Astrophysics and Cosmology (77501): Formulae Page

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Useful constants

Speed of light	c	2.998 10^{10}	cm sec^{-1}
Planck's constant	h	6.626 10^{-27}	erg s
Rationalized Planck's constant	$\hbar = h/2\pi$	1.055 10^{-27}	erg s
Boltzmann's constat	k_B	1.381 10^{-16}	erg $^\circ\mathrm{K}^{-1}$
Electron charge	e	4.803 10^{-10}	esu
Electron rest mass	m_e	9.110 10^{-28}	g
Gravitational Constant	G	6.673 10^{-8}	dyne cm^2 g^{-2}
Avogadro's number	N_A	6.022 10^{23}	mole^{-1}
Atomic mass unit	a.m.u.	1.661 10^{-24}	g
Gas constant	R	8.314 10^{7}	erg $^\circ\mathrm{K}^{-1}$ mole^{-1}
Stephan-Boltzmann constant	σ	5.670 10^{-5}	erg cm^{-2} s^{-1} $^\circ\mathrm{K}^{-4}$
Radiation constant	a	7.564 10^{-15}	erg cm^{-3} $^\circ\mathrm{K}^{-4}$
Thomson cross section	σ_T	6.656 10^{-25}	cm^2
Astronomical unit	a.u.	1.496 10^{13}	cm
Parsec	pc	3.086 10^{18}	cm
Light year	l.y.	9.460 10^{17}	cm
Solar mass	$M_\odot$	1.989 10^{33}	g
Solar radius	$R_\odot$	6.960 10^{10}	cm
Solar luminosity	$L_\odot$	3.826 10^{33}	erg s^{-1}
Electron volt	eV	1.602 10^{-12}	erg

Black body Radiation

Flux, energy density and pressure of a black body radiation

$$F = \sigma T^4 \qquad \epsilon = a T^4 \qquad P = \frac{1}{3} a T^4 \tag{1}$$

Potential Energy and Virial theorem

Gravitational potential energy of spherically symmetric mass M:

$$E_{grav} = -G \int_0^M \frac{m dm}{r} \tag{2}$$

Virial theorem (spherical system in hydrostatic equilibrium with vanishing surface pressure) :

$$3 \int_0^M \frac{P}{\rho} dm + E_g = 0 \tag{3}$$

The thermal energy density (*i.e.* per unit mass) is

$$\epsilon_{th} = \frac{P}{(\gamma - 1)\rho}, \tag{4}$$

where $\gamma \equiv \frac{C_p}{C_V} = \frac{n_f + 2}{n_f}$ and n_f is the number of degrees of freedom per particle. Also, recall that for adiabatic processes $P \propto \rho^\gamma$.

Ideal Gas (mono-atomic non-degenerate) Pressure

$$P_{gas} = \frac{\rho}{\mu m_p} k_B T \tag{5}$$

Molecular Weight

For a fully ionized plasma made of H (mass fraction X), H_e (mass fraction Y) and heavier elements (mass fraction $Z = 1 - X - Y$):

$$\frac{1}{\mu} \approx 2X + \frac{3}{4}Y + \underbrace{\left\langle \frac{(1+Z)}{A} \right\rangle}_{\approx 1/2} Z \tag{6}$$

Cold Electron Degeneracy Pressure

The Fermi momentum is $p_F = \left(\frac{3 n_e h^3}{8\pi}\right)^{1/3}$ and the Fermi energy is $\epsilon_F = \frac{p_F^2}{2m}$.

Non-relativistic electrons:

$$P_{e,nr} = \left[\frac{1}{20} \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \right] \rho^{5/3} \tag{7}$$

Extremely relativistic electrons:

$$P_{e,r} = \left[\frac{1}{8} \frac{3}{\pi} \frac{hc}{m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \right] \rho^{4/3} \tag{8}$$

Equations for stellar structure

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{9}$$
$$\frac{dP}{dr} = -\rho \frac{Gm(r)}{r^2} \tag{10}$$
$$\frac{dT}{dr} = -\frac{3k\rho}{16\pi a c r^2 T^3} L \tag{11}$$
$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \tag{12}$$

The opacity (absorption coefficient per unit mass; κ) is approximated at $T > 10^5 K$ by : $\kappa = \kappa_0 \rho T^{-3.5}$ (Kramer opacity)

The energy production rate (per unit mass) ϵ is approximated by $\epsilon = \epsilon_0 \rho^m T^\eta$. For pp burning $m = 1$, $\eta \approx 5$. For CNO burning $m = 1$, $\eta \approx 20$. For 3α burning $m = 2$, $\eta \approx 40$.

Convective energy transfer

Condition for convection:

$$\frac{d \ln P}{d \ln \rho} > \gamma \tag{13}$$

When the energy transfer is dominated by convection:

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{adiab} = \frac{\gamma - 1}{\gamma} \frac{dP}{dr} \tag{14}$$

Nuclear Reactions

Gamov energy:

$$E_G = \left(\pi \alpha Z_A Z_B\right)^2 2mc^2 \tag{15}$$

Gamov factor:

$$g(E) = \exp\left[- \sqrt{\frac{E_G}{E}}\ \right] \tag{16}$$

Galaxies & Clusters

Rotation curve:

$$V_{circ}^2 \equiv \frac{Gm(r)}{r} \tag{17}$$

Virial theorem for stellar systems:

$$\frac{\ddot{I}}{2} = E_g + 2E_k \tag{18}$$

(where I is the moment of inertia, E_g and E_k are the gravitational and kinetic energies, respectively).

Cosmology

Redshift: $z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$

Hubble's constant: $H_0 = 100 h km/s/Mpc$

Robertson-Walker Metric:

$$ds^2 = (cdt)^2 - R(t)^2 \Big[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{19}$$

where $K = 0, \ \pm 1$.
Einstein Field Equations for the RW metrics:

$$\ddot{R} = -\frac{4\pi G}{3}\Big(\rho + 3\frac{p}{c^2}\Big)R \tag{20}$$

$$R\ddot{R} + 2\dot{R}^2 = 4\pi G(\rho - \frac{P}{c^2})R^2 - 2Kc^2 \tag{21}$$

Friedmann Equations:

$$\dot{R}^2 = \frac{8}{3}\pi G \rho R^2 - Kc^2 \tag{22}$$

Definition: $a(t) \equiv \frac{R(t)}{R(t_0)}$
Cosmological parameters:

$$H_0 = \Big(\frac{\dot{R}}{R}\Big)_0 \tag{23}$$

$$\Omega_0 = \Big(\frac{\rho}{\rho_c}\Big)_0 \tag{24}$$

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \tag{25}$$

$$q_0 = -\Big(\frac{\ddot{R}R}{\dot{R}^2}\Big)_0 \tag{26}$$

'Cosmological' equation of state:

$$p = w\rho c^2 \tag{27}$$

Cosmological constant:

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \tag{28}$$

$$P_\Lambda = -\frac{\Lambda c^4}{8\pi G} \tag{29}$$

Black Body Radiation:

$$\epsilon(\nu) = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\exp(\frac{h\nu}{k_B T}) - 1} \tag{30}$$

$$n_\gamma = \frac{2.4 k_B^3}{\pi^2 \hbar^3 c^3} T^3 \tag{31}$$

$$(h\nu)_{mean} \approx 2.7 k_B T \tag{32}$$

Luminosity distance:

$$d_L = \Big(\frac{L}{4\pi F}\Big)^{1/2} \tag{33}$$

Proper distance (at time t_0) to a distance located at coordinate r:

$$d_p(t_0) = R(t_0)\int_0^r \frac{d\tilde{r}}{\sqrt{1 - K\tilde{r}^2}} \tag{34}$$

Lookback time:

$$t_0 - t(z) = \frac{1}{H_0}\Big[z - (1 + \frac{q_0}{2})z^2 + O[z^3]\Big] \tag{35}$$

The r - z relation:

$$R_0\ r(z) = c\frac{1}{H_0}\Big[z - (\frac{1 + q_0}{2})z^2 + O[z^3]\Big] \tag{36}$$