AUTORESONANT EXCITATION AND CONTROL OF NONLINEAR WAVES

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Abstract. Plasmas and fluids are extended systems described by nonlinear differential equations frequently having a multiplicity of nontrivial solutions far from equilibrium. The question of reaching and controlling a particular stable solution (pattern) in this set of solutions by starting from simple initial/boundary conditions is fundamental to many applications. A simple (and realizable) approach to nonlinear wave formation based on capturing the system into resonance with slow external perturbations, followed by a continuing self-synchronization (autoresonance) in space and/or time is discussed. The synchronization means excursion in the system solution's space with possible emergence of the desired coherent state. Recent applications to formation of nontrivial plasma/vortex configurations in 2D ideal fluid dynamics and generation of multi-phase waves in the Korteweg-de-Vries (KdV) system are discussed. A similar approach is applied to excitation of large amplitude standing waves (plasma oscillations) in the context of driven sin-Gordon equation.

I. INTRODUCTION

A. PFS paradigm. Nonlinear wave equations in fluids and plasmas frequently have many classes of solutions. Suppose, one desires to generate (experimentally or in simulations) a waveform u of a particular class in a system governed by wave equation $N(u) = 0$, N being a nonlinear spatio-temporal differential operator. The standard procedure for achieving this goal requires accurate realization of some nontrivial initial/boundary conditions. Alternatively, in searching for a more realizable approach, one can start from simple initial/boundary conditions, but consider a perturbed problem $N(u) = \varepsilon f$, where $\varepsilon << 1$ and f is a function on space-time. Can one find a *simple f*, such that u in the perturbed system arrives at a vicinity of the desired nontrivial solution in the process of evolution? More counterintuitivly, is it possible to excite (and control) a *large* amplitude wave from zero by using a *perturbation*? These questions were addressed previously for a class of nonlinear waves [1], in applications to the nonlinear Shroedinger equation [2,3], and, most recently, in the context of formation of nontrivial vorticity waves (V-states) in 2D-fluid dynamics [4] and multiphase waves of the Korteweg-de-Vries (KdV) equation [5]. In all these cases, the answer was positive, while the perturbing function yielding the desired solutions was either a simple wave or a superposition of simple waves having slowly varying frequencies and/or

wave vectors passing through resonances with the unperturbed system. Furthermore, successful excitations were always accompanied by a persisting phase-locking (synchronization) with the driving perturbations, yielding efficient controllability of the excitation process. Therefore, the approach can be referred to as *pattern formation by synchronization* (PFS) by passage through resonances. We describe two recent applications of the PFS in subsections B and C below in order to demonstrate the main ideas of the approach and illustrate the broad range of possible applications. These examples will be followed (in subsection D) by an introduction to the main subject of this study, the generation of large amplitude standing waves in the context of the driven sin-Gordon equation.

B. Formation of m-fold symmetric V-states. Two-dimensional dynamics of ideal fluids is governed by Euler's equations

$$\frac{\partial \omega}{\partial t} + \mathbf{V} \cdot \nabla \omega = 0; \quad \Delta \psi = \omega, \tag{1}$$

where ψ is the stream function, $\mathbf{V} = -\nabla \psi \times \mathbf{e}_z$ is the fluid velocity field and $\boldsymbol{\omega} = \nabla \times \mathbf{V} = \omega \mathbf{e}_z$ is the vorticity. The same system of equations describes
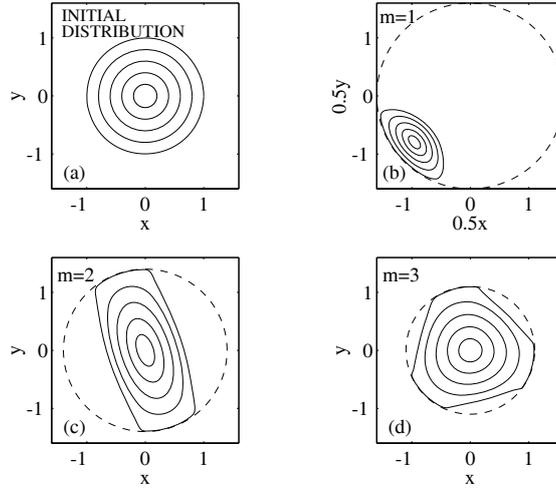

Fig.1 Nonuniform V-states in bounded space (the stream functions vanish on dashed circles). (a): Initial vorticity is stepwise-constant $\omega_n = \omega_0(1 - 0.6 r_n^2)$, $r_n = 0.2n, \, n = 1,2,...,5$; (b): *m=1* diocotron mode rotating due to interaction with the image charge; (c): *m=2*; (d): *m=3*. All these V-states emerge by capture into resonance, starting from the same vorticity distribution shown in (a).

azimuthal dynamics of magnetized electron clouds [6], where ψ is the electric potential, while ω is proportional to the electron density. These equations allow a variety of solutions. For example, Deem and Zabusky [7] discovered *m*-fold symmetric, steadily rotating vortex patch (uniform ω on a compact support) solutions (uniform V-states) in this system. The related question is that of existence *nonuniform* V-states. Also, in the PFS context, can one create and control both uniform and nonuniform V-states by a small forcing? A positive answer to these questions was found recently [4]. It was shown that m-fold symmetric V-states can be formed by subjecting an initially axisymmeric vortex to a weak external strain, having stream function $\psi_{ext} \sim \varepsilon(t) r^m \cos(m\varphi)$, (r, φ being polar coordinates) and oscillating strain rate $\varepsilon = \varepsilon_0 \cos(\int \Omega dt)$. Frequency $\Omega(t)$ of the strain was a slowly varying function of time, passing through a resonance with the Kelvin mode of the initial vortex. We illustrate this process in Fig.1, showing the final nonuniform V-states (Figs.1b,c,d) in a bounded space, as obtained by starting with a nonuniform axisymmetric vortex (Fig.1a) and using chirped frequency strains of *m=1,2,* and *3* symmetries.

C. Synchronized multi-phase KdV waves. Our second illustration deals with multi-phase solutions of spatially periodic KdV equation. The ideal (unperturbed) KdV equation has a variety of solutions, including traveling wave trains $u = u(\theta), \theta = \kappa x - \nu t$ and solitary waves. If one adds periodic boundary condition $u(x,t)=u(x+L,t)$, a class of multiphase (N-gap) solutions exist of form $u = u(\mathbf{\Theta}), \mathbf{\Theta} = \{\theta_1,...,\theta_N\}$, where $\theta_m = \kappa_m x - \nu_m t$ have constant ν_m and κ_m (a multiple of $k_0 = 2\pi / L$). Posing the question of generating N-gap waves from trivial initial conditions (PFS paradigm), we consider the driven system

$$u_t + 6uu_x + u_{xxx} = \varepsilon f(x,t) \qquad (2)$$

The goal is achieved by using a superposition $\varepsilon f = \sum \varepsilon_m \sin \varphi_m$ of eikonal waves, where the driving phases $\varphi_m = mk_0 x - \int \omega_m(t) dt$ have slowly varying frequencies $\omega_m(t)$, all passing at, say *t=0*, through linear resonances $[-(mk_0)^3]$ of the unperturbed problem. Fig.2 shows an example of excitation of a *3*-gap solution by using a combination of three drives as described above, while Fig.3 presents our diagnostics of the excited solution. The spectral approach of the Inverse Scattering Transform (IST) was used for finding the number of phases in the wave and their frequencies in the process of evolution. The IST method associates an N-gap solution with two linear eigenvalue

problems, yielding the main and auxiliary spectra $\{E_i\}$ and $\{\mu_i\}$ [8]. The main spectrum components remain constant in the unperturbed problem, and evolve
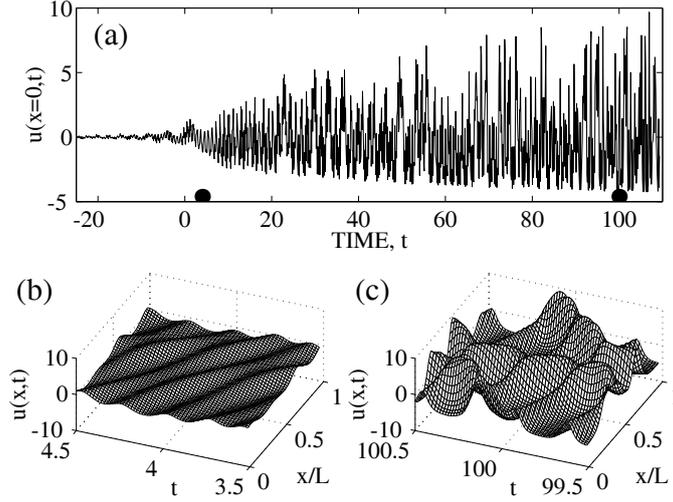

Fig.2 Emergence of a 3-phase waveform u in configuration space [5]. (a): $u(x=0,t)$; (b), (c): $u(x,t)$ in two time windows $3.5<t<4.5$ and $99.5<t<100.5$ [indicated by full circles on the t-axis in (a)].

slowly in the driven problem. *N open gaps* $[E_{2n}, E_{2n+1}]$ in the main spectrum indicate N-gap solution (three such gaps are seen in Fig.3a, i.e., a 3-gap solution). In contrast to $\{E_i\}$, the auxiliary spectrum components $\{\mu_i\}$ oscillate, each inside one of the open gaps. The frequencies of N-gap solutions can be calculated from its main spectrum [8]. Fig.3 shows the results of these calculations for the example of Fig.2. The three linearly chirped driving frequencies in this example are also shown in Fig.3. One observes a continuing phase-locking of all degrees of freedom in the driven problem beyond the linear resonance ($t=0$). This persisting resonance (synchronization) in the system despite the variation of the driving frequencies will be refereed to as *wave autoresonance* [9] in the following. The threshold phenomenon is one of the most important signatures of autoresonance. For example, one finds that the synchronization in Fig.3b requires driving amplitudes ε_n to exceed certain threshold values, all scaling as $\varepsilon_m^{th} \sim \alpha_m^{3/4}$, $\alpha_m = d\omega_m / dt$ being the frequency chirp rate of the corresponding driving component. One can see the departure of one of the frequencies of the 3-gap solution from resonance in Fig.3b, when

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the associated driving amplitude is slightly below the threshold. The autoreso-nance threshold phenomenon was first discovered in experiments with mag-netized
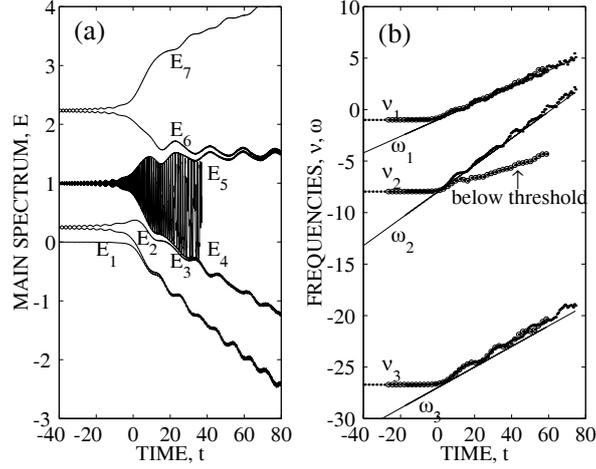

Fig.3 The spectral analysis of the driven 3-gap solution [5]. (a) The opening of three main spectrum gaps $[E_{2n}, E_{2n+1}]$ by adiabatic synchronization (oscillations of the auxiliary spectrum component μ_2 are shown in the gap $[E_4, E_5]$; (b): The fre-quencies ν_n of the excited wave (dots) and chirped driving frequencies ω_n (straight lines). The circles illustrate escape from resonance for ε_2 below the threshold.

electron clouds [10], where rotating *m=1*-type electron/vortex structures (dio-cotron modes), analogous to that shown in Fig.1b, were formed by synchroniz-ing external potentials. We shall further discuss this phenomenon in our new application in Sec.III.

D. Autoresonant standing waves. Following illustrations of the PFS in vortex and KdV systems, we proceed to our main problem of autoresonant excitation of large amplitude standing waves of the periodic sine-Gordon (SG) equation. We consider the driven system

$$u_{tt} - u_{xx} + \sin u = \varepsilon f(x,t), \qquad (3)$$

where both *u* and driving function *f* are subject to periodic boundary condi-tions *u(x,t)=u(x+L,t)*, *f(x,t)=f(x+L,t)*. The sine-Gordon equation (ε =0) is one of the most important equations of nonlinear physics and describes many physical applications [11]. It has a variety of solutions, a *single* phase wave,

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$u(x,t) = u(\theta)$, $\theta = \kappa x - \nu t$, being the simplest example. If one adds periodic boundary condition, standing SG wave solutions exist of form [12]

$$u(x,t) = 4\tan^{-1}[F(x)G(t)], \qquad (4)$$

where $F(x)$ and $G(t)$ are expressed in terms of Jacobian elliptic functions, which are periodic in x and t respectively. Well known realizations of Eq.(4) are *breather* and *plasma* oscillations. In the breather case $F(x) = A dn[\beta x, m_1]$ and $G(t) = cn[\gamma t, m_2]$, while, for plasma oscillations, $F(x) = A cn[\beta x, m_1]$ and $G(t) = cn[\gamma t, m_2]$. There are *5* constants in the definition of each of these standing waves, i.e., β, γ amplitude A and moduli $m_{1,2}$ of the elliptic functions, but only *two* of these constants (say $m_{1,2}$) are independent, while others are related algebraically. For example, in the breather case [13], $\beta = \gamma A$ (the dispersion relation) and $A^2(1 - m_1^2) = m_1^2 / A^2 = (\gamma^2 + \beta^2)^{-1} - 1$. For plasma oscillations [13], the dispersion relation is $\gamma^2 - \beta^2 = (1 - A^2)/(1 + A^2)$, while $\beta^2(m_1^2 - A^2) = \gamma^2(A^2 - m_2^2) = A^2(1 + A^2)^{-2}$. Note that Eq.(4) represents simplest *two-phase* solutions of SG equation. The two-phase structure is emphasized by using notation $u = U(\theta_1, \theta_2; m_1, m_2)$, where U is 2π-periodic with respect to phase variables θ_1, θ_2, describing oscillations of either *F(x)* or *G(t)*. The wave numbers associated with the spatial oscillations are $\kappa = \theta_{1x} = \beta /[2K(m_1)]$ and $\beta /[4K(m_1)]$ in the breather and plasma cases respectively, K is the complete elliptic integral of the first kind, while the frequency of temporal oscillations in both cases is $\nu = \theta_{2t} = \gamma /[4K(m2)]$.

The main goal of the present work is to find simple perturbations εf in (3), yielding excitation of large amplitude plasma oscillations, starting from *zero* initial conditions. We outline our idea of control of plasma oscillations by adiabatic synchronization. Let the perturbing function in Eq.(3) be an oscillation $f = f_0(x)\cos \int \omega(t)dt$ having slowly varying frequency ω and periodically modulated amplitude $f_0(x) = f_0(x + L)$. Then, assume that the perturbed system allows an approximate *synchronized* solution of form (4), where parameters are slow functions of time, such that the solution is phase-locked with the perturbing function at *all* times. In other words, we assume the perturbed solution $u \approx U(\theta_1, \theta_2; m_1, m_2)$, where $m_{1,2}$ and $\nu = \theta_{2t}$ are slow functions of time, and the wave number κ is constant due to the periodic boundary conditions, i.e., $\theta_1 = \kappa x$ and $\theta_2 \approx \int \nu dt$. The phase-locking assumption translates into the equality of the wave numbers and frequencies, i.e., $\kappa = 2\pi / L$ (as-

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suming lowest order spatial mode) and $\nu \approx \omega(t)$. This, in turn, means existence of two additional algebraic constrains on parameters $m_{1,2}$ of the standing wave, while its phases $\theta_{1,2}$ are known by phase-locking assumption. Therefore, if the above-mentioned synchronized solution exists, it can be controlled by a single external parameter [the frequency $\omega(t)$ of the driving perturbation]. We shall see below that under certain conditions, the passage through resonances in the driven SG system yields the desired persistent phase-locking between the driving perturbation and the emerging standing waves. The analysis of this excitation process comprises the main goals of the present work.

The ac-driven periodic SG system was studied in the past for *constant* frequency drives [14]. The idea of passage through resonance and subsequent synchronization for exciting *single* phase SG waves was suggested more recently [15]. The theory in this case used Whitham's single phase averaged variational principle [16] describing slow evolution of the synchronized state. A similar approach was also applied to nonlinear mode conversion in the system of two weakly coupled SG equations with slow parameters [17]. The present work is an extension of the idea of control of SG waves by synchronizing perturbations to simplest multiphase solutions, the plasma oscillations. Our presentation will be as follows. We will proceed from numerical illustrations of excitation and control of large amplitude standing waves by synchronizing perturbations in Sec.II. Section III will describe our theory of thresholds for synchronization of plasma oscillations in the driven SG system. A weakly nonlinear version of Whitham's two-phase averaged variational principle will be developed for this application. Finally, Sec.IV will present our conclusions.

II. ADIABATIC SYNCHRONIZATION VIA SIMULATIONS

We start by illustrating the idea of controlling plasma oscillations by synchronization in simulations. We solve Eq.(3) numerically by a standard spectral method [18], subject to zero initial (at $t = t_{in}$) and periodic boundary conditions. Our driving perturbation is a small amplitude standing wave

$$\varepsilon f(x,t) = \varepsilon \cos(2\pi x / L) \cos(\int \omega dt). \qquad (5)$$

The driving frequency in Eq.(5) decreases linearly in time, $\omega = \omega_0 - \alpha t$ ($\alpha \ll 1$), passing, at $t=0$, the linear resonance, $\omega = \omega_0 = (k_0^2 + 1)^{1/2}$, $k_0 = 2\pi / L$ with small amplitude plasma oscillations of the unperturbed system.

The simulation results presented in Fig.4a-d illustrate a large amplitude standing wave emerging from zero after passage through resonance. The thick line in Fig.4a shows evolution of "energy" $E_1 = < 0.5(u_t^2 + u_x^2) + \cos u >_x - 1$,

where $<...>_x = L^{-1}\int_0^L (...)dx$ means averaging over one spatial period. The energy is time independent for plasma oscillations of perfect SG system and its growth in the driven case indicates excitation of a growing amplitude wave. We used parameters $L=7, \alpha = 1.5 \times 10^{-4}, \varepsilon = 8.25 \times 10^{-4}$ and initial time $t_{in} = -1000$ in our simulations and switched off the driving function at $t_f = 2500$ (note that E_1
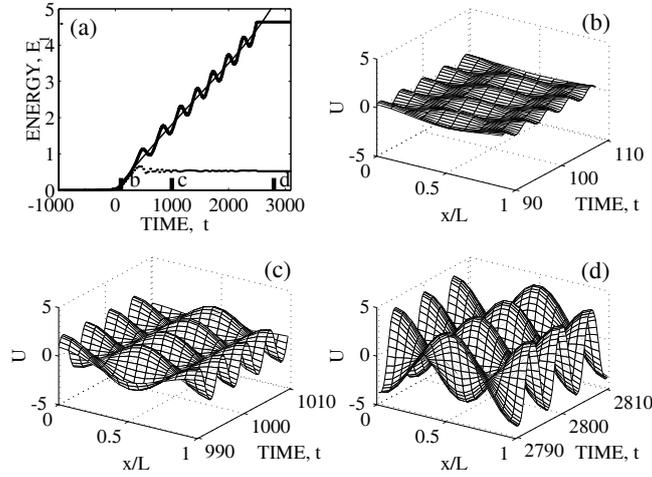

Fig.4 The emergence of plasma oscillations by synchronization. (a): Energy E_1 versus time (thick line). The ideal synchronized state is shown by thin line. Dotted line represents evolution of E_1 in simulations for ε slightly below threshold. (b), (c), and (d): The numerical waveforms in three time windows of duration $\Delta t = 20$ at different stages of evolution. Location of these windows is shown by bars on the t-axis in (a).

Remains constant beyond this time). We also added a small, $O(10^{-4})$ spatial modulation in the initial conditions, in checking stability of the excited solution. Figures 4b-d show the actual numerical waveforms as observed in three narrow time windows\ of duration $\Delta t = 20$ at different stages of excitation. The short bars on the t-axis in Fig.4a indicate positions of these windows, i.e., just beyond the linear resonance ($t=100$, Fig.4b), at some intermediate stage (t=1000, Fig.4c), and at $t_{in} = 2800 > t_f$ (Fig.4d). These results show that the

excited wave is indeed continuously phase-locked in both space and time with the driving standing wave. The spatial phase-locking (the location of the wave maxima remains at *x=L/2* at all times) is obvious, while one can also see that the frequency of temporal oscillations of the waveform decreases as one passes from Fig.4b, through Fig.4c to Fig.4d, following the decrease of the driving frequency. One also finds numerically that the growing amplitude wave (as in Figs.4b-d) beyond the linear resonance emerges only if the driving amplitude exceeds a threshold. The driving amplitude above is just slightly higher than the threshold value $\varepsilon_{th} = 8 \times 10^{-3}$ for synchronization in this case [see Eq.(24) in Sec.III]. The dotted line in Fig.4a shows a similar simulation, but for $\varepsilon = 7.95 \times 10^{-3}$, i.e., just below the threshold. We see that the energy in this case saturates at some relatively low value. The simulations also show that the phase-locking between the wave and the drive in this case is destroyed near the linear resonance. Furthermore, we find that if the driving field is switched off as in Fig.1, the resulting solution remains numerically stable for times much longer than those shown in the Figure. Nevertheless, if the drive is present beyond t_f, but the driving frequency remains constant, the excited wave stays phase-locked with the drive for some time, and then becomes unstable, destroying the phase-locking at later times. This instability growth rate increases with ε, so reaching larger plasma oscillation energies by synchronization approach requires smaller ε and, therefore, smaller driving frequency chirp rate due to the threshold phenomenon. In order to qualitatively test the form of the excited wave and synchronization in the system, we have assumed existence of an ideal synchronized state beyond the linear resonance and found parameters of this state by solving the system of algebraic equations

$$\gamma^2 - \beta^2 = (1 - A^2)/(1 + A^2), \tag{6}$$

$$\beta^2(m_1^2 - A^2) = \gamma^2(A^2 - m_2^2) = A^2(1 + A^2)^{-2}, \tag{7}$$

$$\beta = 4K(m_1)/L; \gamma = 4K(m_2)\omega(t). \tag{8}$$

We substitute these parameters in the expression for the energy of plasma oscillations of the perfect SG equation to get

$$E_1 = 8\beta^2[m_1^2 - 1 + E(m_1)/K(m_1)] \tag{9}$$

(*E* is the complete elliptic integral of the second kind), and present the results in Fig.1a by a thin line. We observe that beyond linear resonance, E_1 in simulations (the thick line) performs small oscillations around a monotonically growing energy predicted via the ideal persistent synchronization assumption. These oscillations comprise an additional characteristic signature of phase-locking in the driven system and their frequency scales as $\varepsilon^{1/2}$ (see Sec.III).

This completes our numerical illustration and we proceed to the problem of thresholds for autoresonant excitation of plasma oscillations.

III. THE THRESHOLD PHENOMENON

The threshold for synchronization for capture into resonance is a weakly nonlinear phenomenon and, in the case of plasma oscillations in the SG system, can be conveniently studied by using a weakly nonlinear version of the Whitham's averaged variational principle [16], adopted to our driven system. To this end, we formulate the driven problem (3) via the variational principle

$$\delta \int L \, dx \, dt = 0, \qquad (10)$$

where the Lagrangian of the weakly nonlinear limit is

$$L(u, u_x, u_t; x, t) = \tfrac{1}{2}(u_t^2 - u_x^2 - u^2) + \tfrac{1}{24}u^4 + \varepsilon u f(x,t) \qquad (11)$$

and $f(x,t) = \cos(k_0 x)\cos(\int \omega(t)dt)$. We assume a slow variation of the driving frequency $\omega(t)$, and, seeking adiabatically phase-locked standing wave solution of this problem, adopt the following two-scale representation:

$$u = u_0(t) + u_1(t)\cos(\theta_1)\cos(\theta_2) + u_2(t)\cos^2(\theta_1)\cos^2(\theta_2), \quad (12)$$

where the time dependence in the amplitudes $u_i(t)$ is slow, the phases $\theta_1(x) = k_0 x$, $\theta_2 = \theta_2(t)$ are fast, but the associated wave number k_0 is constant and frequency $\nu(t) = d\theta_2/dt$ is a slow function of time. Furthermore, since the standing wave solution for the linear unperturbed problem is $u = u_1\cos(\theta_1)\cos(\theta_2)$, $u_1 = const$, we order $u_{0,2} \sim O(u_1^2)$ in Eq.(12). We have already assumed phase-locking between θ_1 and the spatial driving phase k_0. We shall also assume that the second phase, θ_2, is locked with the driving phase $\int \omega(t)dt$, but allow a *small* and *slow* mismatch $\Phi(t) = \theta_2 - \int \omega(t)dt$ in the problem. The goal is to set up a procedure for finding *slow* amplitudes $u_i(t)$ and phase $\theta_2(t)$ in the problem.

Following Whitham's approach [16], we substitute representation (12) into Eq.(11) and average the result with respect to the fast phase variables, viewing the slow time in the problem fixed. This yields averaged Lagrangian

$$\Lambda(u_0, u_1, u_2; \theta_2, \theta_{2t}) = <L>_{\theta_{1,2}} = \Lambda_a + \Lambda_b + \tfrac{1}{2}\varepsilon u_1 \cos\Phi, \qquad (13)$$

Where $\Lambda_a = \tfrac{1}{8}(\nu^2 - k_0^2 - 1)u_1^2$ and

$$\Lambda_b = \tfrac{3}{512}u_1^4 + \tfrac{3}{32}u_2^2(\nu^2 - k_0^2 - \tfrac{9}{4}) - \tfrac{1}{2}u_0(u_0 + \tfrac{1}{2}u_2).$$

Note that θ_2 enters directly in Eq.(13) through Φ in the driving term, but also via $\nu = d\theta_2 / dt$. By Whitham's approach, the averaged Lagrangian (13) serves in the averaged variational principle

$$\delta \int \Lambda dx dt = 0. \qquad (14)$$

This principle yields the desired evolution equations for phase θ_2 and slow amplitudes u_i. For example, taking variations with respect to $u_{0,2}$, we obtain

$$u_0 + \tfrac{1}{4} u_2 = 0, \qquad (15)$$

$$\tfrac{3}{16} u_2^2 (\nu^2 - k_0^2 - \tfrac{3}{4}) - \tfrac{1}{4} u_0 = 0. \qquad (16)$$

The lowest order of (16) gives $u_0 = 0$ and, therefore, from (15), $u_2 = 0$. Similarly, the variation with respect to u_1 yields

$$\tfrac{1}{4}(\nu^2 - \omega_0^2) u_1 - \tfrac{3}{128} u_1^3 + \tfrac{1}{2}\,\varepsilon \cos \Phi = 0. \qquad (17)$$

where $\omega_0 = (1 + k_0^2)^{1/2}$ is the linear response frequency of the unperturbed system. Assuming $\nu \approx \omega_0$, we rewrite Eq.(17) as

$$\nu = d\theta_2 / dt \approx \omega_0 - \tfrac{3}{64} u_1^2 / \omega_0 - \tfrac{1}{2}\,(\varepsilon / \omega_0 u_1) \cos \Phi = 0, \qquad (18)$$

or, for the phase mismatch,

$$d\Phi dt \approx d\theta_2 / dt - \omega = \alpha t - \tfrac{3}{64} u_1^2 / \omega_0 - \tfrac{1}{2}\,(\varepsilon / \omega_0 u_1) \cos \Phi = 0, \qquad (19)$$

where $\omega = \omega_0 - \alpha t$ is used for the driving frequency. Finally, we take variation in (14) with respect to θ_2, yielding the lowest order evolution equation for u_1:

$$du_1 / dt = -(\varepsilon / 2\omega_0) \sin \Phi \qquad (20)$$

Equations (19) and (20) comprise the desired closed set of two slow equations for the phase mismatch and amplitude of the driven wave. We notice the possibility of having phase-locked, growing amplitude solution of this system beyond linear resonance ($t>0$). Indeed, by requesting

$$d\Phi / dt \approx \alpha t - \tfrac{3}{64} u_1^2 / \omega_0 \approx 0, \qquad (21)$$

one has $u_1 \approx \bar{u}_1 = (8\sqrt{\alpha \omega_0 / 3\alpha}) t^{1/2}$. In studying the stability of this solution, we write $u_1 = \bar{u}_1 + \delta u_1$ ($\bar{u}_1$ is viewed as slow and $|\delta u_1 / u_1| << 1$), differentiate Eq.(21) and substitute Eq.(20) to get

$$d^2 \Phi / dt^2 = \alpha + \frac{3\varepsilon \bar{u}_1}{128 \omega_0^2} \sin \Phi. \qquad (22)$$

For $\alpha << \Omega^2 = (3\varepsilon\bar{u}_1)/(128\omega_0^2)$ this equation predicts solutions oscillating at frequency $\Omega \sim O(\varepsilon^{1/2})$ around a small averaged value, i.e., phase-locking in the system. These oscillations of Φ lead to characteristic oscillating modulations of the energy (see Fig.4). But, how one gets into this adiabatically synchronized state via passage through resonance? This question leads to the problem of thresholds which is discussed next.

Define rescaled time and amplitude, $\tau = \alpha^{1/2}t$ and $A = \frac{1}{8}\alpha^{-1/4}\sqrt{\frac{3}{\omega_0}}u_1$, and introduce complex variable $\Psi = A\exp(i\Phi)$. One finds that Ψ is described by a nonlinear Shroedinger-type equation

$$id\Psi/d\tau + (\tau - |\Psi|^2)\Psi = \mu, \qquad (23)$$

having a single parameter $\mu = \frac{\sqrt{3}}{16}\varepsilon\omega_0^{-3/2}\alpha^{-3/4}$. We seek asymptotic solutions of this equation at large positive τ subject to zero initial Ψ at $\tau = -\infty$. This will describe passage through resonance (at *t=0*) in our system. There exist two such asymptotic solutions, the bounded solution $\Psi = \Psi_0\exp(i\tau^2/2), \Psi_0 = const$, while phase mismatch $\Phi = \tau^2/2$ increases in time, and the unbounded solution $\Psi = \tau^{1/2}$, with $\Phi = 0$. It is this phase-locked, growing amplitude solution, which describes capture into resonance and autoresonance in our system. But, how the system chooses between the saturated (bounded) and autoresonant solution by starting from zero initial conditions at $\tau = -\infty$? The answer is simple: the bifurcation is controlled by *single* parameter μ in the problem. Indeed, the analysis of Eq.(23) for a different application [19] shows that the autoresonant solution is obtained when $\mu = 0.411$. Finally, by transforming back to our original parameters, we obtain the threshold for synchronization by passage through resonance in the driven plasma oscillations problem:

$$\varepsilon > \varepsilon_{th} = 3.8\omega_0^{3/2}\alpha^{3/4} \qquad (24)$$

We find $\varepsilon_{th} = 0.008$ for parameters of simulations in Fig.4 ($k_0 = 0.9$, $\alpha = 0.00015$), which is in an excellent agreement with simulations.

IV. CONCLUSIONS

(a) We have reviewed recent examples of pattern formation by synchronization (PFS) by capture into resonances in two-dimensional vortex and periodic KdV systems and discussed a new application to excitation of standing waves (plasma oscillations) in the context of sin-Gordon equation.

(b) In all cases, driving perturbations in the form of simple waves or a superposition of simple waves allowed excitation of both single and multiphase

solutions from trivial (realizable) initial conditions. The threshold driving amplitude $\varepsilon_{th} \sim \alpha^{3/4}$ (α being the driving frequency chirp rate) for capture into resonance, and slow oscillating modulations of parameters of driven waves [the frequency of the modulations scales as $O(\varepsilon^{1/2})$] are the main signatures of the persistent phase-locking (autoresonance) in the driven systems.

(c) We have shown emergence of large amplitude plasma oscillations by synchronization in simulations and studied the thresholds in this system via a weakly nonlinear Whitham's two-phase averaged variational principle.

(d) The simplicity of both the forcing perturbations and the initial conditions used in generating nontrivial solutions in different nonlinear wave systems via the PFS approach may bridge between physics and pure mathematics in the field, making formation and control of nontrivial multiphase waves experimentally realizable. Application of a similar, multi-frequency control, leading to emergence of nontrivial multi-phase solutions in other fundamental nonlinear systems, seems to be an interesting direction for future research.

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