the figure. We have driven this system by a chirped frequency strain and varied the driving frequency (amplitude) between 0.9 and 1.61 (0 and 0.03). Figures 13(b)–13(d) show the final vorticity distributions of the three largest initially circular patches. One can see that the vorticity holes were formed in
each of these patches (at the time shown in the figure the holes were just formed in patch III). Furthermore, the holes in all the three slices of the vortex in the figure have the same azimuthal positions being locked to the external drive. At the same time, the two smallest initially circular patches did not develop a significant azimuthal deformation and played the role of the line vortex core in our previous examples. This example completes our discussion of adiabatically driven vorticity hole systems.

V. CONCLUSIONS

(a) We have studied the formation and control of $m$-fold symmetric 2D vorticity hole structures in ideal fluids. It was shown that such structures emerge by starting from a trivial, axisymmetric vortex patch with a line vortex core and using the autoresonance (continuous dynamical phase-locking) with a small amplitude, chirped frequency external flow. Numerical simulations based on the fast multipole method showed stability of these driven phase-locked states.

(b) We have developed an analytic, contour dynamics-type theory of driven uniformly rotating $m$-fold symmetric vorticity hole equilibria. Such equilibria exist with or without the external driving flow. Our numerical simulations show that while a single unstrained hole structure is stable, $m \geq 2$ cases without an external drive are unstable and decay to lower $m$ states by hole merger process. Nevertheless, all $m$-hole vortex states can be stabilized by an oscillating $m$-fold symmetric external flow.

(c) We have generalized our contour dynamics theory of driven vorticity holes to include slow variation of the driving parameters. This adiabatic extension allowed us to describe all stages of formation and evolution of the vortex structure, starting from a simple axisymmetric vortex configuration, development of an $m$-fold symmetric singly connected $V$-state, passage through the resonance at the vortex boundary, followed by the formation of vorticity holes drifting toward the vortex core. The theory used adiabatic invariants in the problem (action integrals of the limiting trajectories) and allowed calculation of the vortex shape by solving a set of six equations (two ODEs, three algebraic equations, and a 1D integral equation). The results of the theory were in a good agreement with numerical simulations.

(d) Two conditions were singled out for a successful formation and adiabatic control of a stable $m$-fold symmetric vorticity hole structure. One of the conditions is the departure of the vortex hole boundaries from separatrix after formation of the holes. This departure is characterized by parameter $\rho$ [see Eq. (23)] which would increase after the formation of the holes. The second condition is the sufficient smallness of the ratio $\alpha/\varepsilon$ between the driving frequency chirp rate and the driving amplitude [see the formal inequality in Eq. (29)]. This condition guarantees nearly ideal phase-locking in the driven system, as assumed in our adiabatic theory.

(e) We have shown numerically that more complex $m$-fold symmetric driven vorticity structures with local minima in vorticity distributions can be formed by chirped frequency driving flows by starting from axisymmetric radially decreasing vorticity distributions. The formation of these vortices can be dealt with by a multicontour dynamics extension of the theory developed for a driven flat-top distribution. This extension comprises an interesting goal for future research. Finally, it seems interesting to include a finite phase mismatch between the driven and driving flows in the theory, as well as to explain the observed stability of the driven vorticity hole structures.

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