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Astrophysics and Cosmology (77501 : Fall Term 2007/8)

Problem Set 2

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Problem 1:
The RW metric and coordinate transformation:
Define the a new 'radial' coordinate by:

$$\chi = \int_0^r \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \tag{1}$$

(Note that the proper distance is given by $d_p = R\chi$.) Show that the RW metric can be rewritten as:

$$ds^2 = dt^2 - R^2\Big(d\chi^2 + S_k(\chi)^2 d\Omega^2\Big), \tag{2}$$

where $S_k(\chi) = \Big[\sin(\chi),\ \chi,\ \sinh(\chi)\Big]$ for $k = 1,\ ,-1$.
Note that in "Introduction to Cosmology" (B. Ryden) the notations are reversed, namely $r_{\rm class} = \chi_{\rm Ryden}$ and $\chi_{\rm class} = r_{\rm Ryden}$.

Problem 2:
Consider a homogenous and isotropic 3D space, hence its metric is RW of a radius R. Show that the volume of this space is finite for $k = 1$ and infinite for $k = 0,\ -1$.

Problem 3:
The angular distance:
Suppose we have a cosmic standard yard stick of length l. We observe it from a distance and it spans an angle θ over the sky, where θ (measured in radians) is small, $\theta \ll 1$. Define the angular distance as

$$d_A \equiv \frac{l}{\theta}. \tag{3}$$

a. Suppose that the yard yard stick is located at r and is observed at a redshift z. Express d_A in terms of z and r.
b. What is the relation between the luminosity and angular distances (d_L and d_A)?
c. Expand d_A to 2nd order in z. Write it in terms of H_0 and q_0.

Problem 4:
Cosmological Expansion:
a. Consider a (homogenous and isotropic) universe made of one kind of component whose density is ρ and its equation of state is $P = w\rho c^2$. Show that the cosmological evolution of the density is given by

$$\rho = \rho_0\Big(\frac{R}{R_0}\Big)^{-3(1+w)}, \tag{4}$$

where R is the scale factor of the universe.

b. Show that for cold matter ("dust") this implies mass conservation and that for black-body radiation (CMB) it implies conservation of the number of photons, and hence the entropy per photon of the radiation field.

Problem 5:
Friedmann Equation - Newtonian derivation:
Consider a finite sphere of radius R at time t in an expanding homogenous and isotropic universe of density ρ. Assume that the radius of the sphere is small enough such that its dynamics is governed by Newtonian dynamics.
a. Write the equation of motion of $R(t)$ (i.e. the equation for $\ddot{R}$).
b. Integrate the equation and find its integral of motion. (Hint: multiply the equation by $\dot{R}$ and integrate)
c. Rewrite the resulting equation and obtain the Friedmann Equation. What is the physical meaning of the critical density?
As the sphere expands with the rest of the universe its equation of motion describes also the expansion of the universe. Note that the Newtonian derivation coincides with the full GR treatment upon assuming that the density is the mass-energy density.