Removal of resonances by rotation in linearly degenerate two-dimensional oscillator systems

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A system of two nonlinearly interacting, resonant harmonic oscillators is investigated, seeking transformation to approximate action-angle variables in the vicinity of the equilibrium via the canonical perturbation theory. A variety of polynomial perturbations dependent on parameters is considered. The freedom of choice of the zero-order approximation characteristic of a linearly degenerate (resonant) system is used to cancel lower-order resonant terms in the canonical perturbation series. It is found that the cancellation of the resonant terms is only possible for particular values of parameters of the interaction term. These special sets of parameters include all the cases with the Panlevé property. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2719145]

I. INTRODUCTION

In various applications of nonlinear dynamics it is desirable to obtain action-angle (AA) description[1] of nonlinear oscillatory systems. For example, the theories of nonlinear resonance[2] and autoresonance[3] use AA representation of the driven system explicitly. It is well known that any *linear* system of oscillators admits transformation to AA variables.[4] Transformation to AA variables in a generic *nonlinear* system is possible only if the system is integrable, i.e., there exists a sufficient number of conserved quantities.[1] It is well known that a nonlinear system of more than one degree of freedom is generally not integrable. Nonetheless, if *approximate* integrals of motion exist in the vicinity of the system equilibrium, a transformation to approximate AA variables may be found.

The theory of normal forms[1] established a number of results concerning approximate integrability of nonlinear systems in the vicinity of the stable equilibrium. This includes the theorem stating that any nonlinear oscillatory system with *nonresonant linear* frequencies is approximately integrable.[6] Another theorem states that any two-dimensional (2D) nonlinear oscillatory system can be approximately integrated in the vicinity of the equilibrium, i.e., it can be approximated by a completely integrable nonlinear system in a sufficiently small neighborhood of the equilibrium.[7]

The goal of the present work is to investigate the possibility of transforming to *approximate* AA variables in a 2D nonlinear system of oscillators near the equilibrium. We shall focus on finding a formal transformation to approximate AA variables, but leave out the question of convergence of the canonical perturbation series. The desired transformation can be conveniently obtained via the canonical Poincaré-Von Zeipel perturbation procedure,[8,9] provided the linear frequencies of the oscillatory system are *nonresonant*. Thus, we shall focus on the *resonant* case and consider linearly degenerate two-dimensional oscillator systems, i.e., nonlinear oscillator systems with equal linear frequencies (the 1:1 resonance) governed by Hamiltonian $H(\mathbf{p},\mathbf{q})=\frac{1}{2}(p_1^2+p_2^2+q_1^2+q_2^2)+V(\mathbf{q})$ with a variety of polynomial perturbations $V(\mathbf{q})$ depending on parameters. In this case a straightforward implementation of Poincaré-Von Zeipel procedure cannot be carried out.

The technical reason for the failure of the canonical perturbation theory to provide the lowest order transformation to AA variables in the 1:1 resonance case is the appearance of zero denominators in the first order terms (first order *resonant terms*) of the canonical perturbation series. Can we overcome this difficulty? We shall see below that in some cases the answer to this question is positive via a special regularization procedure. The procedure will be based on the fact that a resonant linear system is *superintegrable*, i.e., the number of independent integrals of motion is larger than the number of degrees of freedom.[5] This will allow choosing different zero-order AA variables in the problem. We shall use this freedom of choice to cancel lowest order resonant terms in the canonical perturbation series furnishing the formal transformation to first order AA variables. It will be demonstrated that generally, the aforementioned freedom of choice is not sufficient to cancel the resonant terms in the perturbation expansion, but the cancellation is possible for particular families of parameters in the perturbation.

Our presentation will be as follows. Section II will proceed from a short outline of the canonical, time-independent perturbation theory. Later in this section we shall show that the degeneracy of a 2D harmonic oscillator results in a particular freedom of choice of the zero-order approximation for the perturbation series. In Sec. III we shall examine the possibility of removal of the lowest order resonant terms in the perturbation series for a variety of perturbations $V(\mathbf{q})$. In particular, we shall analyze the generalized Henon-Heiles system,[10] oscillators with quartic and sextic homogeneous polynomial perturbations, and the truncated three-particle periodic Toda lattice.[11] Similar examples were analyzed by other authors for the Panlevè property[12–15] and the Panlevè-positive (P-positive) cases are well known.[12] We shall see that the families of parameters satisfying our criterion include all P-positive cases. Finally, Sec. IV will summarize our results.

II. PERTURBED, 2D HARMONIC OSCILLATOR PROBLEM

Here we consider a perturbed, 2D linear oscillator governed by Hamiltonian

$$H(\mathbf{p},\mathbf{q}) = H_0(\mathbf{p},\mathbf{q}) + \varepsilon H_1(\mathbf{q}) + \varepsilon^2 H_2(\mathbf{q}), \tag{1}$$

where

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\sum_{i,j=1}^{2} a_{ij} q_i q_j, \tag{2}$$

a_{ij} is symmetric and positive definite, $\varepsilon \ll 1$ is the perturbation parameter, and $H_{1,2}$ are homogeneous polynomials in $q_{1,2}$ of orders 3 and 4, respectively. Since the problem associated with the unperturbed Hamiltonian H_0 is integrable, the convenient approach in dealing with the perturbed problem is via the canonical, time-independent perturbation theory. We shall use this approach later in analyzing the degenerate problem, but, first, for completeness, present a brief outline of the Poincaré-Von Zeipel perturbation procedure.[8,9] We shall follow the lines of the classical textbook[4] and all missing points can be filled in from this reference.

A. Canonical perturbation theory

The starting point of the perturbation theory is an unperturbed completely integrable system with Hamiltonian $H_0(\mathbf{p},\mathbf{q})$, admitting transformation to the AA variables $(\mathbf{I},\boldsymbol{\Theta})$. The perturbed Hamiltonian (1) is then assumed to be of form

$$H = H_0(\mathbf{I}) + \varepsilon H_1(\boldsymbol{\Theta},\mathbf{I}) + \varepsilon^2 H_2(\boldsymbol{\Theta},\mathbf{I}) + \cdots. \tag{3}$$

The goal of the canonical perturbation theory is to find the near identity generating function $S(\boldsymbol{\Theta},\mathbf{I}')$ of the canonical transformation $CT:(\mathbf{I},\boldsymbol{\Theta}) \to (\mathbf{I}',\boldsymbol{\Theta}')$, where the new canonical variables $(\mathbf{I}',\boldsymbol{\Theta}')$ are such that the Hamiltonian (3) can be written as a function of $\mathbf{I}'$ alone. Should such transformation exist, the set $(\mathbf{I}',\boldsymbol{\Theta}')$ is the set of the AA variables of the perturbed problem (3). Let

$$S(\boldsymbol{\Theta},\mathbf{I}') = \boldsymbol{\Theta}\cdot\mathbf{I}' + \varepsilon S_1(\boldsymbol{\Theta},\mathbf{I}') + \varepsilon^2 S_2(\boldsymbol{\Theta},\mathbf{I}') + \cdots \tag{4}$$

and

$$H(\mathbf{I}') = K_0(\mathbf{I}') + \varepsilon K_1(\mathbf{I}') + \varepsilon^2 K_2(\mathbf{I}') + \cdots . \tag{5}$$

Then, it can be shown that

$$K_0(\mathbf{I}') = H_0(\mathbf{I}'),$$

$$K_1(\mathbf{I}') = \Omega_0 \cdot \frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} + H_1(\boldsymbol{\Theta}, \mathbf{I}'), \tag{6}$$

$$K_2(\mathbf{I}') = \boldsymbol{\Omega}_0 \cdot \frac{\partial S_2(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} + V_2(\boldsymbol{\Theta}, \mathbf{I}'),$$

where

$$V_2(\boldsymbol{\Theta}, \mathbf{I}') = H_2 + \frac{\partial H_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \mathbf{I}'} \cdot \frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} + \frac{1}{2}\frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} \cdot \frac{\partial^2 H_0(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \mathbf{I}' \, \partial \mathbf{I}'} \cdot \frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}}, \tag{7}$$

and components of $\boldsymbol{\Omega}_0$ are the frequencies of the unperturbed system H_0, taken at $\mathbf{I}=\mathbf{I}'$. Since $S_k(\boldsymbol{\Theta}, \mathbf{I}')$ is periodic in $\boldsymbol{\Theta}$, one can expand

$$S_k(\boldsymbol{\Theta}, \mathbf{I}') = \sum_{\mathbf{m}} s_{\mathbf{m}}^{(k)}(\mathbf{I}')\exp(i\boldsymbol{\Theta} \cdot \mathbf{m}), \tag{8}$$

where $s_{\mathbf{m}}^{k}(\mathbf{I}')$ is given by

$$s_{\mathbf{m}}^{(k)}(\mathbf{I}') = -i\frac{v_{\mathbf{m}}^{(k)}(\mathbf{I}')}{\boldsymbol{\Omega}_0 \cdot \mathbf{m}}, \tag{9}$$

and $v_{\mathbf{m}}^{k}(\mathbf{I}')$ is found from

$$\langle V_k(\boldsymbol{\Theta}, \mathbf{I}')\rangle_{\boldsymbol{\Theta}} - V_k(\boldsymbol{\Theta}, \mathbf{I}') = \sum_{\mathbf{m}} v_{\mathbf{m}}^{(k)}(\mathbf{I}')\exp(i\boldsymbol{\Theta} \cdot \mathbf{m}). \tag{10}$$

Here $V_1(\boldsymbol{\Theta}, \mathbf{I}')=H_1(\boldsymbol{\Theta}, \mathbf{I}')$ and $\langle\ \rangle_{\boldsymbol{\Theta}}$ denotes averaging over the old angle variables.

The zero denominators problem in the perturbation scheme is seen clearly. If the unperturbed system is degenerate, i.e., an integer vector $\mathbf{m} \neq 0$ exists such that $\boldsymbol{\Omega}_0 \cdot \mathbf{m}=0$, the perturbation theory blows up, unless $v_{\mathbf{m}}^{(k)}(\mathbf{I}')$ vanishes identically for all $\mathbf{I}'$. Now, let us assume that the perturbed system admits *approximate* (i.e., first order) AA variables $(\mathbf{I}', \boldsymbol{\Theta}')$. Then, we expect that the canonical transformation $CT:(\mathbf{I}, \boldsymbol{\Theta}) \rightarrow (\mathbf{I}', \boldsymbol{\Theta}')$ can be Taylor-expanded to the first order around the identity transformation, i.e.,

$$\mathbf{I} = \frac{\partial S(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} = \mathbf{I}' + \varepsilon\frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \boldsymbol{\Theta}} + O(\varepsilon^2),$$

$$\boldsymbol{\Theta} = \frac{\partial S(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \mathbf{I}'} = \boldsymbol{\Theta}' + \varepsilon\frac{\partial S_1(\boldsymbol{\Theta}, \mathbf{I}')}{\partial \mathbf{I}'} + O(\varepsilon^2), \tag{11}$$

with finite first-order terms. In other words, we expect that all the resonant contributions (with $\boldsymbol{\Omega}_0 \cdot \mathbf{m}=0$ in the denominator) to the first-order terms vanish identically. We shall see in Sec. III that such a regularization of the perturbation procedure is indeed possible in a variety of systems by using of the freedom of choice of the AA variables in the degenerate, zero order approximation. As a preparatory step, let us discuss this freedom of choice in more detail.

B. Zero-order approximation

We start with the normal mode representation[16] of the coordinates of the 2D oscillator associated with Eq. (2)

$$\mathbf{q} = \mathrm{Re}[A_1\mathbf{e}_1 e^{i\theta_1} + A_2\mathbf{e}_2 e^{i\theta_2}]. \tag{12}$$

Here $A_{1,2}$ are *real* amplitudes, $\theta_{1,2}=\omega_{1,2}t+\theta^0_{1,2}$, $\omega^2_{1,2}$, and $\mathbf{e}_{1,2}$ are the eigenvalues (real and positive) and the corresponding eigenvectors (generally complex) of matrix a_{ij}. The eigenvectors in Eq. (12) are orthonormal, i.e., $\mathbf{e}_i^\dagger\cdot\mathbf{e}_j=\delta_{ij}$ and the normal mode representation is valid for both the nondegenerate ($\omega_1\neq\omega_2$) and degenerate ($\omega_1=\omega_2=\omega$) problems. Nevertheless, in the nondegenerate problem, the eigenvectors can be written as $\mathbf{e}_i=\tilde{\mathbf{e}}_i\exp(i\phi_i)$, where $\tilde{\mathbf{e}}_i$ is real and uniquely defined by a_{ij}. Since arbitrary phase factors ϕ_i can be always included in the definitions of $\theta_{1,2}$ in Eq. (12), we can view $\mathbf{e}_{1,2}$ in this expression as real and unique. In contrast, in degenerate problems, the choice of the orthonormal eigenvectors is not unique and an arbitrary normalized superposition of a pair of eigenvectors can be taken as a new eigenvector. In our case [see Eq. (2)], the degeneracy means $a_{12}=a_{21}=0$, $a_{11}=a_{22}$ and, therefore, unit vectors $\hat{\mathbf{q}}_i$ along the axis in the (q_1,q_2) plane comprise a particular choice of orthonormal eigenvectors in the problem. More generally, $\mathbf{e}_{1,2}$ in the degenerate case can be written as

$$\mathbf{e}_1=\alpha_1\hat{\mathbf{q}}_1+\tilde{\beta}_1\hat{\mathbf{q}}_2,$$

$$\mathbf{e}_2=\alpha_2\hat{\mathbf{q}}_1+\tilde{\beta}_2\hat{\mathbf{q}}_2, \tag{13}$$

where $\alpha_{1,2}$ may be taken positive and $\tilde{\beta}_{1,2}=\beta_{1,2}\exp(i\varphi_{1,2})$, where $\beta_{1,2}$ are positive. The orthonormality condition $\mathbf{e}_i^\dagger\cdot\mathbf{e}_j=\delta_{ij}$ yields four equations for the six parameters $\alpha_{1,2}$, $\tilde{\beta}_{1,2}$, $\varphi_{1,2}$

$$\alpha_1^2+\beta_1^2=1,\quad \alpha_2^2+\beta_2^2=1,$$

$$\alpha_1\alpha_2+\beta_1\beta_2\cos(\varphi_1-\varphi_2)=0, \tag{14}$$

$$\beta_1\beta_2\sin(\varphi_1-\varphi_2)=0,$$

and, therefore, two parameters remain undetermined. It is this freedom in the definition of the normal modes which can be used in the canonical perturbation theory to cancel, if possible, the lowest order resonant terms.

Next, in view of the first two equations in Eq. (14), we introduce new angles $0<\psi_{1,2}<\pi/2$ related to $\alpha_{1,2}$ and $\beta_{1,2}$ via $\alpha_{1,2}=\cos\psi_{1,2}$, $\beta_{1,2}=\sin\psi_{1,2}$, reducing Eq. (14) to just two equations

$$\cos\psi_1\cos\psi_2+\sin\psi_1\sin\psi_2\cos(\varphi_1-\varphi_2)=0, \tag{15}$$

$$\sin\psi_1\sin\psi_2\sin(\phi_1-\phi_2)=0. \tag{16}$$

There are two cases described by this system. In the first case $\psi_{1,2}\neq 0$, yielding $\varphi_1-\varphi_2=\pi$ and, consequently, $\psi_1+\psi_2=\pi/2$. In the second case $\psi_1=0$, $\psi_2=\pi/2$, and $\varphi_{1,2}$ are arbitrary. Thus, we arrive at two possible normal mode representations of $q_{1,2}$: for the first case

$$q_1=A_1\cos\psi_1\cos\theta_1+A_2\sin\psi_1\cos\theta_2,$$

$$q_2=A_1\sin\psi_1\cos\bar{\theta}_1-A_2\cos\psi_1\cos\bar{\theta}_2, \tag{17}$$

where $\bar{\theta}_{1,2}=\theta_{1,2}+\varphi_1$ and for the second case (the *decoupled* case in the following), including φ_2 in the definition of θ_2:

$$q_1 = A_1 \cos\,\theta_1, \;\; q_2 = -A_2 \cos\,\theta_2. \tag{18}$$

Formally, Eq. (18) is obtained from Eq. (17) by setting $\psi_1 = \varphi_1 = 0$. Finally, we calculate the action variables

$$I_{1,2} = \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1,2} p_i \frac{\partial q_i}{\partial \theta_{1,2}} d\theta_{1,2} = \frac{\omega}{2}A_{1,2}^2, \tag{19}$$

introduce $s=\tan\,\psi_1$ and $r=\sqrt{2/\omega}\cos\,\psi_1=[\omega(1+s^2)/2]^{-1/2}$, and rewrite Eq. (17) in the form

$$q_1 = r(\sqrt{I_1}\cos\,\theta_1 + s\sqrt{I_2}\cos\,\theta_2),$$

$$q_2 = r(s\sqrt{I_1}\cos\,\bar{\theta}_1 - \sqrt{I_2}\cos\,\bar{\theta}_2). \tag{20}$$

As expected Eq. (20) has *two* free parameters s and φ_1 and therefore, can be viewed as a *generalized rotation*. We shall use this freedom in applying Eq. (20) as the zero order ansatz in the canonical perturbation theory in Sec. III.

III. EXAMPLES

The following examples comprise perturbed, linearly degenerate 2D oscillators. We shall consider polynomial perturbations of order higher than two in $q_{1,2}$ and show that only special sets of parameters in the perturbations allow us to find the zero-order AA variables, removing leading order resonant terms in the canonical perturbation theory. We shall refer to these AA variables as *proper* variables and use the term *regularization* by rotation to describe our procedure of removal of resonances in the following. The special cases of oscillators allowing the regularization will be compared to the Painlevé positive cases.[12]

Painlevé analysis is one of the most powerful indirect methods of detecting integrability of dynamical systems.[12,17,18] In this method one investigates the solutions of the equations of motion in the complex time plain. If the singularities are well behaved (the only movable singularities are poles) there is good reason to expect that the model is integrable. This property of the solutions is called the Painlevé property. The Painlevé positive cases correspond to the values of parameters of the system such that its solutions possess the Painlevé property.

A. Generalized Henon-Heiles oscillator

The Hamiltonian of the generalized Henon-Heiles system is given by

$$H = H_0 + q_2 q_1^2 + \frac{\mu}{3}q_2^3, \tag{21}$$

where $H_0=\frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2)$ and μ is a parameter.

We adopt terminology of Sec. II A and define a small parameter ε by rescaling, $q_{1,2}=\varepsilon\tilde{q}_{1,2}$ and $H=\varepsilon^2\tilde{H}$. Then, omitting tildes in Eq. (3), we have

$$H_1 = q_2 q_1^2 + \frac{\mu}{3}q_2^3, \;\; H_2 = 0. \tag{22}$$

The singular terms in Eq. (8) correspond to $\mathbf{m}=(l,-l)$, $l=\pm 1, \pm 2, \ldots$. The first-order term $S_1(\mathbf{\Theta},\mathbf{I}')$ in series (4) is nonresonant because, according to Eq. (10), the resonant contribution in S_1 must come from $V_1(\mathbf{\Theta},\mathbf{I}')=H_1(\mathbf{\Theta},\mathbf{I}')$. However, H_1 is cubic in coordinates and, thus, does not contain $\mathbf{m}=(l,-l)$ terms. The second-order term $S_2(\mathbf{\Theta},\mathbf{I}')$ in series (4) has resonant terms associated with $V_2(\mathbf{\Theta},\mathbf{I}')$ [see Eq. (7)], where $(\partial H_1/\partial \mathbf{I}')\cdot(\partial S_1/\partial \mathbf{\Theta})$, contributes $\mathbf{m}=(l,-l)$, $l=\pm 1, \pm 2, \pm 3$ resonant terms. In calculating these terms, we must express the perturbation H_1 in terms of the AA variables $(\mathbf{I},\mathbf{\Theta})$ of the unperturbed system H_0 and formally substitute $\mathbf{I}'$ for $\mathbf{I}$.

We start with the simpler, zero order ansatz (20) for the decoupled case (with $s=\varphi_1=0$)

$$q_1 = \sqrt{2I_1}\cos\theta_1, \quad q_2 = \sqrt{2I_2}\cos\theta_2. \tag{23}$$

We substitute these expressions into Eq. (22) for H_1 and find the resonant terms in V_2. The straightforward, but lengthy calculation (made with the help of MATHEMATICA[19]) yields (after the substitution $\mathbf{I}=\mathbf{I}'$)

$$V_2(\mathbf{\Theta}, \mathbf{I}') = -\frac{I_1' I_2'}{6}(6-\mu)\cos(2\theta_1 - 2\theta_2). \tag{24}$$

modulo nonresonant terms. Thus, for this ansatz, the lowest order resonant term vanishes only if $\mu=6$.

More complex ansatz (20) yields

$$V_2(\mathbf{\Theta},\mathbf{I}) = C_1 \sin(\theta_1-\theta_2) + C_2\cos(\theta_1-\theta_2) + C_3\sin(2\theta_1-2\theta_2) + C_4\cos(2\theta_1-2\theta_2) + (\text{nonresonant terms}), \tag{25}$$

where

$$C_1 = \frac{r^4 s}{12}\sqrt{I_1' I_2'}(I_1'-I_2')(1+s^2)(6-\mu)\sin(2\varphi),$$

$$C_2 = -\frac{r^4 s}{12}\sqrt{I_1' I_2'}\{I_1'[(s^2-1)(5\mu^2-6\mu-4+(6-\mu)\cos(2\varphi))-5(1-\mu^2)] + I_2'[(1-s^2)(5\mu^2-6\mu-4+(6-\mu)\cos(2\varphi))-5s^2(1-\mu^2)]\}, \tag{26}$$

$$C_3 = \frac{r^4}{24}I_1' I_2'(s^4-1)(6-\mu)\sin(2\varphi),$$

$$C_4 = -\frac{r^4 s^2}{24}I_1' I_2'\{3+12\mu-5\mu^2[2+(s^{-1}-s)^2+(6-\mu)\cos(2\varphi)]\}.$$

Here and in the following $\varphi \equiv \varphi_1 + \pi/2$.

The singular terms in $V_2(\mathbf{\Theta}, \mathbf{I}')$ vanish if all C_i vanish identically. Condition $C_4 \equiv 0$ yields $\mu \neq 6$ and, therefore, condition $C_1 \equiv 0$ gives $\sin(2\varphi)=0$. Then $C_3 \equiv 0$ and $\cos(2\varphi)=\pm 1$. We still need to satisfy two conditions, $C_2 \equiv 0$ and $C_4 \equiv 0$. This is equivalent to the following three relations:

$$(s^2-1)[5\mu^2-6\mu-4\pm(6-\mu)]-5(1-\mu^2)=0,$$

$$(s^{-2}-1)[5\mu^2-6\mu-4\pm(6-\mu)]-5(1-\mu^2)=0,$$

$$3+12\mu-5\mu^2\pm[2+(s^{-1}-s)^2](6-\mu)=0. \tag{27}$$

Consider the positive sign $[\cos(2\varphi)=+1]$ in Eqs. (27) first. Taking the sum and the difference of the first two equations in Eqs. (27), we arrive at

$$(\mu-1)[2(\mu+1)+(s^{-1}-s)^2(\mu-2/5)]=0,$$

$$(\mu-2/5)(\mu-1)(s^2-s^{-2})=0,$$

TABLE I. Regularizable cases of the Henon-Heiles system $\left(H_0=\frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2)\right)$.

Parametric restriction on the perturbation	Parameters of the ansatz (20)	Weakly nonlinear Hamiltonian H_w	Full Hamiltonian
$\mu=-1$	$s=1$, $\varphi_1=\frac{\pi}{2}$	$H_w=H_0+\frac{1}{6}(I_1^2-12I_1I_2+I_2^2)$	$H=H_0+q_2q_1^2-\frac{1}{3}q_2^3$
$\mu=1$	$s=1$, $\varphi_1=0$	$H_w=H_0-\frac{5}{6}(I_1^2+I_2^2)$	$H=H_0+q_2q_1^2+\frac{1}{3}q_2^3$
$\mu=6$	$s=0$, $\varphi_1=0$	$H_w=H_0-\frac{5}{12}(I_1^2+16I_1I_2+36I_2^2)$	$H=H_0+q_2q_1^2+2q_2^3$

$$(s^{-1}-s)^2(\mu-6)+5(\mu^2-2\mu-3)=0. \tag{28}$$

From the first equation in Eq. (28), $\mu\neq 2/5$. From the third equation $\mu\neq 1$. Therefore, the second equation yields $s=1$ and then, the first equation gives $\mu=-1$.

In the case $\cos(2\varphi)=-1$, we have

$$(\mu+1)[2(\mu-1)+(s^{-1}-s)^2(\mu-2)]=0,$$

$$(\mu-2)(\mu+1)(s^2-s^{-2})=0, \tag{29}$$

$$(s^{-1}-s)^2(6-\mu)+5\mu^2-14\mu+9=0.$$

From the first equation in Eq. (29), $\mu\neq 2$. From the third equation $\mu\neq -1$. Therefore, from the second equation yields $s=1$ and then $\mu=1$.

To summarize, there are three values of μ such that the lowest order resonant terms in the perturbation series (4) can be canceled by a proper choice of the zero order AA variables, i.e., $\mu=-1, 1, 6$. Since, at this stage, we also know $s=1, 1, 0$ and $\varphi_1=\frac{\pi}{2}, 0, 0$ in the three cases, we can calculate the corresponding AA variables. Finally, one can also find the weakly nonlinear Hamiltonian in terms of these AA variables as given by our *regularized* perturbation theory. We summarize all these results in Table I.

Next, we observe that only for cases $\mu=1, 6$ the problem (21) possesses the Painlevé property (for detailed application of the Painlevé analysis in this case see Ref. 13). These cases are also known to be completely integrable. The case $\mu=-1$ corresponds to the classical Henon-Heiles problem.[10] The system does not possess the Painlevé property[13] and is not integrable. Nevertheless, our procedure allowed us to cancel the lowest (second) order resonant terms in the perturbation series and made it possible to formally calculate the first nonlinear corrections to the Hamiltonian.

B. Quartic potential

The Hamiltonian in this example is

$$H=H_0+\frac{1}{4}(q_1^4+\sigma q_2^4)+\frac{\rho}{2}(q_2q_1)^2, \tag{30}$$

where, again, $H_0=\frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2)$. Similarly to the previous example we start with the ansatz (20) for the decoupled case (with $s=\varphi_1=0$), yielding

$$V_2(\mathbf{\Theta}, \mathbf{I}') = \frac{I_1' I_2'}{4} \rho \cos(2\theta_1 - 2\theta_2), \tag{31}$$

modulo nonresonant terms. The solution is $\rho=0$, which corresponds to the separable Hamiltonian.

A more general Eq. (20) gives

$$V_2(\mathbf{\Theta}, \mathbf{I}') = C_1 \sin(\theta_1 - \theta_2) + C_2 \cos(\theta_1 - \theta_2) + C_3 \sin(2\theta_1 - 2\theta_2) + C_4 \cos(2\theta_1 - 2\theta_2) + (\text{nonresonant terms}), \tag{32}$$

where

$$C_1 = -\frac{r^4 s}{8} \sqrt{I_1' I_2'} [-I_1'(1+s^2) + I_2'(s^2+1)] \rho \sin(2\varphi),$$

$$C_2 = \frac{r^4 s}{16} \sqrt{I_1' I_2'} \{ I_1' [(s^2-1)(\rho(2-\cos(2\varphi)) - 3\sigma) + 3(1-\sigma)] + I_2' [(1-s^2)(\rho(2-\cos(2\varphi)) - 3\sigma) + 3s^2(1-\sigma)] \}, \tag{33}$$

$$C_3 = -\frac{r^4}{16} I_1' I_2' (1-s^4) \rho \sin(2\varphi),$$

$$C_4 = -\frac{r^4 s^2}{16} I_1' I_2' \{ 4\rho - 3(1+\sigma) + [2 + (s^{-1} - s)^2] \rho \cos(2\varphi) \}.$$

Condition $C_4 \equiv 0$ yields $\sigma=-1$ if $\rho=0$. But for these values of the parameters

$$C_2 = \frac{3r^4 s}{16} \sqrt{I_1' I_2'} (I_1' + I_2')(s^2+1), \tag{34}$$

which, obviously, does not vanish identically. Therefore, $\rho \neq 0$. Then, from $C_1 \equiv 0$, we get $\sin(2\varphi)=0$, i.e., $C_3 \equiv 0$ and $\cos(2\varphi) = \pm 1$. We still have to satisfy $C_2 \equiv 0$ and $C_4 \equiv 0$. This is equivalent to

$$(s^2-1)[\rho(2-\cos(2\varphi)) - 3\sigma] + 3(1-\sigma) = 0,$$
$$(s^{-2}-1)[\rho(2-\cos(2\varphi)) - 3\sigma] + 3(1-\sigma) = 0, \tag{35}$$
$$4\rho - 3(1+\sigma) + [2 + (s^{-1}-s)^2] \rho \cos(2\varphi) = 0.$$

Consider the case $\cos(2\varphi)=1$ first. Taking the sum and the difference of the first two equations in Eqs. (35), we arrive at

$$6(1-\sigma) + (s^{-1}-s)^2(\rho - 3\sigma) = 0,$$
$$(s^{-2}-s^2)(\rho - 3\sigma) = 0, \tag{36}$$
$$[6 + (s^{-1}-s)^2]\rho - 3(1+\sigma) = 0.$$

The only solution of this system is $s=1$, $\rho=\sigma=1$. Similarly, in the case $\cos(2\varphi)=-1$, the only solution is $s=1$, $\rho=3$, and $\sigma=1$.

To summarize, there are only three sets of parameters ρ and σ, such that the lowest order resonant terms in the perturbation series (4) can be removed. In the case $\rho=\sigma=1$, we have $s=1$

TABLE II. Regularizable cases of the quartic coupling $\left(H_0=\frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2)\right)$.

Parametric restriction on the perturbation	Parameters of the ansatz (20)	Weakly nonlinear Hamiltonian H_w	Full Hamiltonian
$\rho=1$	$s=1$	$H_w=H_0+\frac{1}{4}(I_1^2+4I_1I_2+I_2^2)$	$H=H_0+\frac{1}{4}(q_1^4+q_2^4)+\frac{1}{2}(q_2q_1)^2$
$\sigma=1$	$\varphi_1=\frac{\pi}{2}$		
$\rho=3$	$s=1$	$H_w=H_0+\frac{3}{4}(I_1^2+I_2^2)$	$H=H_0+\frac{1}{4}(q_1^4+q_2^4)+\frac{3}{2}(q_2q_1)^2$
$\sigma=1$	$\varphi_1=0$		
$\rho=0$	$s=0$	$H_w=H_0+\frac{3}{8}(I_1^2+\sigma I_2^2)$	$H=H_0+\frac{1}{4}(q_1^4+\sigma q_2^4)$
Any σ	$\varphi_1=0$		

and $\cos(2\varphi)=1$. In the case $\sigma=1$ and $\rho=3$, we have $s=1$ and $\cos(2\varphi)=-1$. In the third decoupled case, $\rho=0$. It is known[13] that in the first two cases above the problem (30) possesses the Painlevé property. These cases are also completely integrable.[13] The last case is integrable too, because it separates in Cartesian coordinates. We summarize our calculations for the quartic potential in Table II, similar to Table I in the previous example.

C. Sextic potential

The Hamiltonian in this case is given by

$$H = H_0 + aq_1^6 + bq_2^6 + dq_2^2q_1^4 + eq_2^4q_1^2, \tag{37}$$

The analysis proceeds along the same lines as in the previous two examples. Therefore, we shall omit the details of the calculations in this example and present the final results only. The decoupled ansatz (23) yields

$$V_2(\mathbf{\Theta}, \mathbf{I}') = (dI_1'^3I_2' + eI_1'I_2'^3)\cos(2\theta_1 - 2\theta_2), \tag{38}$$

modulo nonresonant terms. Thus, we require $d=e=0$, which corresponds to an integrable system with separable Hamiltonian.

The general ansatz (20) yields two possibilities. The first is

$$s = 1, \quad \cos(2\varphi) = 1, \quad a = b, \quad d = e = 3a, \tag{39}$$

and the second

$$s = 1, \quad \cos(2\varphi) = -1, \quad a = b, \quad d = e = 15a. \tag{40}$$

The system (37) was examined for the Painlevé property.[20] The only two sets of parameters for which the problem becomes Painlevé positive are the same two parameter sets identified above in our analysis. Table III is similar to Tables I and II and summarizes our calculations in the sextic potential case.

D. Periodic Toda lattice

The Hamiltonian in our final example will not have a free parameter. Instead, we shall consider the famous (*integrable*) three particle periodic Toda lattice,[11] which is linearly degenerate. As a consequence, one encounters the problem of vanishing denominators in studying weakly nonlinear excitations of the lattice via the canonical perturbation theory. We shall analyze this case and demonstrate that, again, a proper choice of the zero order AA variables solves this problem.

We start with the Toda Hamiltonian $(Q_{n+3}=Q_n)$

TABLE III. Regularizable cases of the sextic coupling $\left(H_0=\frac{1}{2}(p_1^2+q_1^2)+\frac{1}{2}(p_2^2+q_2^2)\right)$.

Parametric restriction on the perturbation	Parameters of the ansatz (20)	Weakly nonlinear Hamiltonian H_w	Full Hamiltonian
$a=b$ $d=e=3a$	$s=1$ $\varphi_1=\frac{\pi}{2}$	$H_w=H_0+a(I_1^3+9I_1^2I_2+9I_1I_2^2+I_2^3)$	$H=H_0+a(q_1^6+q_2^6+3q_2^2q_1^4+3q_2^4q_2^2)$
$a=b$ $d=e=15a$	$s=1$ $\varphi_1=0$	$H_w=H_0+10a(I_1^3+I_2^3)$	$H=H_0+a(q_1^6+q_2^6+15q_2^2q_1^4+15q_2^4q_2^2)$
Any a, b $d=e=0$	$s=0$ $\varphi_1=0$	$H_w=H_0+\frac{5}{2}(aI_1^3+bI_2^3)$	$H=H_0+aq_1^6+bq_2^6$

$$H=\frac{1}{2}\sum_{n=1}^{3}P_n^2+\sum_{n=1}^{3}e^{Q_n-Q_{n+1}}-3. \tag{41}$$

In the case of zero total momentum and by simple canonical transformation and rescaling, this problem reduces to that described by another Hamiltonian of the form[21]

$$\overline{H}=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{24}(e^{2q_2+2\sqrt{3}q_1}+e^{2q_2-2\sqrt{3}q_1}+e^{-4q_2})-\frac{1}{8}. \tag{42}$$

We expand Eq. (42) to fourth order in coordinates, yielding an approximation

$$\overline{\overline{H}}=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^2+q_2^2)+q_2q_1^2-\frac{1}{3}q_2^3+\frac{1}{2}(q_1^2+q_2^2)^2. \tag{43}$$

Thus, the chain is linearly degenerate and one expects resonant terms in the perturbation series (with a linear solution as the zero order approximation) to blow up, unless the zero order approximation is chosen appropriately. In order to find this proper zero order approximation, one can proceed with the analysis similar to that in our examples above. However, we observe that the cubic nonlinear term in Eq. (43) corresponds to the case $\mu=-1$ of the Henon-Heiles problem (see Table I), while the fourth order term in Eq. (43) belongs to the case $\rho=\sigma=1$ in the example with the quartic perturbations (see Table II). Both terms yield the same order resonant contributions in the perturbation series [see Eqs. (26) and (33)], and the proper zero order approximation should take the combined effect of these terms into account. But, the proper zero order approximations, in the $\mu=-1$ case of the Henon-Heiles problem (Table I) and $\rho=\sigma=1$ quartic perturbation case (Table II) were the same. Therefore, the same zero order approximation will also remove the resonant terms due to both the third and the fourth order terms in Eq. (43). We conclude that the proper choice of the zero order action variables for the linear limit of three-particle periodic Toda lattice corresponds to the choice $s=1$ and $\psi_1=\pi/2$ in the ansatz (20). Furthermore, the weakly nonlinear Hamiltonian (43) in the proper action variables can be found by simply adding the weakly nonlinear corrections for the $\mu=-1$ case of the Henon-Heiles problem (Table I) and the $\rho=\sigma=1$ case of the quartic perturbation (Table II) (the latter contribution must be multiplied by a factor of two, because the quartic term in Eq. (43) is twice the quartic term in the $\rho=\sigma=1$ case of the quartic perturbation example). This yields

$$H_w=I_1+I_2+\frac{2}{3}(I_1^2+I_2^2). \tag{44}$$

Note that in contrast to the Henon-Heiles and the quartic perturbation problems, the weakly nonlinear Hamiltonian of the Toda lattice separates in the proper action variables. This fact (also valid in the general N-particle periodic lattice case) was recently used in analyzing autoresonant excitation and control of multiphase solutions of the Toda system.[22]

IV. CONCLUSIONS

We have studied the possibility of transformation to approximate AA variables in variety of 2D degenerate (i.e., 1:1 resonant) harmonic oscillators perturbed by a polynomial interaction potential dependent on parameters. While in the absence of linear resonances the formal transformation to perturbed AA variables can be regularly performed using the canonical perturbation theory, the straightforward application of the theory fails in a linearly degenerate case. The technical reason is the appearance of zero denominators in the perturbation series. The more fundamental reason is that a linearly degenerate system admits a variety of the zero order transformations to AA variables while a perturbation generically lifts the degeneracy. As a consequence, if the transformation exists, it should be furnished by the canonical perturbation theory with a proper choice of the zero order transformation, which is consistent with the restricted symmetries of the perturbed problem.

We have used the aforementioned freedom of choice in constructing the general ansatz for the zero-order transformation (generalized rotation) to AA variables. The ansatz depends on free parameters, the values of which are determined by the condition that the first order resonant terms in the perturbation series vanish identically (regularizability condition). When such values exist, the formal transformation to the first-order AA variables can be accomplished. Since the latter property is nongeneric of nonlinear systems, it is expected that an arbitrary nonlinear perturbation of the potential will not allow regularization for either choice of the zero order transformation. We have found all values of the parameters in the perturbing potentials for which the regularization is possible. Interestingly, these special values of parameters in the perturbing potentials include *all* the known cases of the Panlevé positive systems in the considered class of perturbations.

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