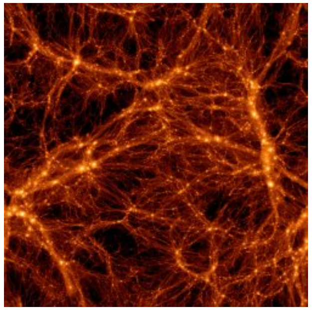
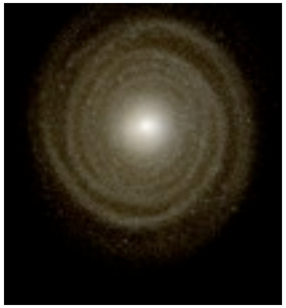
Galaxy formation physics and it numerical implementation

Romain Teyssier

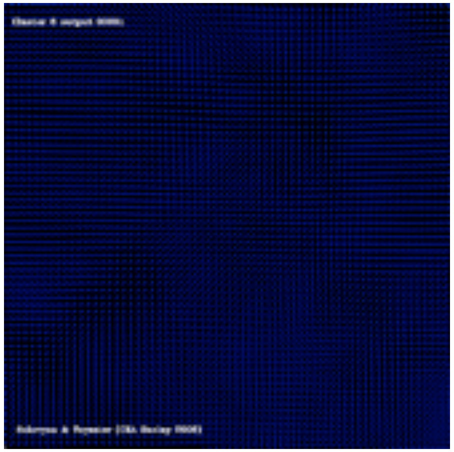
University of Zurich





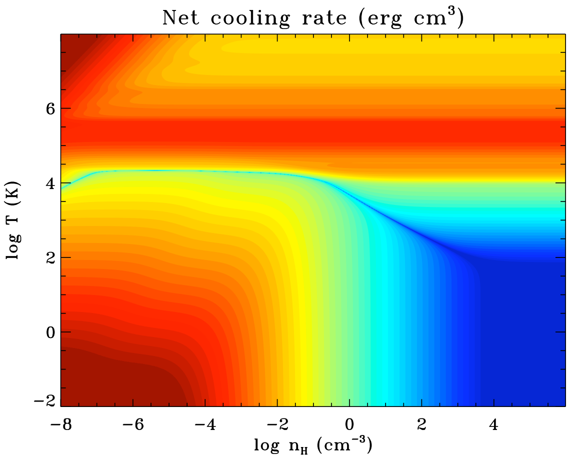
Formation of slowly rotating dark matter halos

- spin from tidal torques
- statistical Virial equilibrium



Hot gas settles in thermal equilibrium

$$\frac{3}{2}\frac{k_{\rm B}T_{gas}}{\mu m_{\rm H}} = \frac{1}{2}\frac{GM_{halo}}{R_{halo}}$$



Radiative cooling dissipates pressure support, dense gas discs settle into centrifugal equilibrium: radiative atomic physics sets the galaxy mass.

$$\mathcal{I}_0 = 13.6 \text{ eV} \qquad M_{\rm galaxies} \simeq 10^{11}\ M_\odot$$

White and Rees (1978); Dekel and Silk (1986)

Disc galaxies form from quiescent gas accretion history, while elliptical galaxies form out of violent mergers.

Outline

- Euler equations in different flavors
- Godunov scheme : numerical implementation
- Self-gravitating fluids : numerical tricks
- High Mach number flows : a problem and a solution
- Radiation hydrodynamics and basic atomic processes
- A Godunov scheme for radiation transfer
- Reducing the speed of light or using GPU ?

Hydrodynamics

The Euler equations in conservative form

Gas is a highly collisional system, so that f is a Maxwell distribution function.

A system of 3 conservation laws (mass, momentum and energy) + the EoS

$$\partial_t \rho + \nabla \cdot \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \nabla \cdot (\rho \mathbf{u} \times \mathbf{u}) + \partial_x P = 0$$

$$\partial_t E + \nabla \cdot \mathbf{u}(E + P) = 0$$

In cosmology, one need to add source terms to these conservation laws:

- gravity
- radiative processes
- star formation and feedback

Euler equations as conservation laws

General system of conservation laws with ***F*** flux vector.

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$$

Examples:

1- Isothermal Euler equations

$$\mathbf{U} = (\rho, m)$$

$$\mathbf{F} = \left(u\rho, um + \rho a^2\right)$$

2- Euler equation

$$\mathbf{U} = (\rho, m, E)$$

$$\mathbf{F} = (u\rho, um + P, u(E + P))$$

3- Ideal MHD equations

$$\mathbf{U} = \left(\rho, m_x, m_y, m_z, E, B_x, B_y, B_z\right)$$

$$\mathbf{F} = \left(v_x\rho, v_x m_x + P_{tot} - B_x^2, v_x m_y - B_x B_y, v_x m_z - B_x B_z, \right.$$

$$\left. 0, v_x B_y - v_y B_x, v_x B_z - v_z B_x\right)$$

The 1D isothermal Euler equations

Conservative form with conservative variables $\mathbf{U} = (\rho, m)$

$$\partial_t \rho + \partial_x m = 0$$

$$\partial_t m + \partial_x \left(\rho u^2 + \rho a^2\right) = 0$$

Primitive form with primitive variables $\mathbf{W} = (\rho, u)$

$$\partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0$$

$$\partial_t u + u \partial_x u + \frac{a^2}{\rho} \partial_x \rho = 0$$

a is the isothermal sound speed

Godunov scheme for hyperbolic systems

The system of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

is discretized using the following exact integral form:

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

The time average flux function is computed using the self-similar solution of the inter-cell Riemann problem:

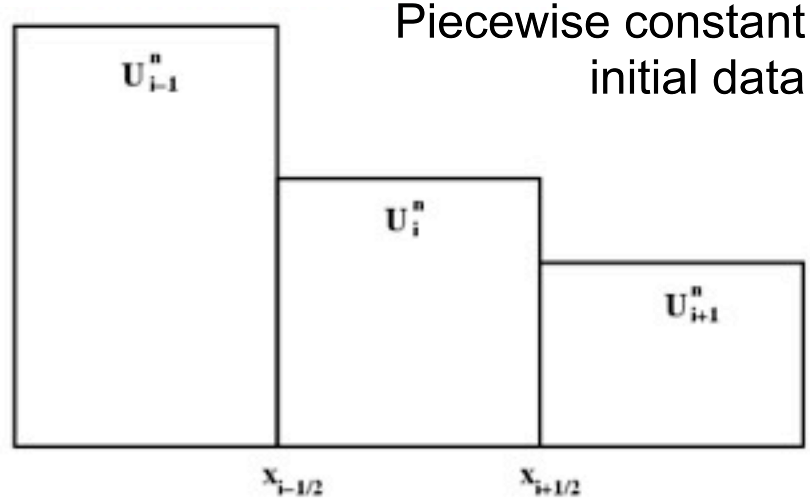
$$\mathbf{U}_{i+1/2}^{*}(x/t) = \mathcal{RP}\left[\mathbf{U}_i^n, \mathbf{U}_{i+1}^n\right]$$

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}(\mathbf{U}_{i+1/2}^{*}(0))$$

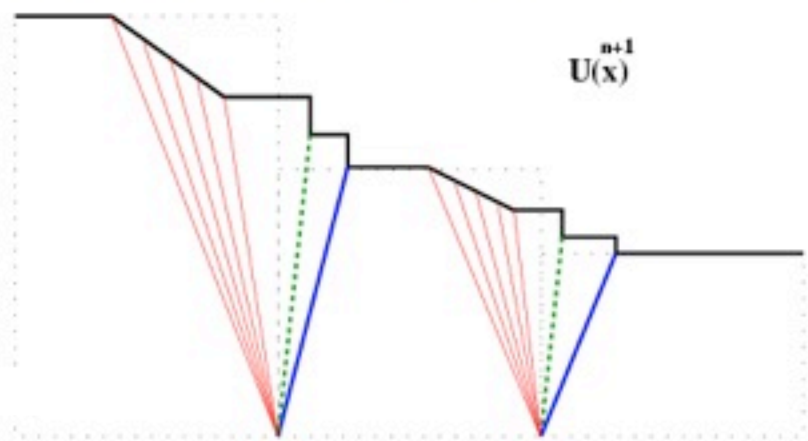
This defines the Godunov flux:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^{*}(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$$

Piecewise constant initial data

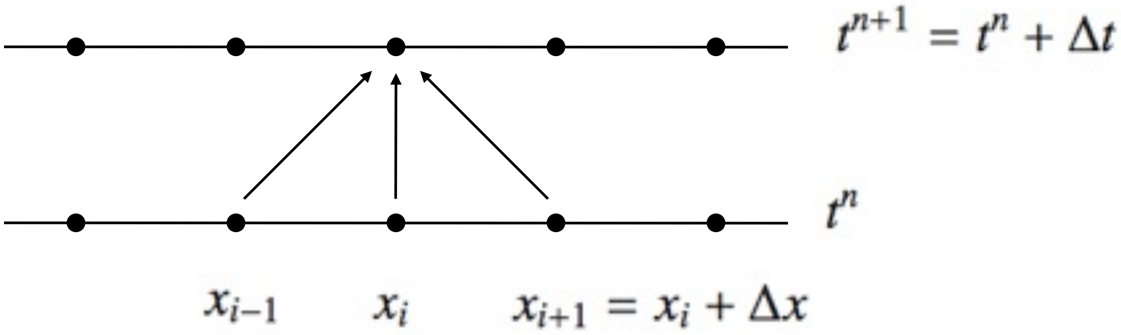


- **Godunov, S. K.** (1959), A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations, *Math. Sbornik*, ***47***, *271-306*, translated US Joint Publ. Res. Service, JPRS 7226, 1969.



Advection: 1 wave, Euler: 3 waves, MHD: 7 waves

Finite difference scheme for the advection equation



$$u_i^n = u(x_i, t^n) \qquad \partial_x u \simeq \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \qquad \partial_t u \simeq \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Finite difference approximation of the advection equation

$$\partial_t u + a\partial_x u = 0 \quad \Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

The modified equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

Taylor expansion in time up to second order

$$u_i^{n+1} = u_i^n + \Delta t\left(\frac{\partial u}{\partial t}\right) + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)$$

Taylor expansion in space up to second order

$$u_{i+1}^n = u_i^n + \Delta x\left(\frac{\partial u}{\partial x}\right) + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$u_{i-1}^n = u_i^n - \Delta x\left(\frac{\partial u}{\partial x}\right) + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

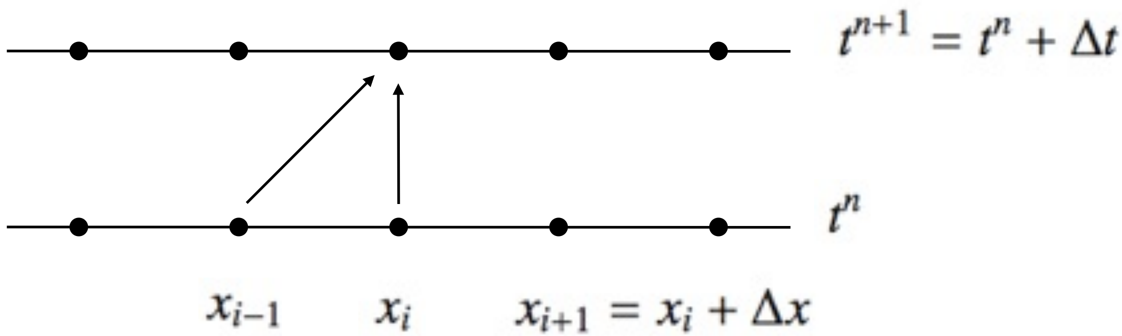
The advection equation becomes the advection-diffusion equation

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = -\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + O(\Delta t^2, \Delta x^2)$$

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = -a^2\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Negative diffusion coefficient: the scheme is *unconditionally unstable*

Upwind scheme for the advection equation



a>0: use only upwind values, discard downwind variables

$$\partial_x u \simeq \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad \Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Taylor expansion up to second order:

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = -\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + a\frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Upwind scheme is stable if C<1, with $C = a\frac{\Delta t}{\Delta x}$

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = a\frac{\Delta x}{2}(1 - C)\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Advection-diffusion type modified equation

Finite difference approximation of the advection equation:

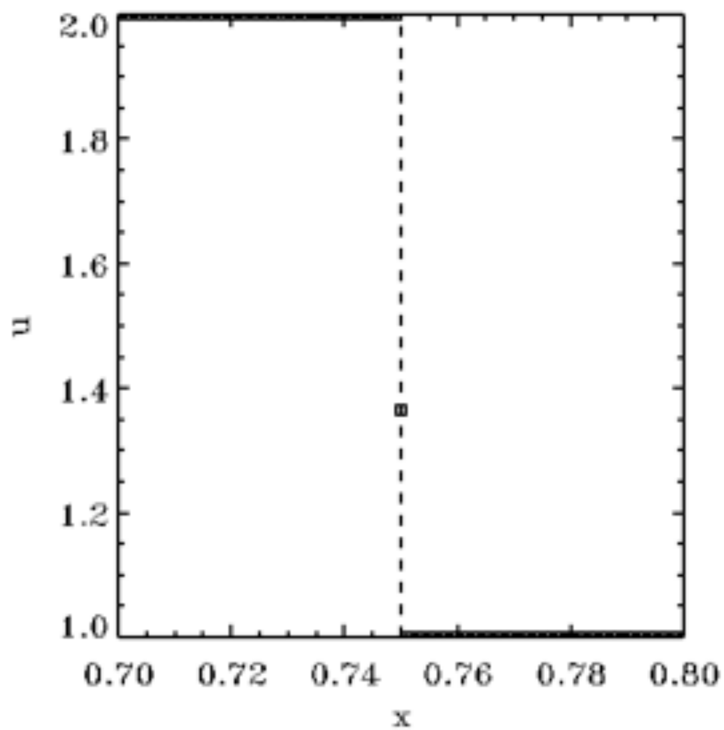
$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = \eta\left(\frac{\partial^2 u}{\partial x^2}\right)$$

Central differencing unstable: $\eta < 0$

Upwind differencing is stable: $\eta > 0$ $\quad \eta = a\frac{\Delta x}{2}(1 - C)$

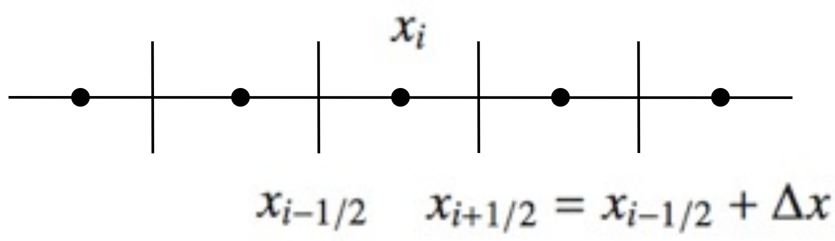
Smearing of initial discontinuity:

"numerical diffusion"



Thickness increases as $\sqrt{\eta t}$

Finite volume scheme for the advection equation



Finite volume approximation of the advection equation:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t^n)\mathrm{dx}$$

Use integral form of the conservation law:

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{t^n}^{t^{n+1}} \mathrm{dxdt}\,(\partial_t u + a\partial_x u) = 0$$

Exact evolution of volume averaged quantities:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

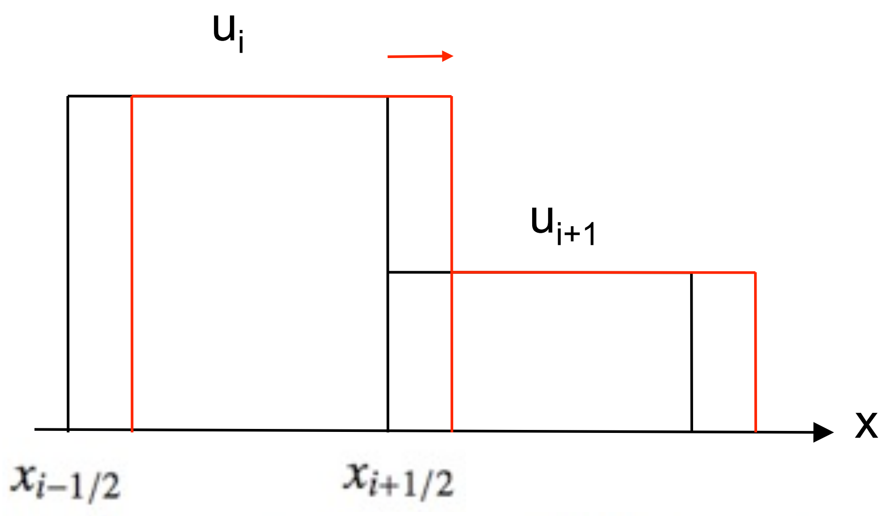
Time averaged flux function: $$u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2},t)\mathrm{dt}$$

Godunov scheme for the advection equation

The time averaged flux function: $u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{d}t$

is computed using the exact solution of the problem defined

at cell interfaces with piecewise constant initial data: the Riemann problem



For all t>0:

$$u(x_{i+1/2}, t) = u_i^n \quad \text{if} \quad a > 0$$
$$u(x_{i+1/2}, t) = u_{i+1}^n \quad \text{if} \quad a < 0$$

The Godunov scheme for the advection equation is identical to the upwind finite difference scheme.

The isothermal wave equation

We linearize the isothermal Euler equation around some equilibrium state.

$$\mathbf{W} = \mathbf{W}_0 + \Delta\mathbf{W}$$

Using the system in primitive form, we get the ***linear*** system:

$$\partial_t \Delta\mathbf{W} + \mathbf{A}_0 \partial_x \Delta\mathbf{W} = 0$$

where the constant matrix has 2 real eigenvalues and 2 eigenvectors

$$\mathbf{A}_0 = \begin{Bmatrix} u & \rho \\ \frac{a^2}{\rho} & u \end{Bmatrix} \qquad \lambda^+ = u + a \qquad \Delta\alpha^+ = \frac{1}{2}\left(\Delta\rho + \rho\frac{\Delta u}{a}\right)$$

$$\lambda^- = u - a \qquad \Delta\alpha^- = \frac{1}{2}\left(\Delta\rho - \rho\frac{\Delta u}{a}\right)$$

The previous system is equivalent to 2 independent *scalar linear* PDEs.

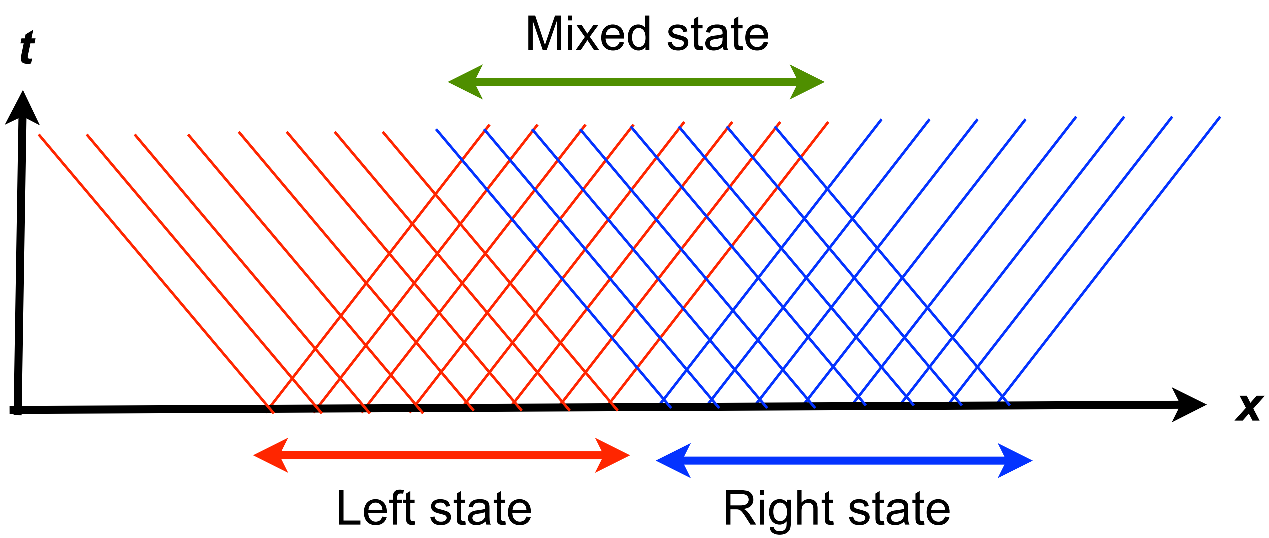
$$\partial_t \Delta\alpha^+ + (u + a)\partial_x \Delta\alpha^+ = 0$$

$$\partial_t \Delta\alpha^- + (u - a)\partial_x \Delta\alpha^- = 0$$

$\Delta\alpha^+$ ($\Delta\alpha^-$) is a Riemann invariant along characteristic curves moving with velocity $u + a$ ($u - a$)

Riemann problem for isothermal waves

Initial conditions are defined by 2 semi-infinite regions with piecewise constant initial states $(\Delta\rho_R, \Delta u_R)$ and $(\Delta\rho_L, \Delta u_L)$



"Star" state is obtained using the 2 Riemann invariants.

$$u - a < \frac{x}{t} < u + a$$

$$\Delta\rho^* = \Delta\alpha_L^+ + \Delta\alpha_R^-$$

$$\Delta u^* = \frac{a}{\rho}(\Delta\alpha_L^+ - \Delta\alpha_R^-)$$

Godunov scheme for the isothermal wave equation

We now explain the Godunov scheme for the density only.

$$\frac{\Delta\rho_i^{n+1} - \Delta\rho_i^n}{\Delta t} + \frac{F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

$$F_{i+1/2}^{n+1/2} = u\ \Delta\rho_{i+1/2}^{n+1/2} + \rho\ \Delta u_{i+1/2}^{n+1/2}$$

Using the Riemann problem defined by $(L,R) = (i, i+1)$ we get

$$\Delta\rho_{i+1/2}^{n+1/2} = \Delta\alpha_i^+ + \Delta\alpha_{i+1}^- = \frac{\Delta\rho_i + \Delta\rho_{i+1}}{2} - \frac{\rho}{2a}(\Delta u_{i+1} - \Delta u_i)$$

$$\Delta u_{i+1/2}^{n+1/2} = \frac{a}{\rho}\left(\Delta\alpha_i^+ - \Delta\alpha_{i+1}^-\right) = \frac{\Delta u_i + \Delta u_{i+1}}{2} - \frac{a}{2\rho}(\Delta\rho_{i+1} - \Delta\rho_i)$$

The final flux function is given by the explicit linear function:

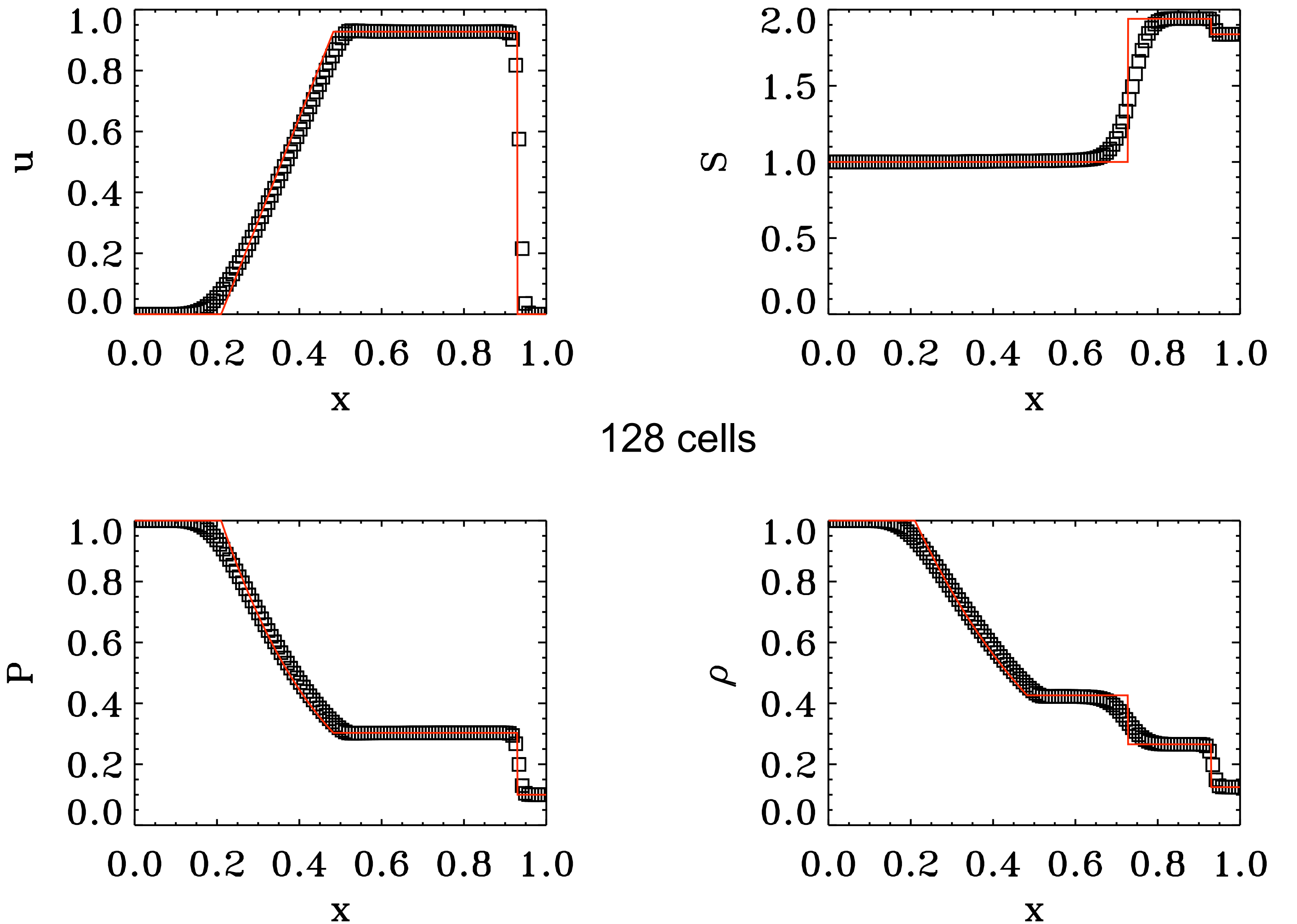
$$F_{i+1/2}^{n+1/2} = u\,\frac{\Delta\rho_i + \Delta\rho_{i+1}}{2} + \rho\,\frac{\Delta u_i + \Delta u_{i+1}}{2} - \frac{a\Delta x}{2}\frac{\partial\rho}{\partial x} - \frac{\rho}{a}\frac{u\Delta x}{2}\frac{\partial u}{\partial x}$$

Unstable centered FD scheme (first two terms) — Diffusive term (last two terms)

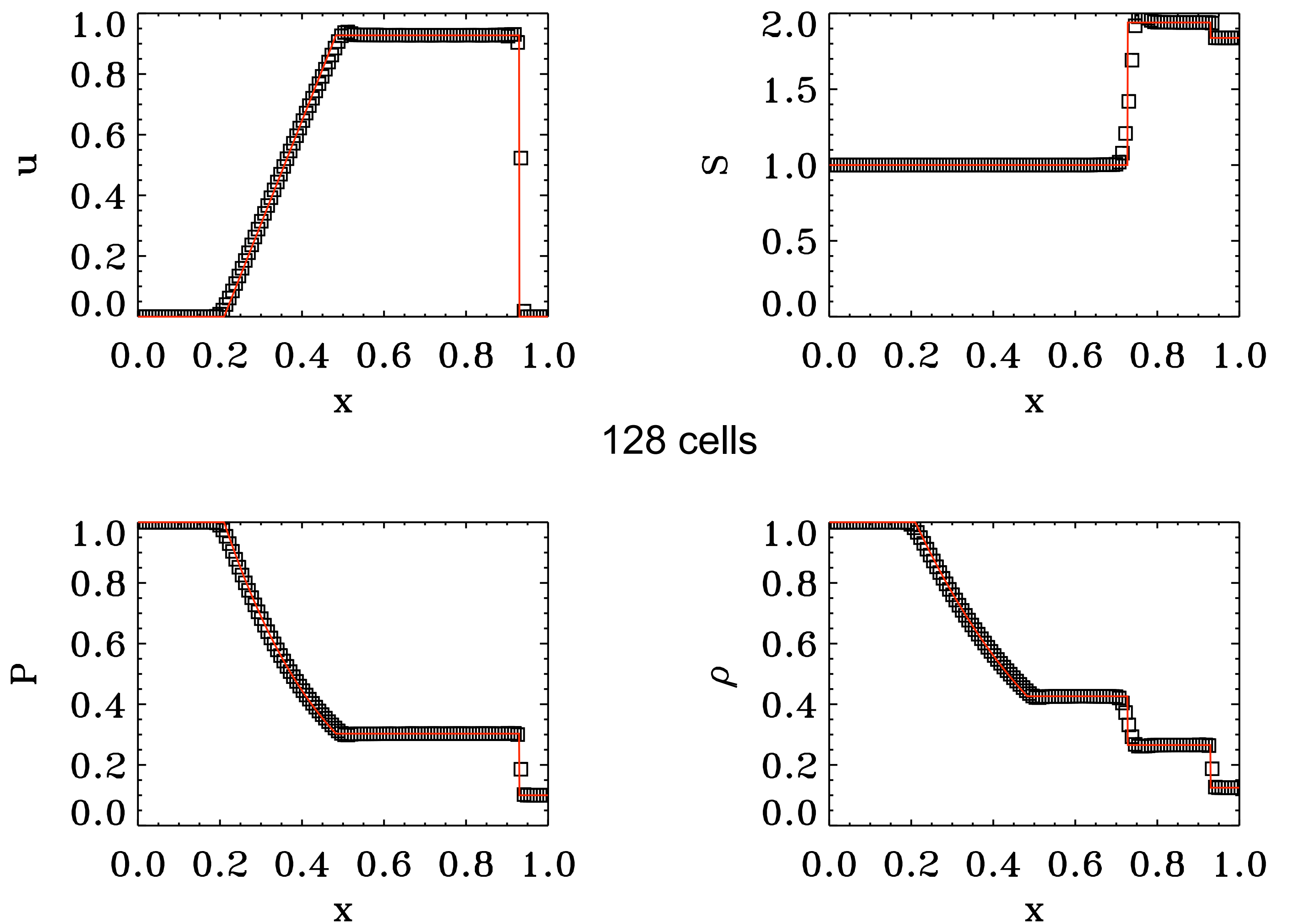
Formally, we have $F = F_{\text{true}} - \nu\nabla\rho$ which results in the modified equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot F_{\text{true}} = \nu\Delta\rho \quad \text{with} \quad \nu = \frac{a\Delta x}{2}$$

Sod test with first order Godunov scheme



Sod test with second order Godunov scheme



Hydrodynamics and gravity

Self-gravitating fluids

The fluids equation in conservative form in presence of gravity write:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}) + \vec{\nabla} P = \rho \vec{F}$$

In a self-gravitating fluid, the gravitational potential follow the Poisson equation

$$\Delta \Phi = 4\pi G \rho \text{ with } \vec{F} = -\vec{\nabla}\Phi$$

We drop the constant 4πG from now on: $\rho = \Delta\Phi = -\vec{\nabla} \cdot \vec{F}$

For each component, we have $\rho F_x = -\left(\vec{\nabla} \cdot \vec{F}\right) F_x = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \left(\vec{F} \cdot \vec{\nabla}\right) F_x$

We then use the relations

$$\partial_x F_x = -\partial^2_{xx}\Phi = \partial_x F_x$$

$$\partial_y F_x = -\partial^2_{xy}\Phi = \partial_x F_y$$

$$\partial_z F_x = -\partial^2_{xz}\Phi = \partial_x F_z$$

The tidal tensor $\partial_j F_i = -\partial_{ij}\Phi$ is symmetric

Finally, we have $\rho F_x = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \vec{F} \partial_x \vec{F} = -\vec{\nabla} \cdot \left(F_x \vec{F}\right) + \partial_x \frac{\left|\vec{F}\right|^2}{2}$

Fully conservative form:

$$\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v} \otimes \vec{v} + P\overline{\overline{1}} + \vec{F} \otimes \vec{F} - \frac{F^2}{2}\overline{\overline{1}}\right) = 0$$

Jeans instability and gravo-acoustic waves

We consider an equilibrium state with $\rho = \rho_0, \;\; \Phi = 0 \text{ and } v = 0$

In this infinite medium, the Poisson equation has to be modified $\Delta\Phi = 4\pi G\left(\rho - \rho_0\right)$

The perturbed state satisfies $\Delta(\delta\Phi) = 4\pi G(\delta\rho)$ $\qquad \delta\rho = \Delta\rho\exp^{i(kx-\omega t)}$

The linearized continuity equation is $\partial_t(\delta\rho) + \rho_0\vec{\nabla}\cdot(\delta\vec{v}) = 0$

The Euler equation becomes $\partial_t(\delta\vec{v}) + \frac{c_0^2}{\rho_0}\vec{\nabla}(\delta\rho) + \vec{\nabla}(\delta\Phi) = 0$

We get the following dispersion relation $\boxed{\omega^2 = c_0^2 k^2 - 4\pi G\rho_0 = c_0^2(k^2 - k_J^2)}$

where we have introduce the Jeans length $k_J = \frac{2\pi}{\lambda_J} = \sqrt{\frac{4\pi G\rho_0}{c_0^2}}$

It is the length a sound wave will travel before it collapses $\lambda_J = c_0 t_{\rm ff}$

where we have defined the collapse time of free-fall time as $t_{\rm ff} = \sqrt{\frac{\pi}{G\rho_0}}$

In order to capture all collapsing modes,

a numerical model must resolve the Jeans length $\Delta x < \lambda_J$

Homogeneous collapse

Consider the isothermal collapse of an self-gravitating gas sphere.

Velocity field: $\mathbf{v} = -H(t)\mathbf{r}$ with $H(t)^2 = \frac{8\pi}{3} G\rho(t) \left(1 - \frac{R(t)}{R_0}\right)$

The momentum flux is given by the Riemann solution at the interface:

$$\mathbf{F}(\rho\mathbf{v}) = \left(\rho v^2 + P\right) - \nu\nabla\left(\rho\mathbf{v}\right) \quad \text{with} \quad \nu = (v + a)\frac{\Delta x}{2}$$

The truncation error is largest at the origin, and has to be smaller than the thermal pressure:

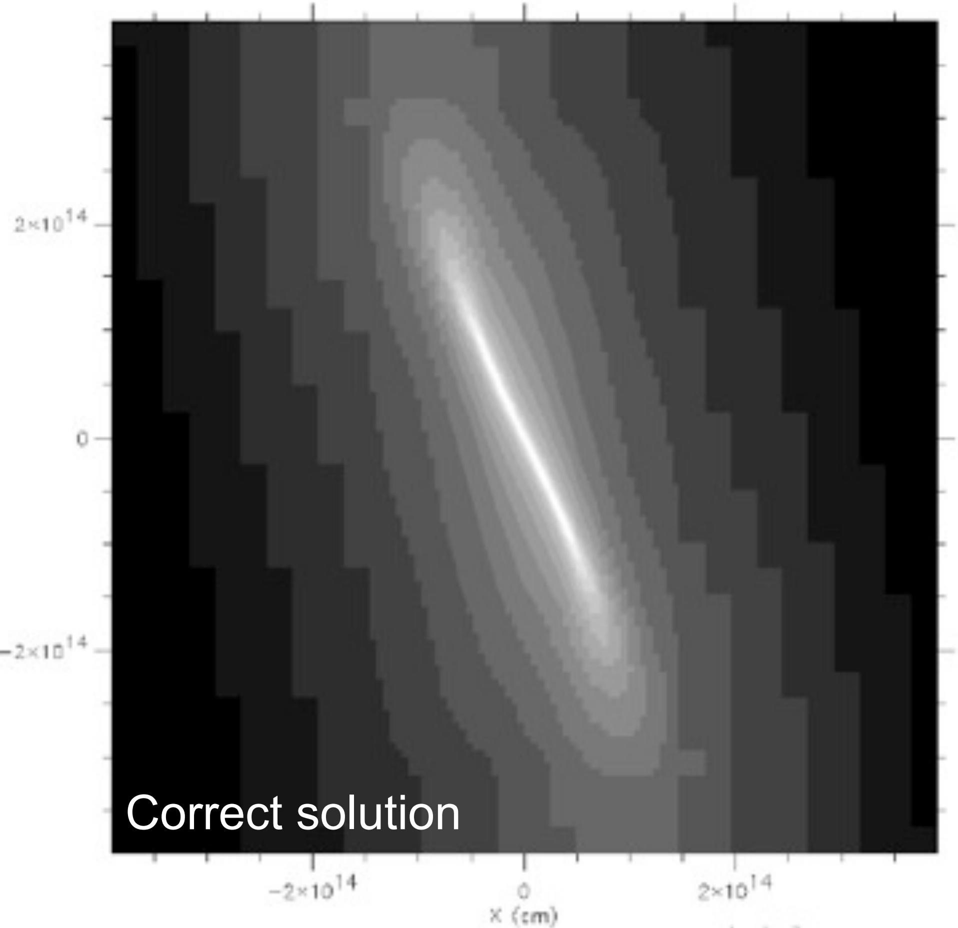
$$a\frac{\Delta x}{2}\rho H(t) < P \quad \text{or} \quad a > \frac{\Delta x}{2} H(t) \simeq \frac{12}{\tau_{\rm ff}}\Delta x$$

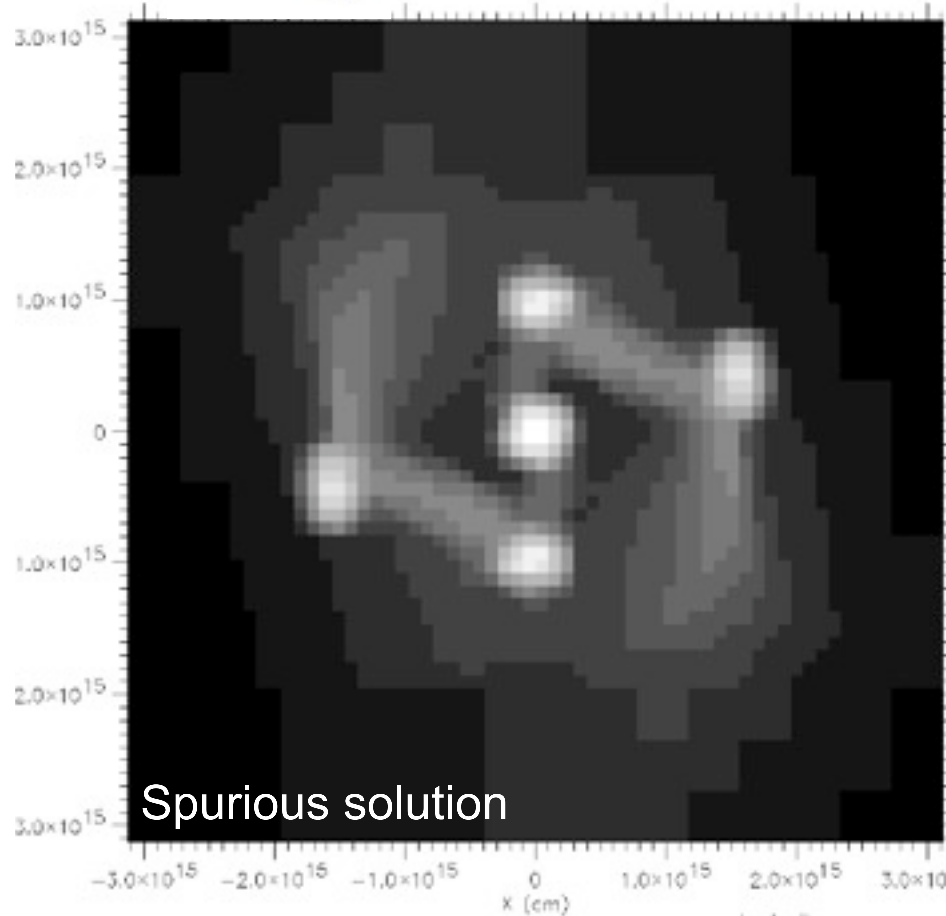
This translates into a condition on the mesh size: $\Delta x < \frac{\lambda_{\rm J}}{12}$

We need to resolve the Jeans length by at least ten cells in order to keep numerical errors below 100%. Otherwise, spurious fragmentation of the cloud occurs before collapse.

Numerical test with a collapsing cloud

Truelove *et al.* (1997) considered an initial m=2 perturbation for the spherical collapse of the homogeneous cloud. Using a PPM solver, they found that spurious fragmentation is avoided for $\Delta x < \frac{\lambda_J}{4}$





J. K. Truelove *et al.*, “The Jeans condition: a new constraint on spatial resolution in simulation of isothermal self-gravitational hydrodynamics”, ApJ, 1997, 489, L179

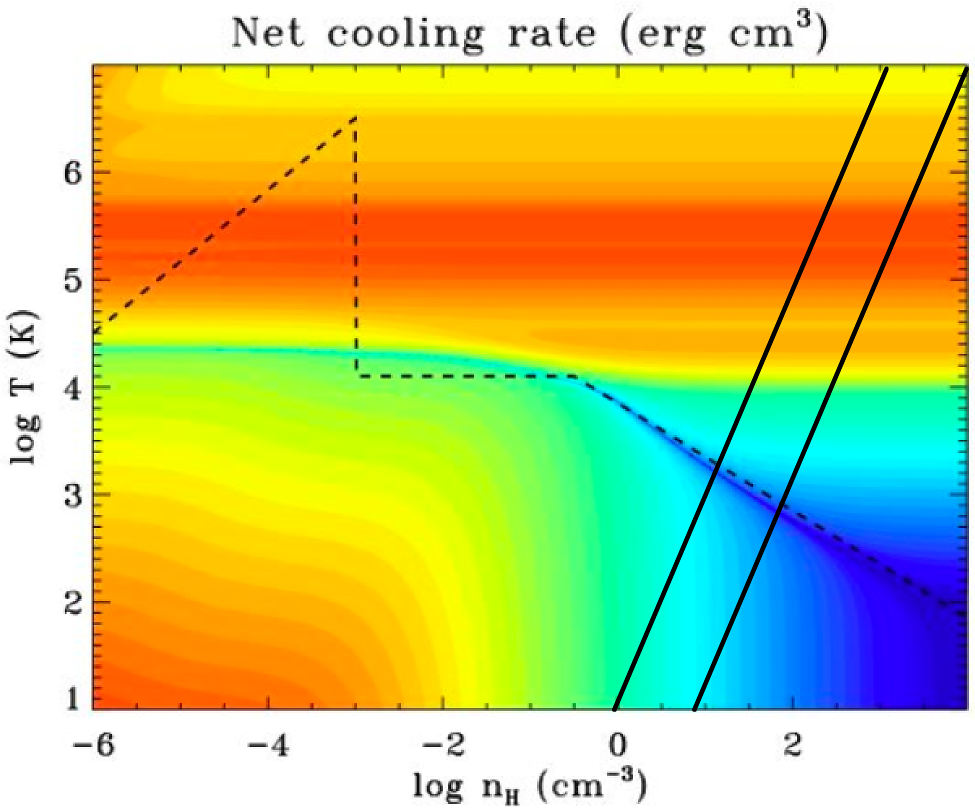
Artificial pressure support

In order to resolve the Jeans length, we add to the thermal pressure a dynamical, Jeans-length related, pressure floor defined as $P_J=16\Delta x^2 G\rho^2$

This sets an artificial thermal Jeans length in the problem.

Keeping a fixed artificial Jeans length, one can then refine the grid and check for convergence.

This artificial Jeans length sets the minimum cloud mass, equal to the thermal Jeans mass M_J.



Truelove *et al.* 1997; Bates & Burkert 1997; Machacek *et al.* 2001, Robertson & Kravtsov 2008

Cold sine wave collapse

Use RAMSES to create a cold sine wave velocity perturbation (Zeldovich pancake)

```
!================================================================
integer::ivar,i,id,iu,ip
real(dp)::twopi
real(dp),dimension(1:nvector,1:nvar),save::q   ! Primitive variables

id=1; iu=2; ip=ndim+2
twopi=2.0*acos(-1.0)
do i=1,nn
   q(i,id)=1.0
   q(i,iu)=sin(twopi*(x(i,1)))
   q(i,ip)=1e-5
end do

! Convert primitive to conservative variables
```

Patch condinit.f90

```
&AMR_PARAMS
levelmin=7
levelmax=7
ngridmax=20000
nexpand=1
boxlen=1.0
/

&INIT_PARAMS
nregion=0
/

&HYDRO_PARAMS
gamma=1.66667
courant_factor=0.8
slope_type=1
scheme='muscl'
riemann='hllc'
/
```

Periodic BCs.

Before shell crossing and shock formation, we know the analytical solution.

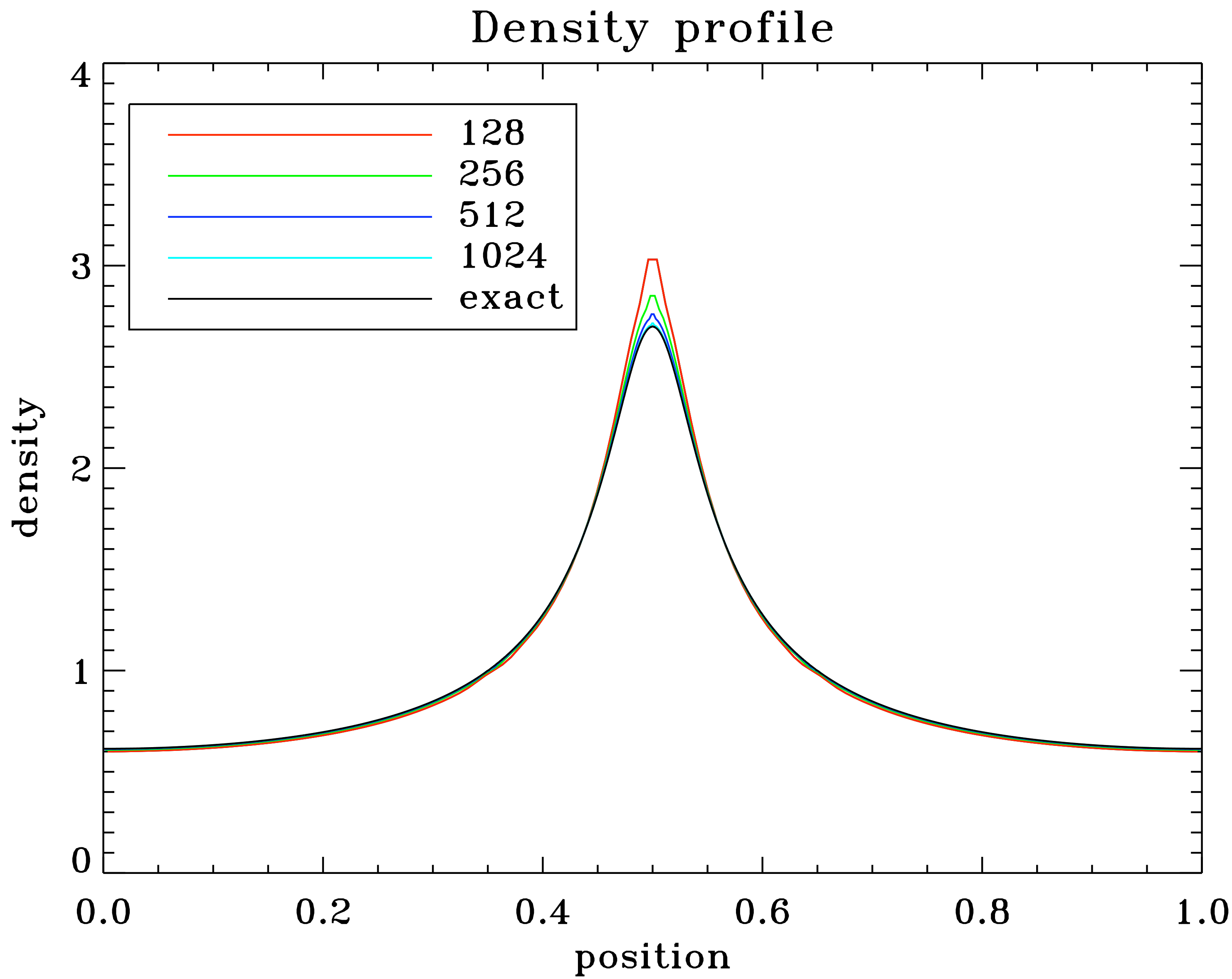
Because the initial temperature is very low, we have spurious numerical heating.

We define a compression time: $$\frac{1}{\tau_{comp}} = \left|\frac{\partial u}{\partial x}\right| \simeq \frac{1}{H(t)}$$

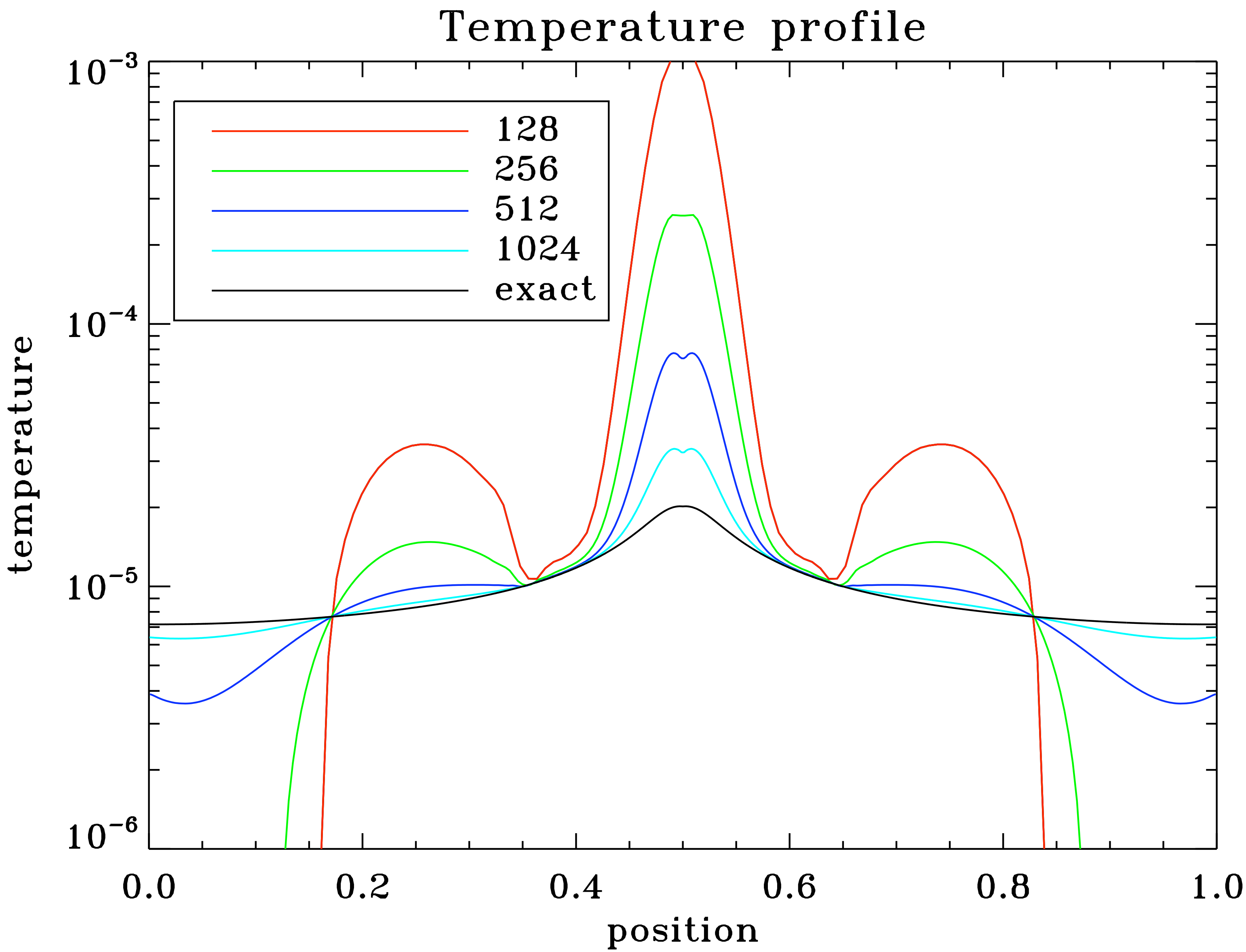
Spurious effects arise if:

$$c\tau_{comp} < \Delta x$$

Cold sine wave collapse at t=0.1



Cold sine wave collapse at t=0.1



Hybrid scheme for high-Mach-number flows

Conservative scheme: total energy flux and pressure evaluation

$$\partial_t(E) + \partial_x(E+P)u = \rho \mathbf{u}\cdot\mathbf{g} \qquad P = (\gamma-1)\left(E - \frac{1}{2}\rho u^2\right)$$

Primitive scheme: internal energy flux and pressure evaluation

$$\partial_t(e) + \partial_x eu = -P\partial_x u \qquad P = (\gamma-1)e$$

For high-Mach-number flows, compression waves are much faster than sound waves. Cold hydrodynamics is better described by decoupling the thermal energy from the kinetic energy.

Jin & Levermore, 1996, JCP, 126, 449 proposed a fix for such stiff problems. We modified it slightly for cosmology and, we derived the following hybrid scheme:

Use total energy update if: $c > \beta\Delta x\,|\partial_x u|$

and internal energy update if: $c < \beta\Delta x\,|\partial_x u|$

See also other implementations by V. Springel, G. Bryan

```
&HYDRO_PARAMS
gamma=1.66667
courant_factor=0.8
slope_type=1
scheme='muscl'
riemann='hllc'
pressure_fix=.true.
beta_fix=1.0
/
```

Hydrodynamics and radiation

The Radiative Transfer Equation

Conservation of photons in phase-space

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = -\kappa_\nu I_\nu + \eta_\nu,$$

$I_\nu(\mathbf{x},\mathbf{n},t)$ radiation specific intensity

$\kappa_\nu(\mathbf{x},\mathbf{n},t)$ absorption coefficient

$\eta_\nu(\mathbf{x},\mathbf{n},t)$ source function

Source term: microscopic collisions with plasma particles.
Absorption and emission processes.

Moments of the Radiative Transfer equation

Radiation energy: $E_\nu(\mathbf{x},t) = \int I_\nu(\mathbf{x},\mathbf{n},t)\frac{\mathrm{d}\mathbf{\Omega}}{c}$

Radiation flux: $\mathbf{F}_\nu(\mathbf{x},t) = \int I_\nu(\mathbf{x},\mathbf{n},t)\mathbf{n}\frac{\mathrm{d}\mathbf{\Omega}}{c}$

Pressure tensor: $\mathbf{P}_\nu(\mathbf{x},t) = \int I_\nu(\mathbf{x},\mathbf{n},t)\mathbf{n}\times\mathbf{n}\frac{\mathrm{d}\mathbf{\Omega}}{c}$

Energy equation: $\frac{\partial E_\nu}{\partial t} + \nabla \mathbf{F}_\nu \quad = \quad -\kappa_\nu c E_\nu + S_\nu\,,$

Flux equation: $\frac{\partial \mathbf{F}_\nu}{\partial t} + c^2 \nabla \mathbf{P}_\nu \quad = \quad -\kappa_\nu c \mathbf{F}_\nu\,.$

Fluid energy equation writes:

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}(E+P)) = \Gamma - \Lambda$$

Heating and cooling functions:

$$\Gamma = \int \kappa_\nu c E_\nu \mathrm{d}\nu \qquad \Lambda = \int S_\nu \mathrm{d}\nu$$

Momentum equation writes:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P) = -\rho \nabla \phi + \mathbf{F}_{\mathrm{rad}}$$

The radiation force: $$\mathbf{F}_{\mathrm{rad}} = \int \kappa_\nu \frac{\mathbf{F}_\nu}{c} \mathrm{d}\nu$$

Atomic Processes

Radiation is emitted or absorbed when electrons make transitions between different states

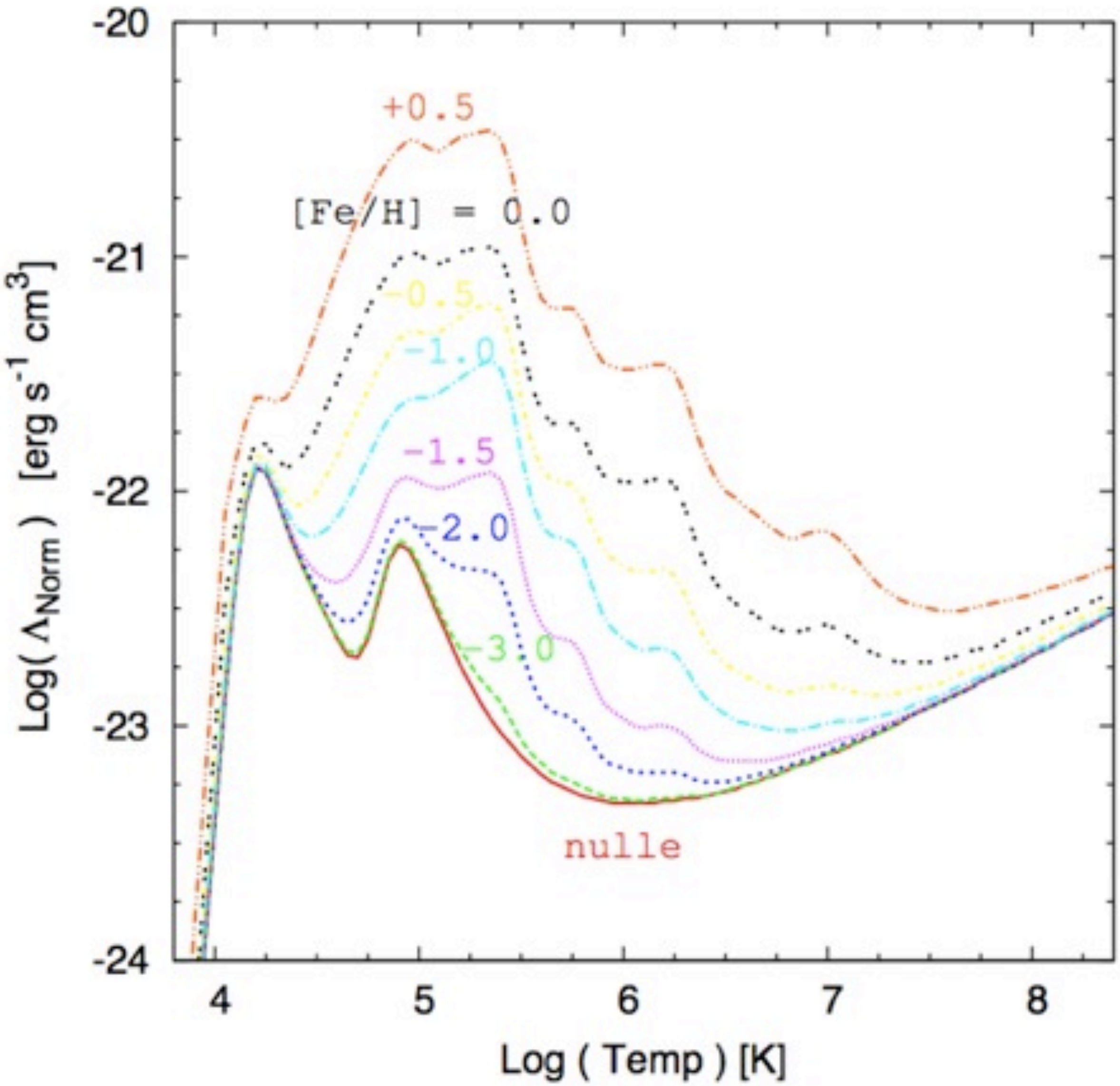
Bound-bound: electrons moves between 2 bound states in an atom or an ion. A photon is emitted or absorbed.

Bound-free: electrons move to the continuum (ionization) or a absorbed from the continuum to a bound state (recombination)

Free-free: electrons in the continuum gain or loose energy (a photon) when orbiting around ions (Bremsstrahlung).

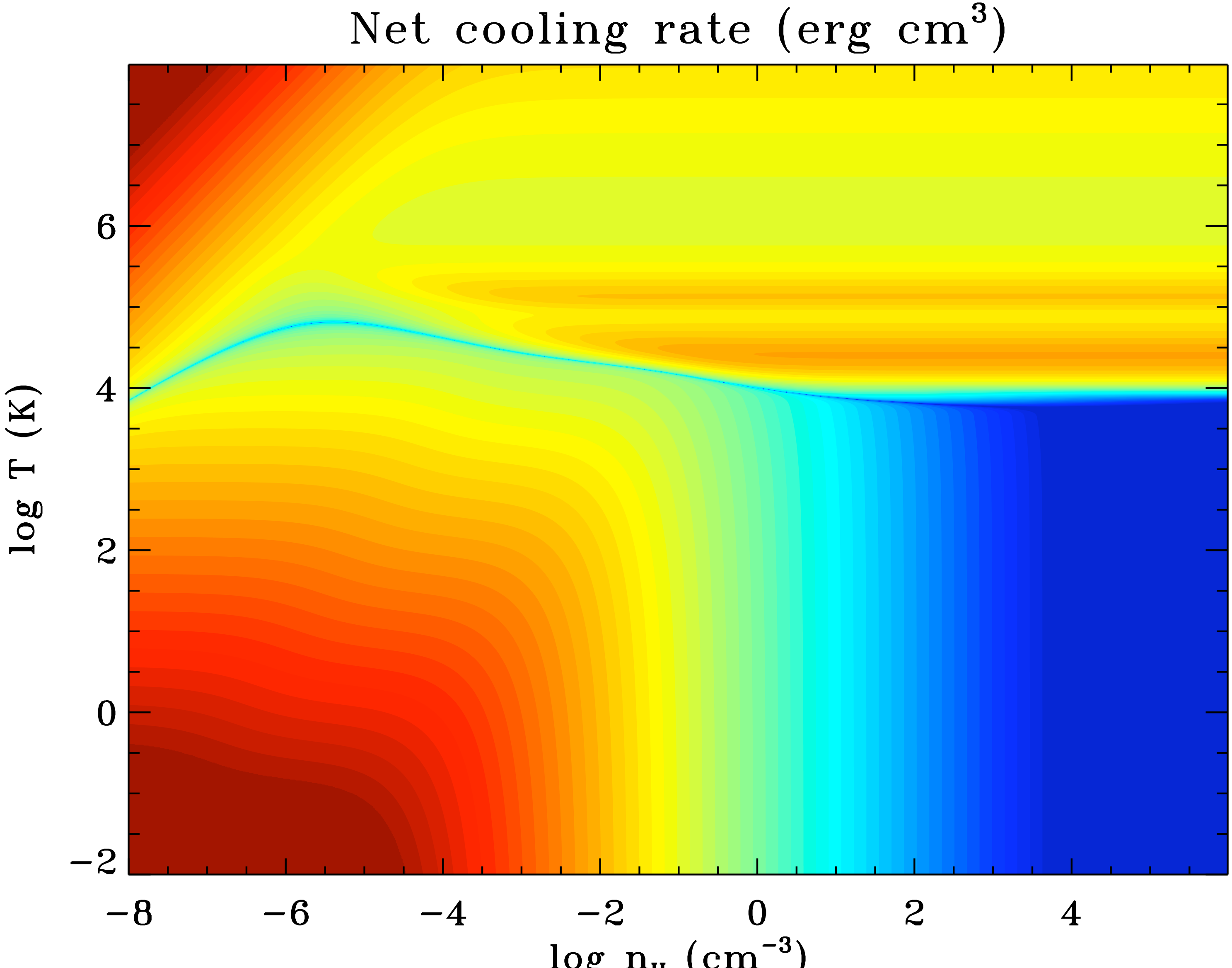
Cooling function for astrophysical plasmas

Collisional Ionization Equilibrium: depends only on T



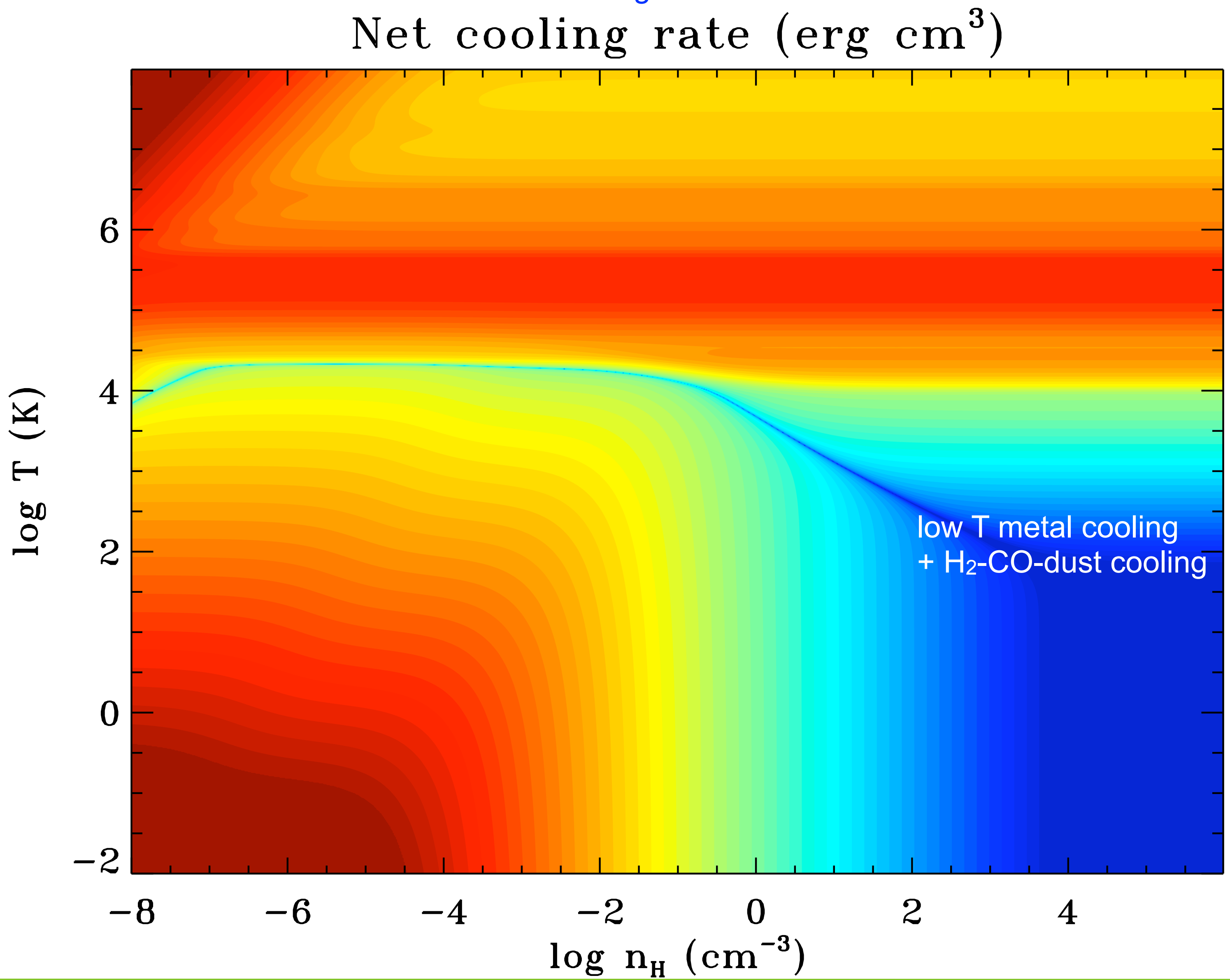
Cooling function for astrophysical plasmas

Photo-Ionization Equilibrium: depends on T and n



Cooling function for astrophysical plasmas

PIE including metals at solar level



Closing the hierarchy : the M1 approximation

Assume that the radiation field is locally a dipole.
The photon distribution function is the Lorentz transform of a Planck distribution function.
Dipole is aligned locally to the radiation flux.

The M1 Eddington tensor writes:

$$\mathbf{D} = \frac{1-\chi}{2}\mathbf{I} + \frac{3\chi-1}{2}\mathbf{u}\otimes\mathbf{u}, \quad \chi = \frac{3+4|\mathbf{f}|^2}{5+2\sqrt{4-3|\mathbf{f}|^2}}.$$

where we define the reduced flux: $\mathbf{f} = \frac{\mathbf{F}}{cN} = f\mathbf{u}.$

Asymptotics: f=0 gives D=1/3 and f=1 gives D=1

Levermore, 1984, JQST, 31, 149

A Godunov scheme for M1 radiation transport

We have a *hyperbolic* system of 4 conservations laws:

$$\frac{\partial N_\gamma}{\partial t} + \nabla \mathbf{F}_\gamma = 0,$$
$$\frac{\partial \mathbf{F}_\gamma}{\partial t} + c^2 \nabla \mathbf{P}_\gamma = 0,$$

Vector of conservative variables: $\mathcal{U} = (N_\gamma, F_\gamma)^T$

Vector of flux functions: $\mathcal{F} = (F_\gamma, P_\gamma)^T$

Godunov's method for M1: Aubert & Teyssier (2008)

$$\frac{(N_\gamma)_i^{n+1} - (N_\gamma)_i^n}{\Delta t} + \frac{(F_\gamma)_{i+1/2}^m - (F_\gamma)_{i-1/2}^m}{\Delta x} = 0,$$
$$\frac{(F_\gamma)_i^{n+1} - (F_\gamma)_i^n}{\Delta t} + c^2 \frac{(P_\gamma)_{i+1/2}^m - (P_\gamma)_{i-1/2}^m}{\Delta x} = 0.$$

Implicit or explicit time integration

Explicit scheme for radiation transport with wave velocity close to c: very restrictive Courant condition: for one large hydro step, we need hundreds or thousands of radiation sub-cycles.

Implicit method is stable for large time steps but requires large sparse matrix solvers (CPU intensive, convergence and parallel computing issues)

Trick 1: reduced speed of light approximation *when valid*.

Trick 2 (brute force strategy): use GPU acceleration to speed-up the explicit solver. Simple Cartesian mesh-based algorithms are maximally accelerated using GPU. Use a local implicit solver for chemistry and cooling.

Reduced speed of light approximation

Consider a Strömgren sphere $r_s = \left(\frac{3\dot{N}_\gamma}{4\pi} \frac{1}{n_H^2 \alpha_B} \right)^{1/3}$.

The solution becomes independent on the value of the speed of light when

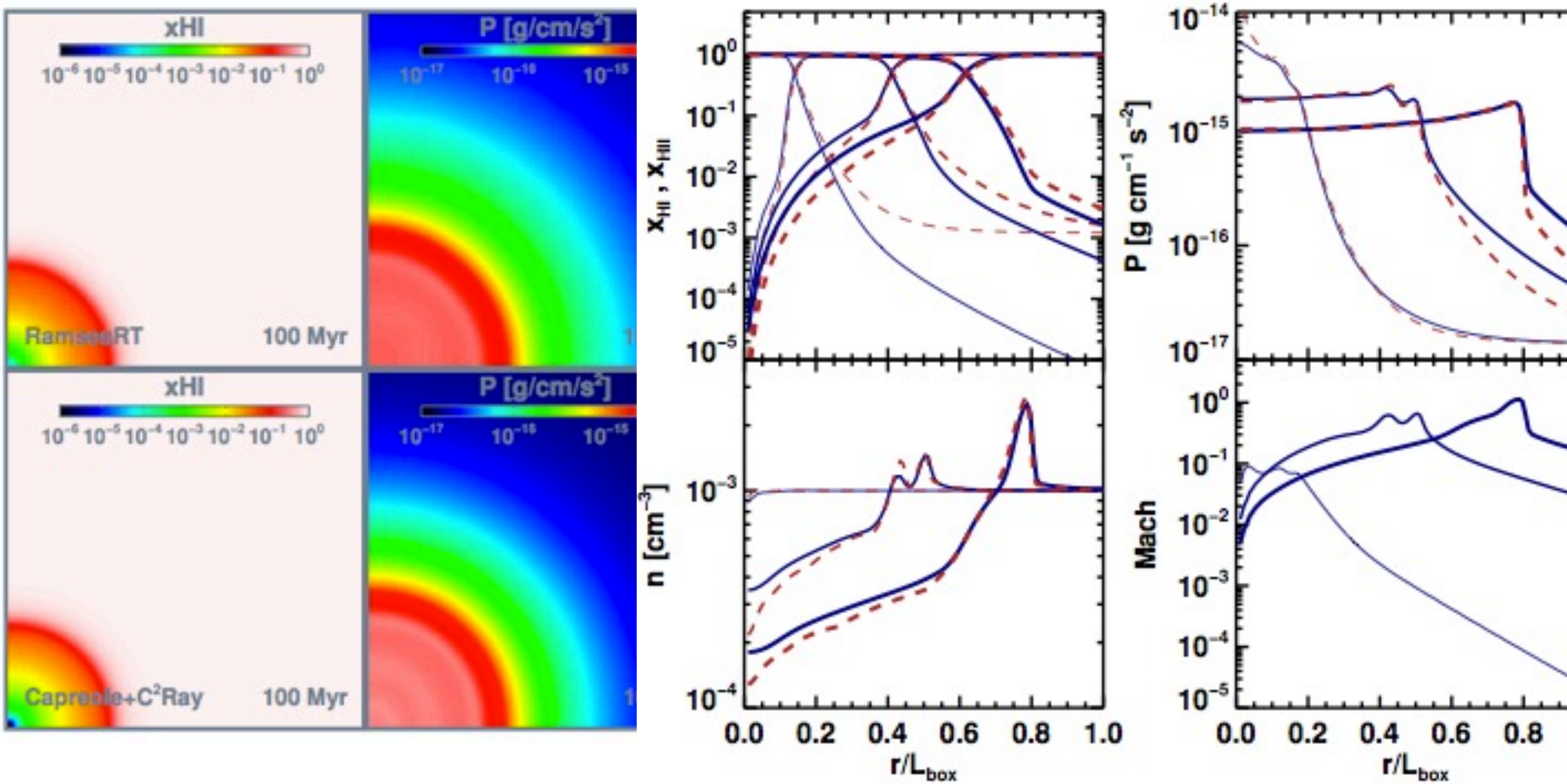
$$t \geq t_{\rm cross} = \frac{r_s}{c}$$

Trick: use $c_{\rm reduced} = \min\left(c, \frac{r_s}{t}\right)$ Rosdahl et al. (2013)

A few important examples:

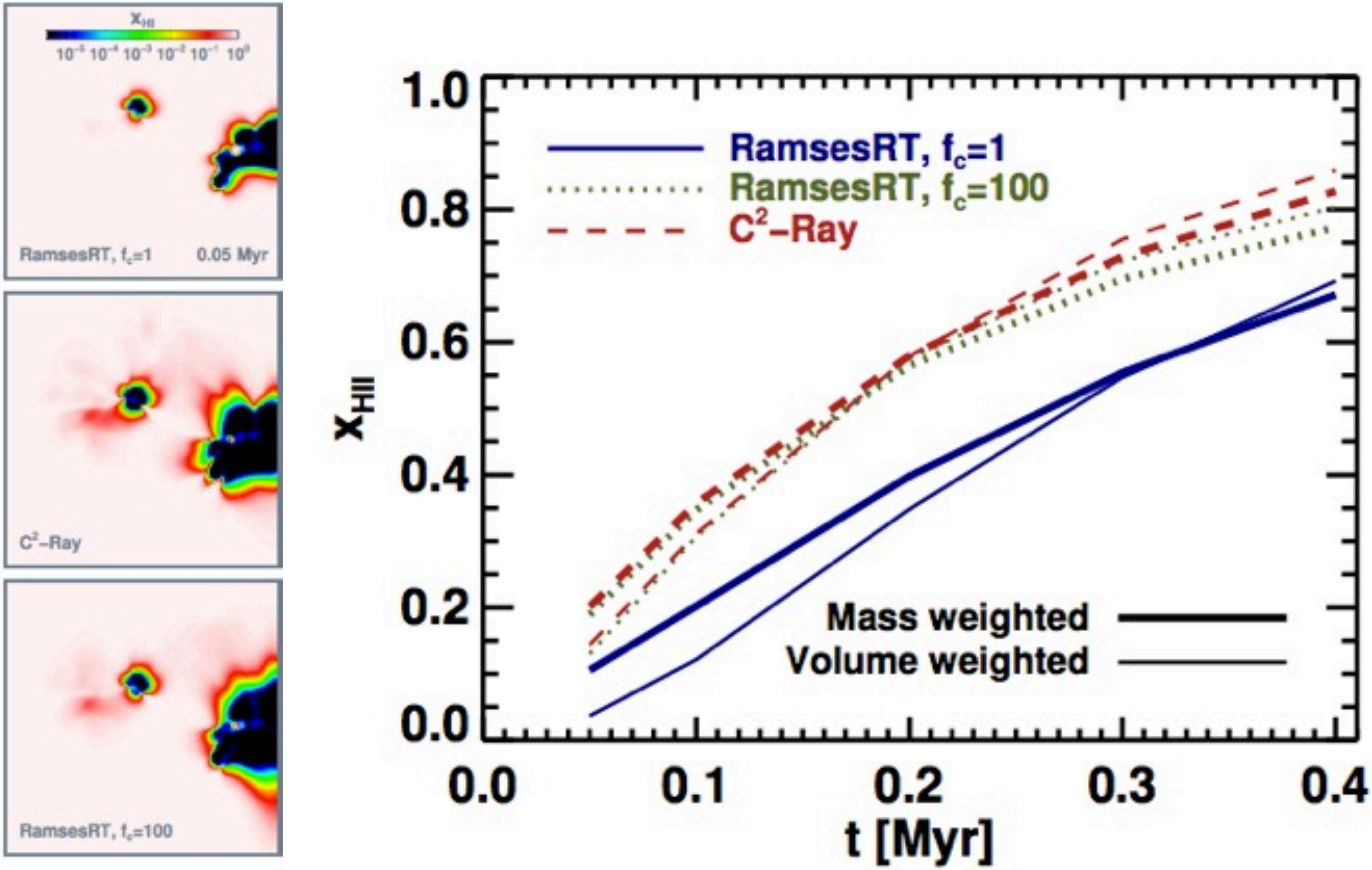
	$n_{\rm H}$ [cm^{-3}]	$\dot{N}_\gamma$ [photons/sec]	r_s [kpc]	$t_{\rm cross}$ [Myr]
OB star in cloud	100	10^{48}	10^{-3}	10^{-5}
Star cluster in disc	0.1	10^{50}	1	1
Milky Way at z=0	10^{-7}	10^{52}	10^{4}	100
Iliev Test 4	10^{-4}	10^{53}	400	1

Test 5 (Iliev et al. 2009) with c=0.01

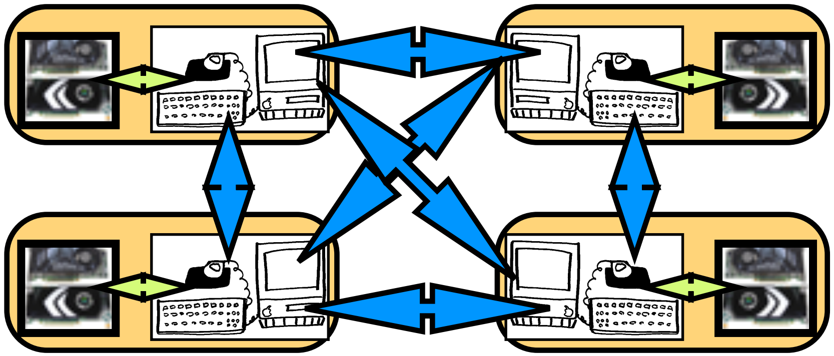


Comparison with the ray-tracing code C^2Ray.

Test 4 (Iliev et al. 2006) with c=1



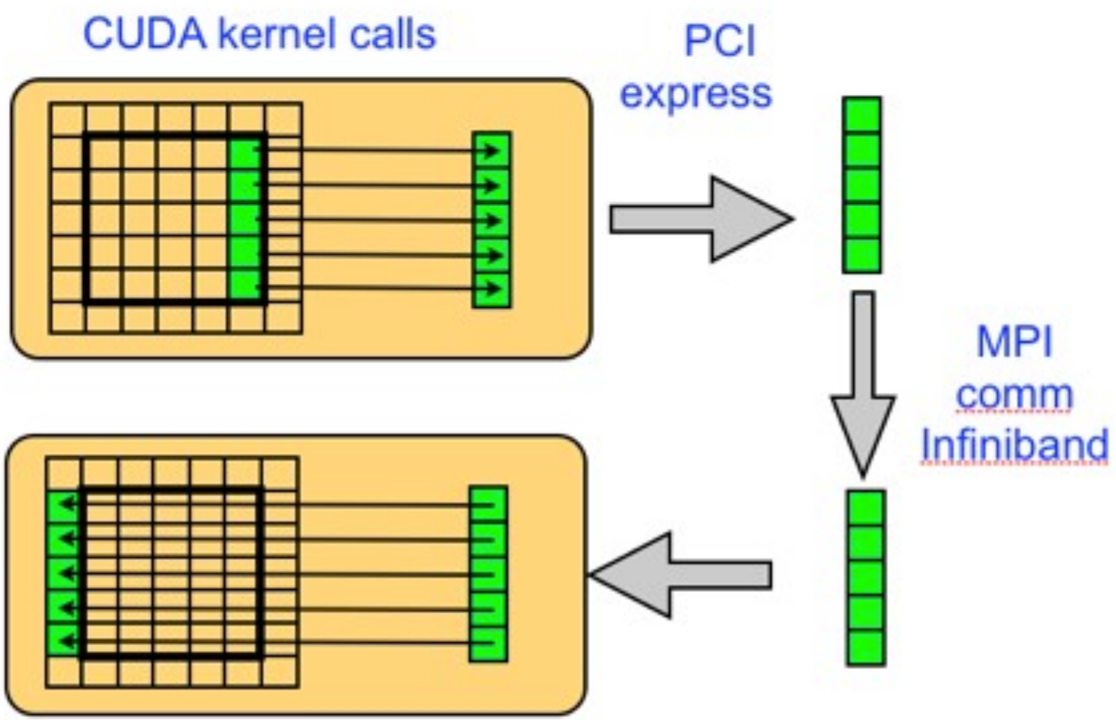
GPU acceleration for large number of time steps



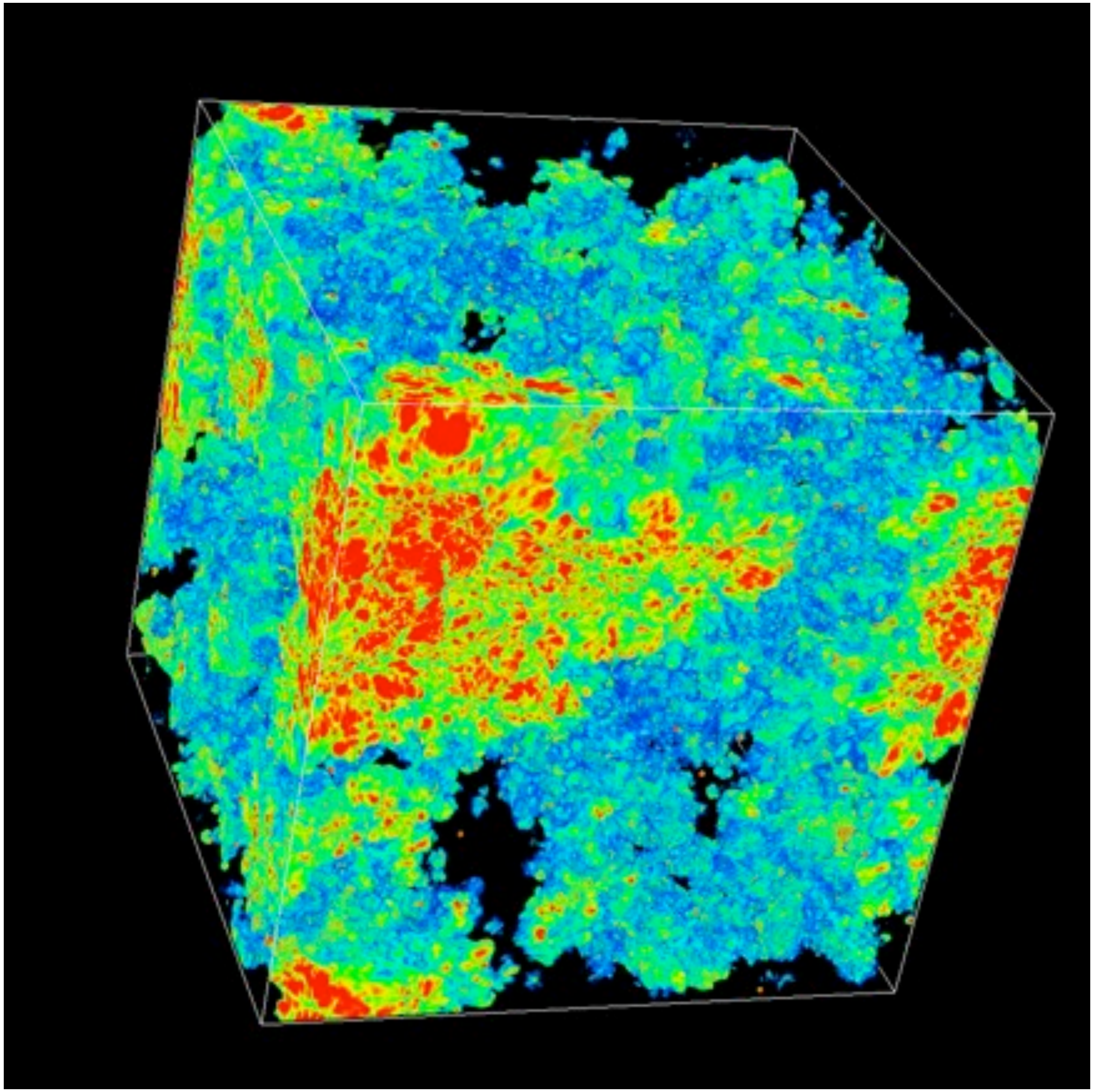
Use graphical device developed mostly by NVidia to speed up scientific computations.
Graphical Processing Units (GPU) have higher memory bandwidth and uses very large number of threads.

High latency and low transfer rate of PCI express port.
Perform 100+ time steps before copying back the results.

Still need parallel GPU computing.
Transfer boundary conditions at every radiation sub-step (PCI express and Infiniband latencies comparable).



Simulating cosmic reionization using GPUs



Aubert & Teyssier (2010)