Three-Wave Mixing of Bogoliubov Quasiparticles in a Bose-Einstein Condensate

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A dressed basis is used to calculate the dynamics of three-wave mixing between Bogoliubov quasiparticles in a Bose condensate. Because of the observed oscillations between different momentum modes, an energy splitting, analogous to the optical Mollow triplet, appears in the Beliaev damping spectrum of the excitations from the oscillating modes.

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Since the experimental realization of Bose-Einstein condensation in trapped atomic gases, which serves as a monoenergetic and dense atomic source, experiments in nonlinear atom optics have become feasible. Atomic four-wave mixing (4WM) [1], superradiance [2], and matter-wave amplification [3] are all examples of nonlinear atom optics, which involve the mixing of several atomic fields or the mixing of atomic and electromagnetic fields.

In the case of atomic 4WM, bosonic amplification directs one of the products of a collision between an excitation atom and a condensate atom into an, initially, largely populated mode [1,4,5]. Thus, a new mode is macroscopically populated through the second collision product. When the energy, ε_k, of an excitation mode $\mathbf{k}$ is low compared to the condensate chemical potential, the excitation can no longer be described as a free atom moving with momentum $\mathbf{k}$, but rather as a collective phonon excitation, which involves a large number of atoms [6]. Atomic 4WM is therefore inadequate for the description of phonon decay. Phonon excitations are described in the framework of Bogoliubov theory [7]. As Bogoliubov excitations are bosons, bosonic amplification will also direct one of the damping products of a phonon into an, initially, largely populated mode, leading to three-wave mixing (3WM) of Bogoliubov quasiparticles.

In this Letter we introduce a basis of states which are dressed by the interaction between three, largely populated, modes of Bogoliubov quasiparticles in a Bose-Einstein condensate at zero temperature. Using this basis of quantum states we calculate the time evolution of the system, which exhibits nonlinear oscillations between the different momentum modes. In contrast to other theoretical approaches to wave mixing, which lead to nonlinear differential equations [4,8], the dressed state approach turns the problem into a linear one, for which propagation in time is trivial. We show that, due to relative number squeezing, the variance in the number difference between the two low momentum modes remains constant.

In analogy to the treatment of spontaneous photon scattering in the atom-laser dressed system [9], we treat damping into the quasicontinuum of empty modes, during the 3WM dynamics, as transfer between dressed state manifolds. Thus, the effects of 3WM on the damping process are calculated. The damping energy spectrum is presented. A transition from elastic to inelastic damping is observed, which, in analogy to the optical Mollow triplet, leads to a splitting of the spectrum into a doublet of resonance energies.

Previously, nonlinear mixing of quasiparticles in a Bose condensate was studied, between the two lowest energy excitation modes in the discrete regime, where the energy of the excited mode is of the order of the energy separation between modes [8,10,11].

Our model system is a homogenous condensate of finite volume V, with N_0 atoms in the ground state. The Hamiltonian for the system in the Bogoliubov basis, taken to the $\sqrt{N_0}$ order, is given by [12]

$$H = H_0 + H_{\text{int}}, \tag{1}$$

where $H_0 = \frac{1}{2}gnN_0 - \sum_{\mathbf{k}} \varepsilon_k v_k^2 + \sum_{\mathbf{k}} \varepsilon_k b_{\mathbf{k}}^+ b_{\mathbf{k}}$ is the part of H which is diagonalized by the Bogoliubov transformation. N_0 is the number of atoms in the $\mathbf{k} = 0$ mode. $\varepsilon_k = \sqrt{\varepsilon_k^0(\varepsilon_k^0 + 2gn)}$ is the recently measured Bogoliubov energy [13] and ε_k^0 is the free particle energy $\varepsilon_k^0 = \hbar^2 k^2/2m$. $g = 4\pi\hbar^2 a/m$ is the coupling constant, a is the s-wave scattering length, and m is the mass of the atoms. $b_{\mathbf{k}}^+$ and $b_{\mathbf{k}}$ are the creation and annihilation operators, respectively, of quasiparticle excitations with momentum $\mathbf{k}$. v_k and u_k are the Bogoliubov quasiparticle amplitudes. This part of the Hamiltonian describes excitations in the condensate with momentum $\mathbf{k}$ and energy ε_k.

H_{int} is the part of the Hamiltonian which is responsible for the interaction between excitations

$$H_{\text{int}} = \frac{g}{2V}\sqrt{N_0}\sum_{\mathbf{k},\mathbf{q}} A_{kq}(b_{\mathbf{k}}^+ b_{\mathbf{q}} b_{\mathbf{k}-\mathbf{q}} + b_{\mathbf{q}}^+ b_{\mathbf{k}-\mathbf{q}}^+ b_{\mathbf{k}}). \tag{2}$$

The first term in parentheses is referred to as Landau damping and is analogous to photonic up-conversion, such as second harmonic generation in optics [14], whereas the second term is referred to as Beliaev damping, and is analogous to photonic down-conversion [15]. A_{kq} is the many-body suppression factor

$$A_{kq} = 2u_k(u_q u_{k-q} - v_q u_{k-q} - u_q v_{k-q}) - 2v_k(v_q v_{k-q} - u_q v_{k-q} - v_q u_{k-q}). \quad (3)$$

The main damping mechanism from a single, largely populated mode $\mathbf{k}$ will be elastic Beliaev damping of the excitations into empty modes, which are on a monoenergetic surface in momentum space [16].

We now consider the case where the condensate is excited with N excitations of momentum $\mathbf{k}$ and M excitations of momentum $\mathbf{q}$, $|N_{\mathbf{k}}, M_{\mathbf{q}}, 0_{\mathbf{k}-\mathbf{q}}\rangle$, such that $\mathbf{k}$ and $\mathbf{q}$ fulfill the Bragg condition, i.e., $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{q}} + \varepsilon_{\mathbf{k}-\mathbf{q}}$. Two-photon Bragg transitions can be used to excite the condensate and populate different momentum modes with a variable number of excitations [13]. The amplitude given by H_{int} for Beliaev damping of the $\mathbf{k}$ momentum excitation into two excitations with momenta $\mathbf{q}$ and $\mathbf{k} - \mathbf{q}$, $|(N-1)_{\mathbf{k}}, (M+1)_{\mathbf{q}}, 1_{\mathbf{k}-\mathbf{q}}\rangle$, will be $\sqrt{N}$ or $\sqrt{M}$ fold larger than any of the other damping channels. This will result in 3WM into a newly populated $\mathbf{k} - \mathbf{q}$ momentum mode. We assume that $N_0 \gg N, M \gg \Gamma t_0$, where Γ is the total Beliaev damping rate of the excitations into the quasicontinuum of empty modes and t_0 is the time of the experiment. Thus, during the experiment time, 3WM dynamics dominates over damping into empty modes. In general the set of $N+1$ excitation Fock states $|(N-i)_{\mathbf{k}}, (M+i)_{\mathbf{q}}, i_{\mathbf{k}-\mathbf{q}}\rangle$, where i varies between 0 and N, spans a degenerate subspace of the eigenstates of H_0. H_{int} couples between pairs of states in this subspace which have a difference of 1 in i, and can be represented by the $(N+1) \times (N+1)$ tridiagonal matrix

$$\langle N-s, M+s, s|H_{\text{int}}|N-i, M+i, i\rangle = \frac{g}{2V}\sqrt{N_0}A_{kq}(\sqrt{N-i}\sqrt{M+i+1}\sqrt{i+1}\delta_{i+1,s} + \sqrt{N-i+1}\sqrt{M+i}\sqrt{i}\delta_{i-1,s}). \quad (4)$$

When H_{int} is diagonalized, we get a new set of $N+1$ eigenstates, $|j\rangle$, where j varies between 1 and $N+1$, that are dressed by the interaction. Note that since the dressed states are superpositions of degenerate eigenstates of H_0, they are eigenstates of the complete Hamiltonian in Eq. (1).

The degeneracy between the excitation Fock states is removed by the interaction. The calculated energy spectrum of the new dressed basis is symmetric around $E = 0$. We find the spectrum to be roughly linear, with an average energy spacing of $dE \simeq 2.48\sqrt{N}$ in units of $(g\sqrt{N_0}/2V)A_{kq}$ [17]. Because of the nonlinearity of the problem, energy differences between each pair of dressed states are slightly different, and vary parabolically with a minimum around the zero energy value.

Figure 1 shows the absolute value squared of the transfer matrix between the bare excitation Fock state basis and the dressed basis for $N = M = 100$ [18].

The main contribution to the dressed state superposition of Fock states comes from the two states which

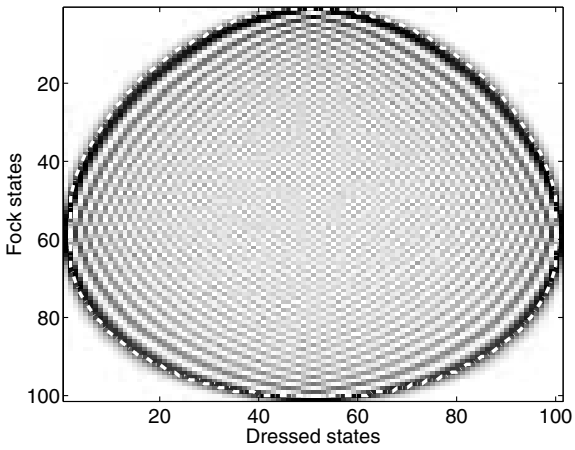

FIG. 1. The square of the transfer matrix between the excitation Fock state basis and the dressed state basis for $N = M = 100$. Darker areas correspond to larger probability. The white dash-dotted line draws the solution to $|E_j| = 2\sqrt{(N^2 - i^2)i}$. The Fock states are numbered by i. The dressed states are numbered by j from lowest to highest energy.

solve $|E_j| = 2\sqrt{(N^2 - i^2)i}$. The dash-dotted line in Fig. 1 indicates the solution of this equation. In order for this equation to have real solutions the resulting spectrum must satisfy $|E_j| < 1.24N^{3/2}$, and therefore has an average energy difference between dressed states of $dE \simeq 2.48\sqrt{N}$, consistent with our numerical observation.

We choose as a model system a condensate of 3×10^5 ^{87}Rb atoms in the $F = 2$, $m_f = 2$ ground state. The condensate, which is similar to the experimental parameters of [13], is homogeneous and has a density of 3×10^{14} atoms/cm^3. The $k = 0.7\hbar/\xi$ mode and $q = k/\sqrt{2}$ mode, where ξ is the healing length of the condensate given by $\xi = (\sqrt{8\pi na})^{-1}$, are populated with 100 excitations each.

We start from the above excitation Fock state, written as a linear superposition of dressed states. The state of the system is then readily propagated in time by evolving each of the dressed states phases according to its energy. Figure 2 shows $N_{\mathbf{k}}$, the expectation value of the number of excitations with momentum $\mathbf{k}$, as a function of time. Excitations oscillate between the $\mathbf{k}$ momentum mode and the $\mathbf{q}$ and $\mathbf{k} - \mathbf{q}$ momenta modes. Inset (a) of Fig. 2 shows N_k during a much longer time. Since the energy spectrum is not precisely linear, at longer times beating between the different oscillation frequencies gives rise to a slow amplitude modulation of the oscillations.

Even though we start from an excitation Fock state, the system immediately evolves into a superposition of Fock states. Inset (b) of Fig. 2 shows the standard deviation of $N_{\mathbf{k}}$ vs time. After a time scale which is set by the nonlinearity of the spectrum, the average value of $N_{\mathbf{k}}$ and its standard deviation are of similar size. However, the expectation value of the number difference between mode $\mathbf{q}$ and mode $\mathbf{k} - \mathbf{q}$ equals M and is constant in time. The

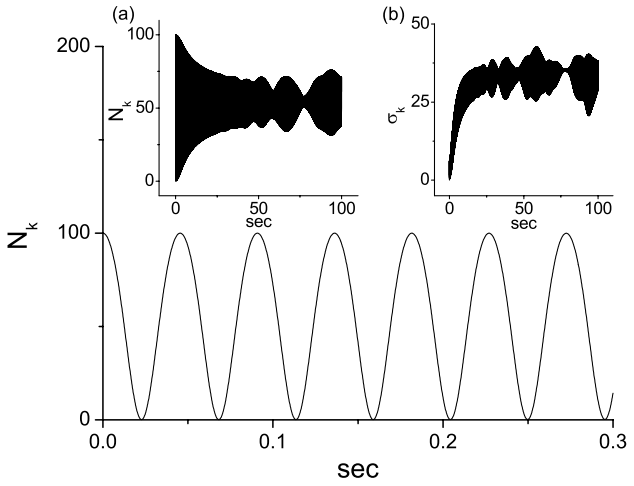

FIG. 2. The expectation value $N_{\mathbf{k}}$, for $k = 0.7\hbar/\xi$ and $q = k/\sqrt{2}$, as a function of time. $N = M = 100$, for a condensate of 3×10^5 ^{87}Rb atoms in the $F = 2$, $m_f = 2$ ground state. The condensate is homogeneous, with a density of 3×10^{14} atoms/cm^3. The oscillation frequency is roughly the average energy difference in the dressed state spectrum. Insets (a) and (b) show $N_{\mathbf{k}}$ and the standard deviation of $N_{\mathbf{k}}$, respectively, during a longer time.

standard deviation of that difference always remains zero, which implies relative number squeezing between the two modes.

In analogy to the treatment of spontaneous photon scattering as a transfer between dressed states manifolds of the dressed atom-laser system [9], we now consider scattering into empty modes as a transfer between dressed state manifolds. Thus, the damping of excitations from mode $\mathbf{k}$ is no longer elastic, but rather carries the energy difference between the dressed states among which it occurred.

Evaluating $H_B = \frac{g}{2V}\sqrt{N_0}\sum_{\mathbf{q}'} A_{kq'}(b^+_{\mathbf{q}'} b^+_{\mathbf{k}-\mathbf{q}'} b_{\mathbf{k}})$, between every pair of states in the two manifolds reveals the spectral structure of the damping process. We find that the spectrum of the $N' = N - 1$, $M' = N$ manifold, which has one less energy eigenvalue, is shifted by roughly half the energy difference with respect to the $N = M$ energy spectrum. H_B significantly couples only dressed states with neighboring energies. This results in a structure of a doublet in the Beliaev damping spectrum.

The finite lifetime of the dressed state results from the fact that there is a quasicontinuum of $N' = N - 1$, $M' = N$ manifolds, all with an identical energy spectrum, to which a dressed state in the $N = M$ manifold can couple to via H_B. To determine the width of each transition between two dressed states, we use the Fermi golden rule. The damping rate between a state $|j\rangle_{N,M}$ in the N, M manifold and a state $|i\rangle_{N-1,M}$ in the $N - 1$, M manifold is then given by

$$\Gamma = \frac{2\pi}{\hbar}\sum_{\mathbf{q}'}\frac{g^2 N_0}{2V^2}|A_{kq'}|^2|_{N-1,M}\langle i|b_{\mathbf{k}}|j\rangle_{N,M}|^2\delta(\varepsilon_{\mathbf{k}} + \varepsilon_0 - \varepsilon_{\mathbf{q}'} - \varepsilon_{\mathbf{k}-\mathbf{q}'}), \tag{5}$$

where ε_0 is the energy difference between $|j\rangle_{N,M}$ and $|i\rangle_{N-1,M}$. Using momentum conservation, the energy conservation δ function becomes a geometrical condition on the angle θ between $\mathbf{k}$ and $\mathbf{q}'$

$$\cos(\theta) = \frac{1}{2kq'}[k^2 + q'^2 + 1 - \sqrt{(\varepsilon_k + \varepsilon_0 - \varepsilon_{q'})^2 + 1}], \tag{6}$$

where momentum is in units of $\hbar/\xi$, and energy is in units of gn. Equation (5) is then simplified to

$$\Gamma = \frac{g}{2\xi^3 h}\int |A_{kq'}|^2|_{N-1,M}\langle i|b_{\mathbf{k}}|j\rangle_{N,M}|^2\frac{q'}{2k}\frac{(\varepsilon_k + \varepsilon_0 - \varepsilon_{q'})}{\sqrt{(\varepsilon_k + \varepsilon_0 - \varepsilon_{q'})^2 + 1}}dq'. \tag{7}$$

The spectrum of each transition is taken as a Lorenzian with a width of $\Gamma/2\pi N_{\mathbf{k}}$ and a normalization of $\Gamma/N_{\mathbf{k}}$ around ϵ_0. $N_{\mathbf{k}}$ is the average occupation of mode $\mathbf{k}$ for the two dressed states involved. Averaging over all of the possible transitions between the manifolds, Fig. 3 shows the damping spectrum between the $N = M = 5 \times 10^3$ and the $N = 5 \times 10^3 - 1$, $M = 5 \times 10^3$ manifolds, for the same model system as in Fig. 2 for $k = 3.2\hbar/\xi$, $1.6\hbar/\xi$, and $0.7\hbar/\xi$, and $q = k/\sqrt{2}$ [19]. A clear doublet structure is evident. The inset of Fig. 3 shows the energy-conserving surfaces for the two center energies of the $k = 0.7\hbar/\xi$ curve (solid line), and the energy-conserving surface for elastic damping from the same mode (dashed line).

The separation between the peaks in Fig. 3, which is equal to the oscillation frequency, decreases with k as A_{kq}. The decrease in the width of each peak as a function of k comes from two contributions. First, since damping into empty modes is incoherent, it decreases as $|A_{kq'}|^2$. Second, as k decreases there are less allowed empty modes on the energy-conserving surface. Therefore, the doublet structure is more resolved for the lower k values. Another contribution to the width of the resonance in Fig. 3 is due to the nonlinearity of the dressed state spectrum, which gives a slightly different energy for transitions between different dressed state pairs.

Several other mechanisms contribute to the broadening of the two peaks which are not included in Fig. 3. The fact

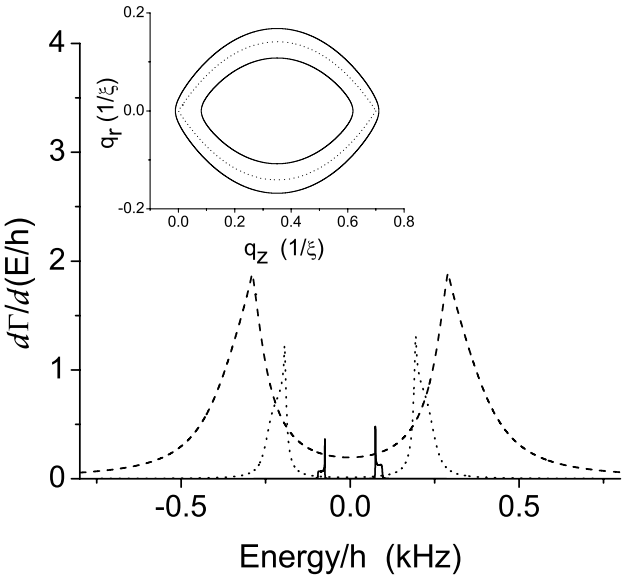

FIG. 3. Damping spectrum between the $N = M = 5 \times 10^3$ manifold and the $N = 5 \times 10^3 - 1$, $M = 5 \times 10^3$ manifold, for the same condensate as in Fig. 2. The three curves are for $k = 3.2\hbar/\xi$ (dashed line), $k = 1.6\hbar/\xi$ (dotted line), and $k = 0.7\hbar/\xi$ (solid line), where $q = k/\sqrt{2}$. The inset shows the energy-conserving surfaces for the two center frequencies of the $k = 0.7\hbar/\xi$ curve (solid line) and the energy-conserving surface for elastic damping from mode $\mathbf{k}$ (dashed line).

that only the first scattering event occurs between the $N = M = 5 \times 10^3$ and the $N = 5 \times 10^3 - 1$, $M = 5 \times 10^3$ manifolds will further broaden the resonances. Since the energy splitting scales as $\sqrt{N}$, in an experiment where one scatters dN atoms from mode $\mathbf{k}$, this will result in a relative broadening of $\frac{dN}{2N}$. According to the same scaling, an initial coherent, rather than Fock, state will cause a relative broadening of $\frac{\sqrt{N}}{2N}$. The condensate finite size or inhomogeneous density profile will further contribute to the width of the resonance. We estimate that for the experimental parameters of [13] a doublet structure in the Beliaev damping spectrum can be resolved. Experimentally, the energy doublet can be observed by computerized tomography analysis of time of flight absorption images of the 3WM system [20].

In conclusion, we calculate the wave-mixing dynamics between three, low k, Bogoliubov quasiparticles in a Bose condensate. The Hamiltonian of this system is diagonalized to the next order in $\sqrt{N_0}$. The resulting basis of dressed states allows for the efficient, linear, propagation of the system in time. Nonlinear oscillations between the different momentum modes are observed. Relative number squeezing between the $\mathbf{q}$ and the $\mathbf{k} - \mathbf{q}$ momentum modes is shown. Beliaev damping of excitations from these modes is treated as a transfer between dressed state manifolds. The damping process is shown to become inelastic and, similar to the optical Mollow triplet, exhibits a doubletlike energy spectrum, which is more resolved for the low k values.

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[17] This scaling law was verified numerically over several $N = M$ values. A similar scaling law can be written for the $N \neq M$ case.
[18] Since excitations will usually bear an uncertainty of $\sqrt{N}$ in their population, the choice of an excitation Fock state as the system's initial condition is somewhat unrealistic. This will smear the system over an ensemble of manifolds. However, for large N, M the relative spread in the excitation population becomes less and less significant.
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