QUANTUM OPTICS SEMINAR

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WIGNER REPRESENTATION AND HOMODYNE DETECTION

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RACAH INSTITUTE OF PHYSICS
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Outline

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THE WIGNER FUNCTION

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- Definition and properties

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- ➢ Definition and properties
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  - ➢ The vacuum state

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  - Selected experiments

Wigner Representation - Properties

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$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dpW(q,p) = 1$$

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$$|\langle q|\psi\rangle|^2 = \int_{-\infty}^{\infty} dp W(q,p) \geq 0$$

$$\text{check} \Rightarrow \int_{-\infty}^{\infty} dp W(q,p) = \int_{-\infty}^{\infty} dy \langle q-y|\psi\rangle \langle\psi|q+y\rangle \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{2ipy/\hbar} dp$$
$$= \int_{-\infty}^{\infty} dy \langle q-y|\psi\rangle \langle\psi|q+y\rangle \delta(y) = \langle q|\psi\rangle\langle\psi|q\rangle \checkmark$$

and also for p

$$|\langle p|\psi\rangle|^2 = \int_{-\infty}^{\infty} dq W(q,p) \geq 0$$

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- There are states for which $W(q,p)$ is *negative* at certain values of q and p (for orthogonal states the trace vanishes).

Wigner Representation - Examples

THE VACUUM STATE

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$$\begin{aligned} W_0(Q,P) &= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \psi_0^*(Q+y)\psi_0(Q-y)e^{2iPy/\hbar}dy \\ &= \frac{1}{\pi\hbar}\frac{1}{\left(\sqrt{\pi\hbar}\right)}\int_{-\infty}^{\infty} e^{-\frac{1}{2\hbar}\left[(Q-y)^2+(Q+y)^2\right]}e^{2iPy/\hbar}dy \end{aligned}$$

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$$= \frac{\mathrm{Exp}\{-\frac{Q^2+P^2}{\hbar}\}}{\pi\hbar}\frac{1}{\left(\sqrt{\pi\hbar}\right)}\int_{-\infty}^{\infty} e^{-\frac{1}{\hbar}\left(y-\frac{iP}{\hbar}\right)^2}dy$$

Wigner Representation - Examples

THE VACUUM STATE

$$\Rightarrow W_0(Q,P) = \frac{1}{\pi\hbar} e^{-\left(\frac{Q^2+P^2}{\hbar}\right)}$$

Wigner Representation - Examples

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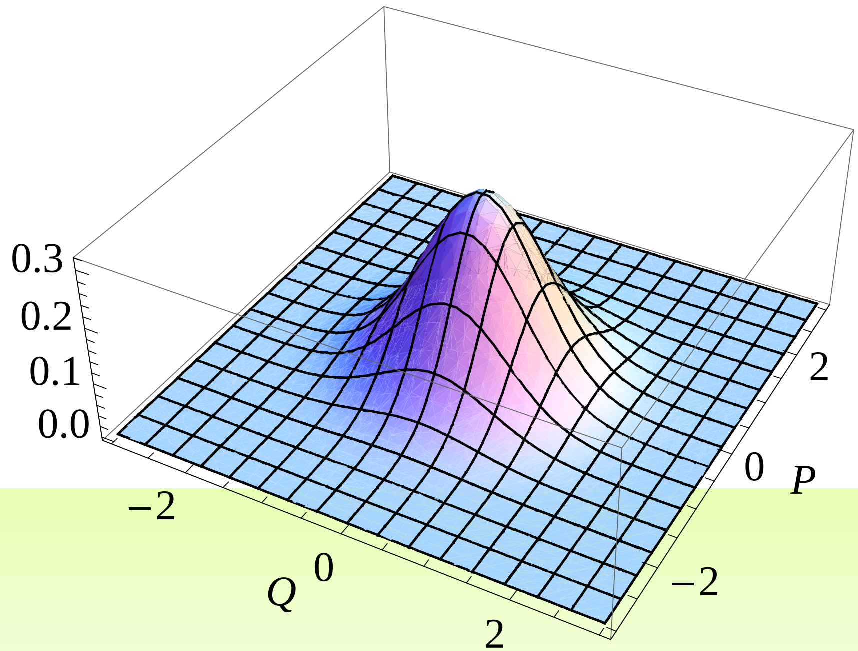
Figure 1

A Plot of $W_0(Q,P)$ as a function of Q and P. The behavior is stationary since the state is an Eigenstate of the Hamiltonian.

Figure 1



Wigner Representation - Examples

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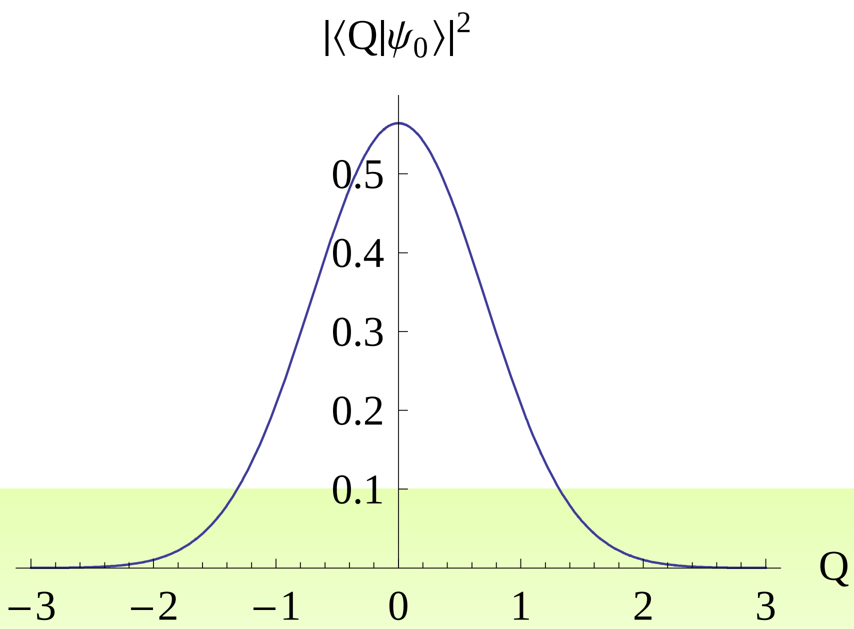
Figure 2

A plot of the marginal probability density $|\langle Q|\psi\rangle|^2$. This is obtained by integrating $W_0(Q,P)$ over P:

$$|\langle Q|\psi_0\rangle|^2 = \int_{-\infty}^{\infty} W_0(Q,P)\, dP = \frac{1}{\sqrt{\pi}} e^{-\frac{Q^2}{\hbar}}$$

Figure 2



Wigner Representation - Examples

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Wigner Representation - Examples

EXCITED FOCK STATES

$$n = 1 \Longrightarrow W_1(Q, P) = -\frac{2}{\pi\hbar} e^{-\frac{P^2+Q^2}{\hbar}} \left(\frac{1}{2} - \frac{P^2 + Q^2}{\hbar} \right)$$

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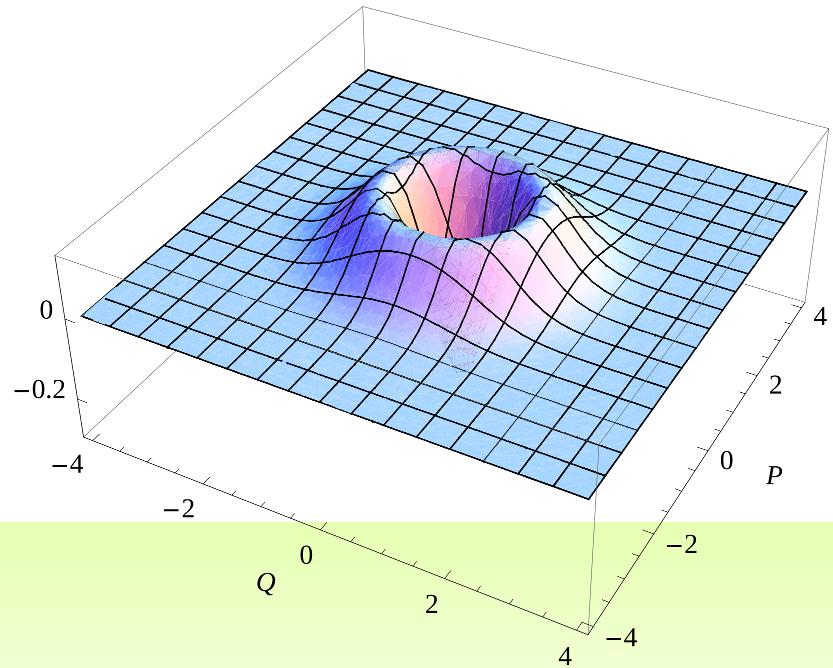
Figure 3

A Plot of $W_1(Q,P)$ as a function of Q and P (a number state with $n = 1$). The behavior is stationary since the state is an Eigenstate of the Hamiltonian.

Figure 3



Wigner Representation - Examples

EXCITED FOCK STATES

$$n = 1 \Longrightarrow W_1(Q,P) = -\frac{2}{\pi\hbar} e^{-\frac{P^2+Q^2}{\hbar}} \left(\frac{1}{2} - \frac{P^2+Q^2}{\hbar} \right)$$

Figure 3

A Plot of $W_1(Q,P)$ as a function of Q and P (a number state with $n=1$). The behavior is stationary since the state is an Eigenstate of the Hamiltonian.

Figure 4

A section at $P = 0$ where the negative part of $W_1(Q,P)$ is evident

Figure 4

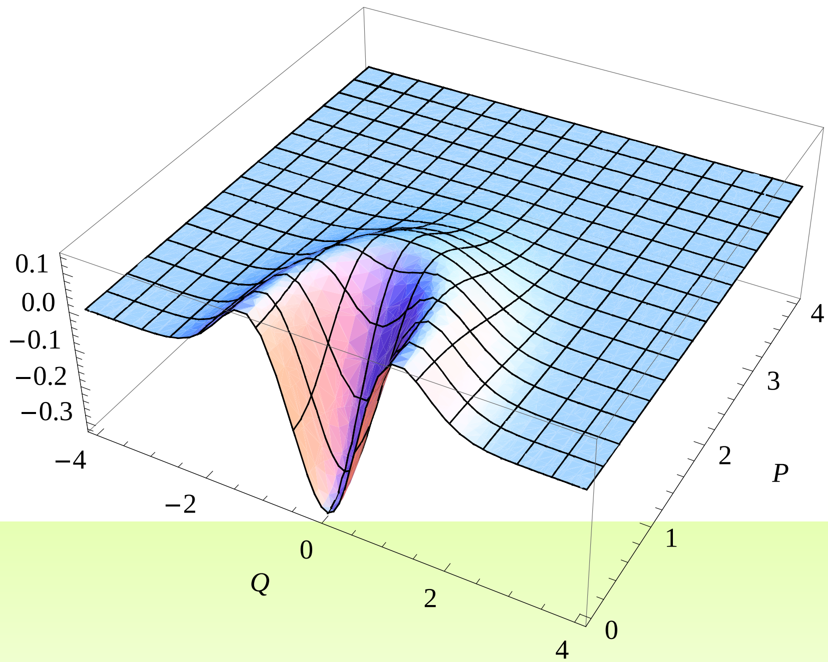


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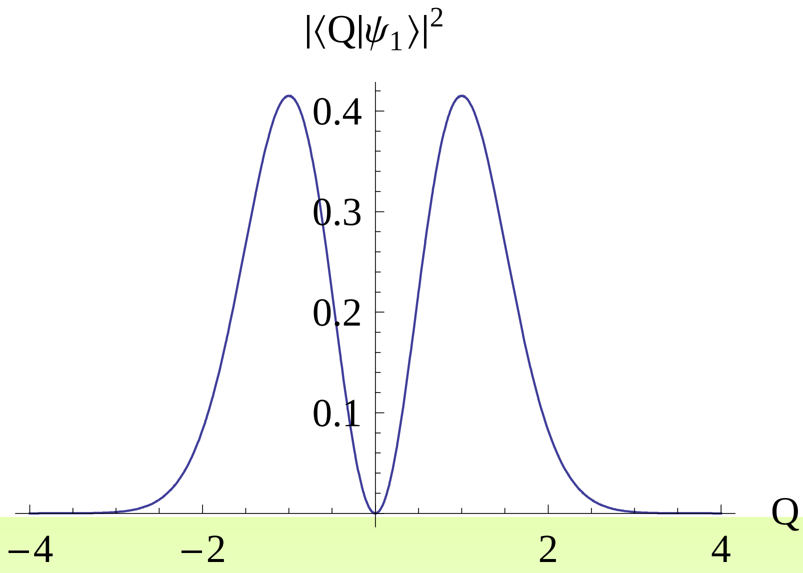
Figure 5

A plot of the marginal probability density $|\langle Q|\psi_1\rangle|^2$. This is obtained by integrating $W_1(Q,P)$ over P:

$$|\langle Q|\psi_1\rangle|^2 = \int_{-\infty}^{\infty} W_1(Q,P)\, dP = \frac{2e^{-Q^2}Q^2}{\sqrt{\pi}}$$

Figure 5



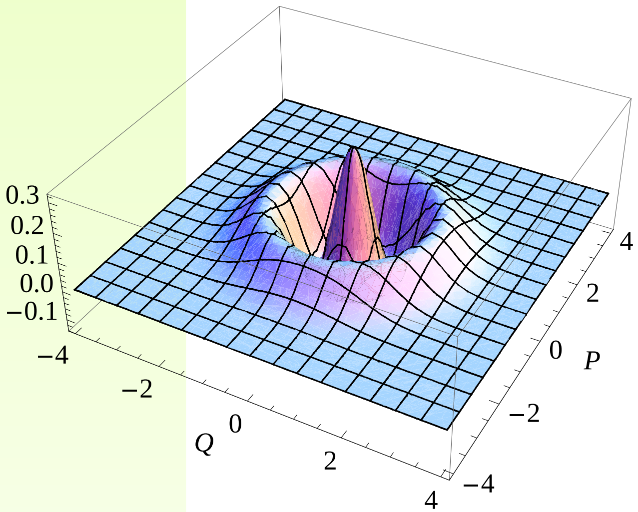
Wigner Representation - Examples

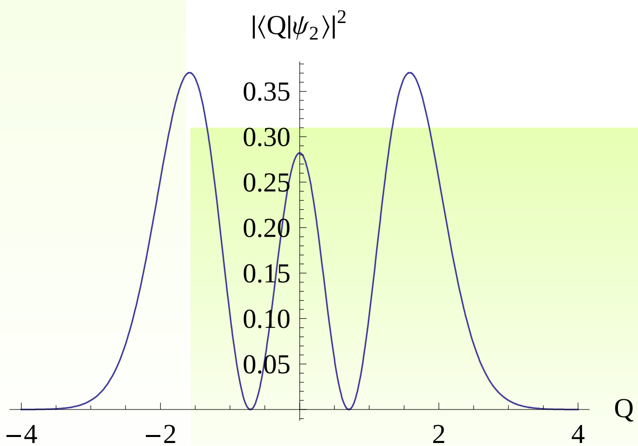
EXCITED FOCK STATES

Wigner Representation - Examples

EXCITED FOCK STATES

$n = 2$





Wigner Representation - Examples

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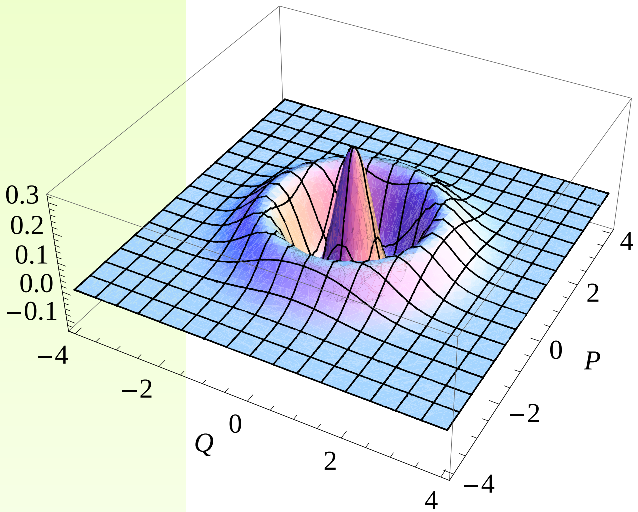
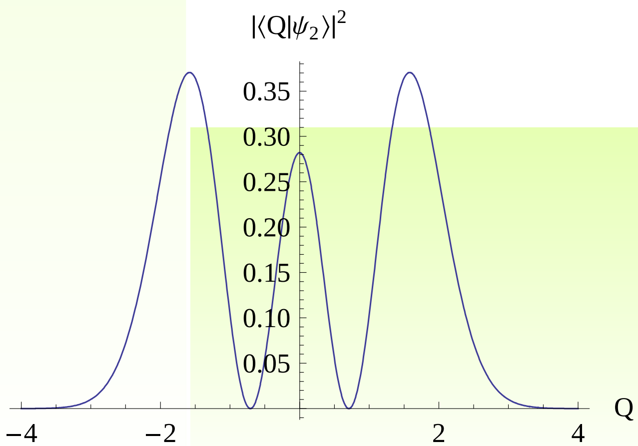
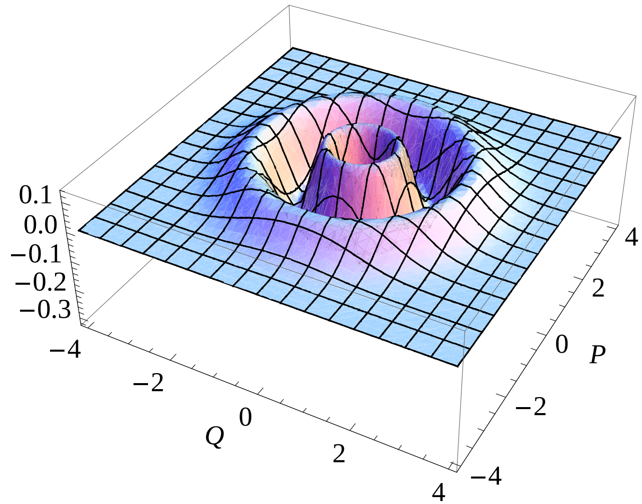
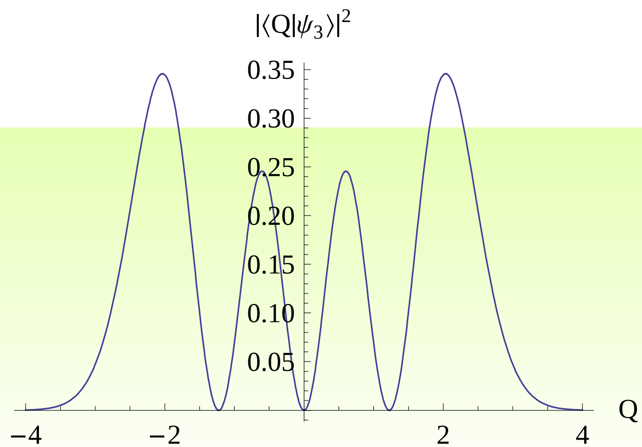




$n = 3$





Wigner Representation - Examples

COHERENT STATE - DEFINITION

Wigner Representation - Examples

COHERENT STATE - DEFINITION

- A coherent state is a linear superposition of number states in the form

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

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\end{aligned}$$

Wigner Representation - Examples

COHERENT STATE - WIGNER FUNCTION

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Wigner Representation - Examples

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  - i.e., it is a Gaussian centered at the phase space point (Q_0, P_0).

Wigner Representation - Examples

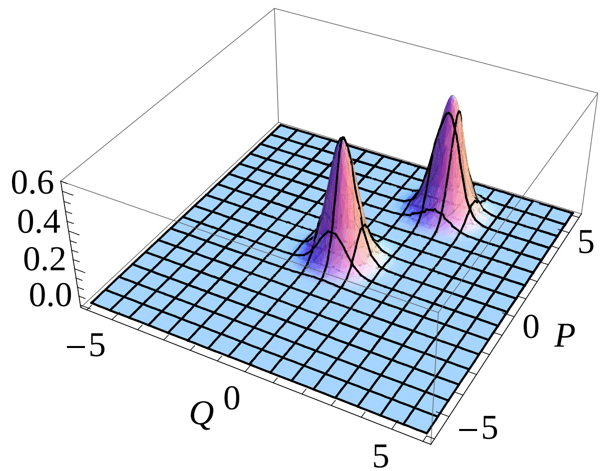
COHERENT STATE - WIGNER FUNCTION

A plot of $W_\alpha(Q, P)$ with

$$Q_0 = P_0 = 0,$$

and with

$$Q_0 = 2, \ P_0 = 4.$$



Wigner Representation - Examples

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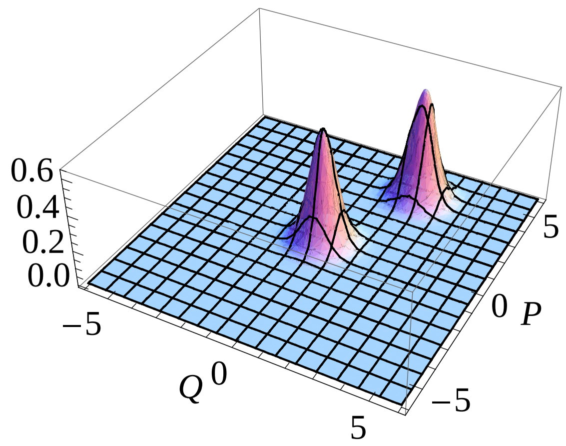
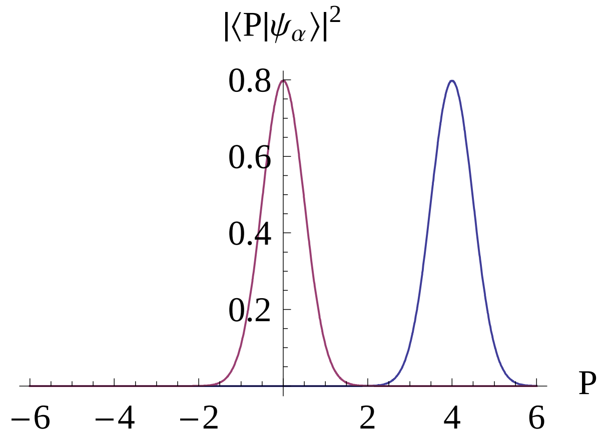
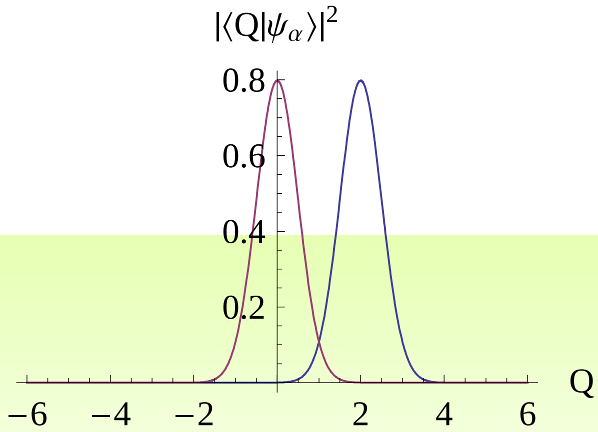
The right and bottom figures correspond to marginal probabilities of P and Q, respectively.







Wigner Representation - Examples

COHERENT STATE - TIME EVOLUTION

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Wigner Representation - Examples

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Wigner Representation - Examples

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|\psi(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}\,|\alpha\rangle \\
&= e^{-i\omega(\hat{N}+\frac{1}{2})t}e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle
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&= e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\,|n\rangle \\
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Wigner Representation - Examples

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&= e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-i\omega(n+\frac{1}{2})t}\,|n\rangle \\
&= e^{-\frac{i\omega t}{2}}e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\,|n\rangle \\
&= e^{-\frac{i\omega t}{2}}\,|\beta\rangle
\end{aligned}
$$

- i.e., the evolved state is still a coherent state, but with a different amplitude

$$\beta = \alpha e^{-i\omega t}, \;\; |\beta|^2 = |\alpha|^2$$

Wigner Representation - Examples

COHERENT STATE - TIME EVOLUTION

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Movie 1	Movie 2
$W_t(Q,P)$ in 3D	Contour of $W_t(Q,P)$ and marginal probabilities

Wigner Representation - Examples

SQUEEZED STATE

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- The annihilation and creation operators transform as

$$\hat{S}^\dagger \hat{D}^\dagger \hat{a} \hat{D} \hat{S} = \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta}\sinh(r) + \alpha_0$$

$$\hat{S}^\dagger \hat{D}^\dagger \hat{a}^\dagger \hat{D} \hat{S} = \hat{a}^\dagger\cosh(r) - \hat{a}e^{-i\theta}\sinh(r) + \alpha_0^*$$

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- The Wigner function may be obtained by making the replacement

$$\alpha \rightarrow \cosh(r)e^{-i\theta}(\alpha - \alpha_0) + \sinh(r)e^{i\theta}(\alpha^* - \alpha_0^*)$$

in the Wigner function of the vacuum state

$$W_0(\alpha, \alpha^*) = \frac{1}{\pi\hbar} e^{-|\alpha|^2}$$

Wigner Representation - Examples

SQUEEZED STATE

- The Wigner function of the squeezed coherent state becomes

$$W_{\xi,\alpha_0}(Q,P) = \frac{1}{\pi\hbar}\text{Exp}\Big(|\cosh(r)e^{-i\theta}(\alpha-\alpha_0)+\sinh(r)e^{i\theta}(\alpha^*-\alpha_0^*)|^2\Big)$$

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- where $\alpha = \frac{1}{\sqrt{2\hbar}}(Q+iP)$. The displacement parameter is $\alpha_0 = \frac{1}{\sqrt{2\hbar}}(Q_0+iP_0)$ and the squeezing parameter is $\xi = re^{i\theta}$

Wigner Representation - Examples

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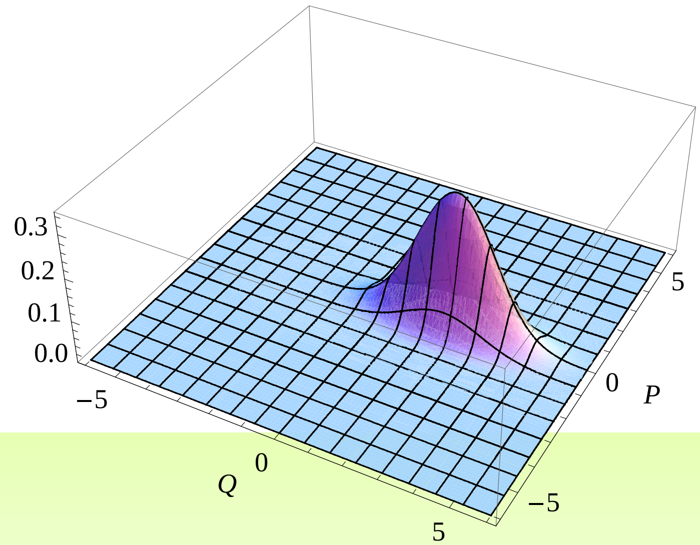
Figure 1
A Plot of $W(Q,P)$ as a function of Q and P for the squeezed coherent state, with parameters values

$$\alpha_0 = 2 \ \text{ and } \ \xi = \frac{1}{2}$$

Figure 1



Wigner Representation - Examples

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Figure 1

A Plot of $W(Q,P)$ as a function of Q and P for the squeezed coherent state, with parameters values

$$\alpha_0 = 2 \text{ and } \xi = \frac{1}{2}$$

Figure 2

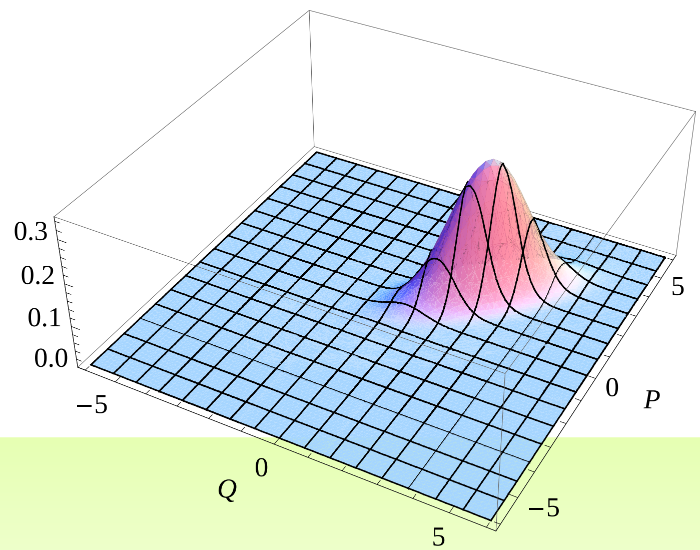

Figure 2

Same as Figure 1 but with complex values

$$\alpha_0 = 2+2i \text{ and } \xi = \frac{1}{2}e^{i\frac{\pi}{4}}$$

Wigner Representation - Examples

SQUEEZED STATE

- The Wigner function of the squeezed coherent state becomes

$$W_{\xi,\alpha_0}(Q,P) = \frac{1}{\pi\hbar}\mathrm{Exp}\Big(|\cosh(r)e^{-i\theta}(\alpha-\alpha_0)+\sinh(r)e^{i\theta}(\alpha^*-\alpha_0^*)|^2\Big)$$

Movie 1	Movie 2	Movie 3
Displacing the vacuum	Squeezing the displaced state	Rotating the squeezed displaced state

Wigner Representation - Examples

SQUEEZED STATE - TIME EVOLUTION

- The time dependent Wigner function of the squeezed coherent state is obtained by the replacements

Wigner Representation - Examples

SQUEEZED STATE - TIME EVOLUTION

- The time dependent Wigner function of the squeezed coherent state is obtained by the replacements

$$Q \to Q\cos(\tau) - P\sin(\tau)$$
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Wigner Representation - Examples

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Wigner Representation - Examples

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Movie 4	Movie 5
$W(Q,P,\tau)$ for $0 \leq \tau \leq 2\pi$	Marginal distribution of Q

Wigner Representation - Examples

WIGNER FUNCTION OF LOCALIZED STATES

Wigner Representation - Examples

WIGNER FUNCTION OF LOCALIZED STATES

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- where $\alpha = re^{i\theta}$, $\beta = \rho e^{i\phi}$ and N is a normalization constant ($N \longrightarrow 1$ when α and β are far apart) .

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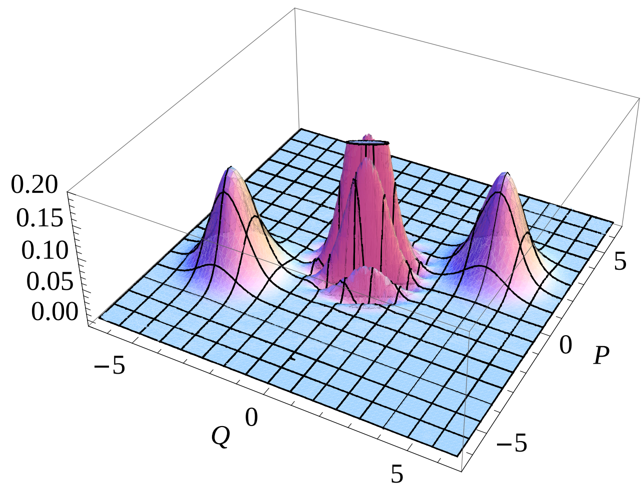

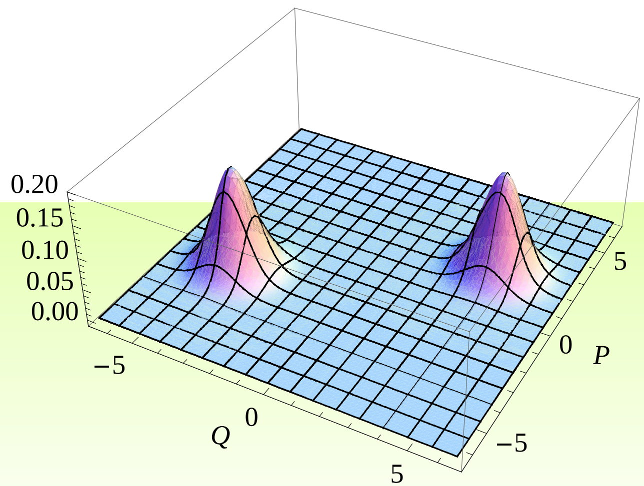

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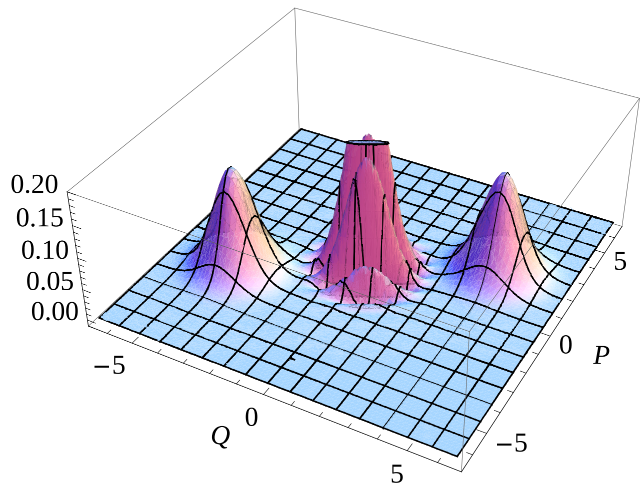

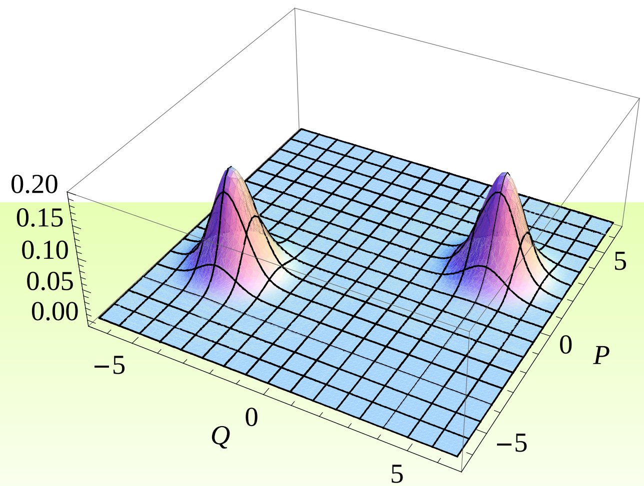

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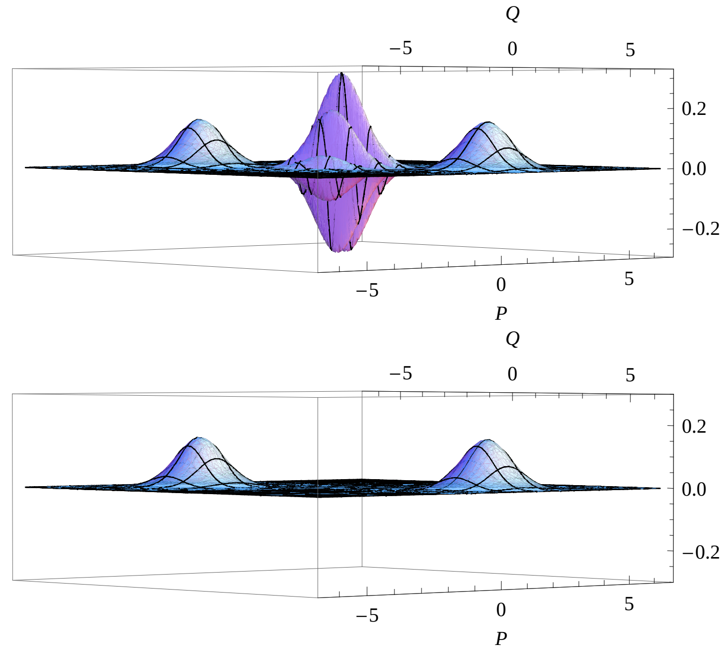

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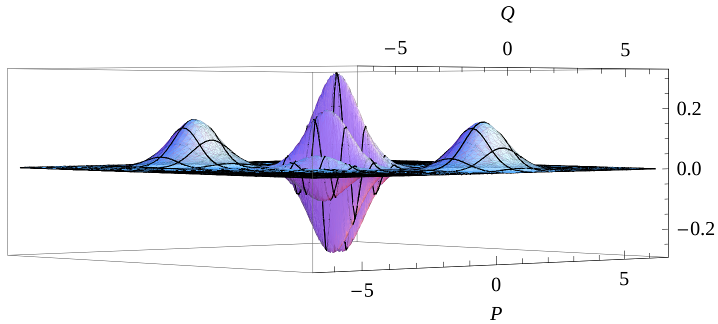

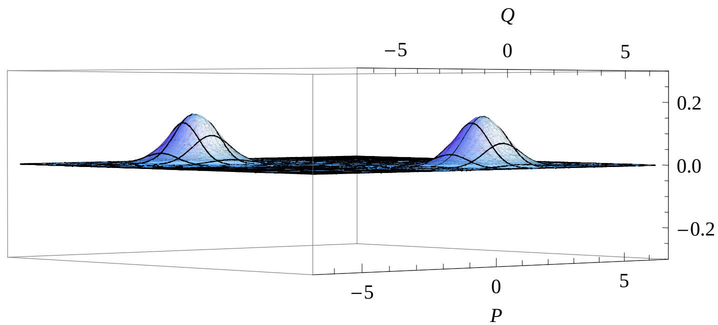

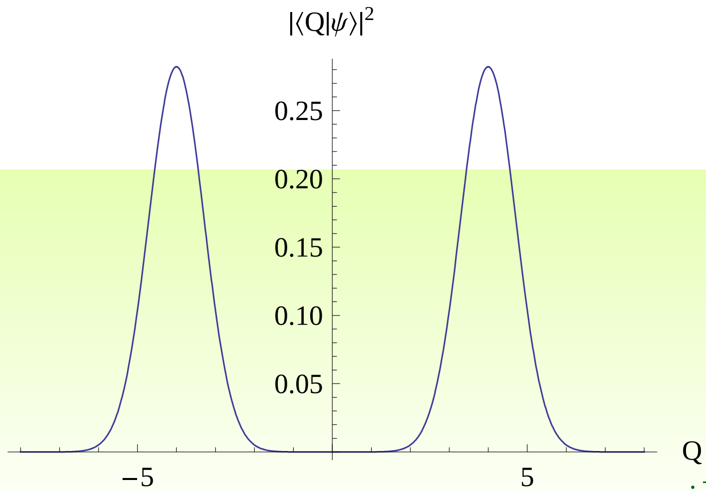

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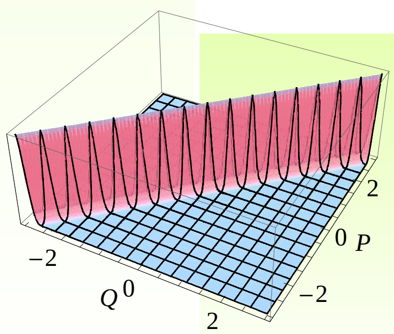

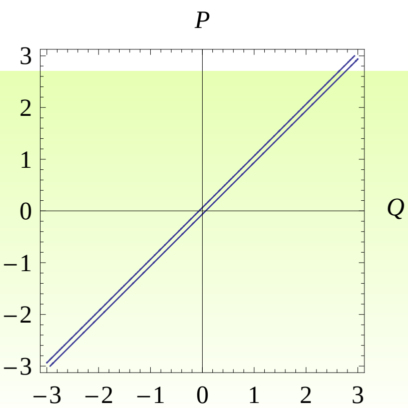

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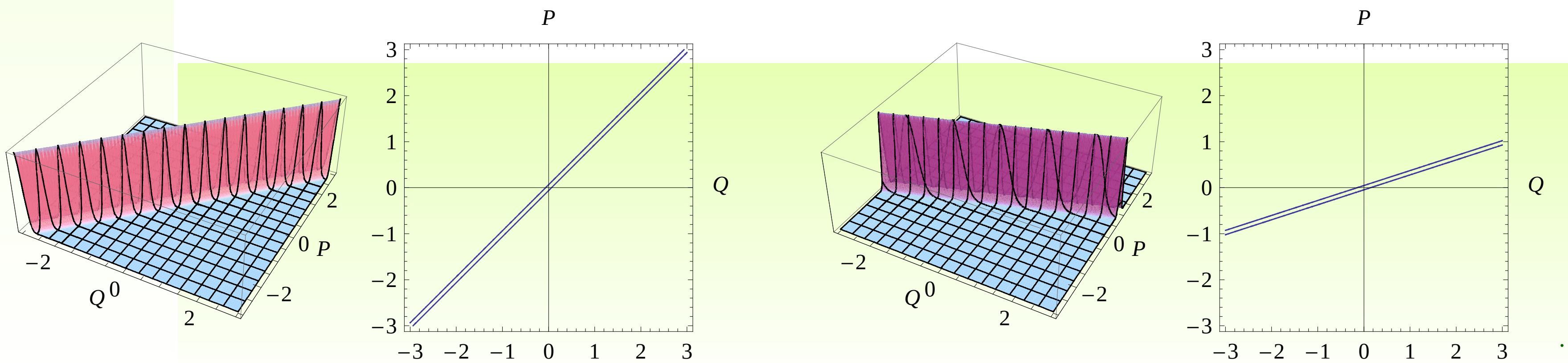

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- The Wigner function of the state $\hat{\rho}$ is the inverse transform of the last equation

$$W_{\hat{\rho}}(Q,P) = \frac{1}{4\pi^2\hbar} \int_{-\infty}^{\infty} dt|t| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{-\infty}^{\infty} dX_\theta \, e^{it(X_\theta - (\cos\theta Q + \sin\theta P))} |\langle X_\theta | \psi \rangle|^2$$

Generation of squeezed states

PARAMETRIC DOWN CONVERSION

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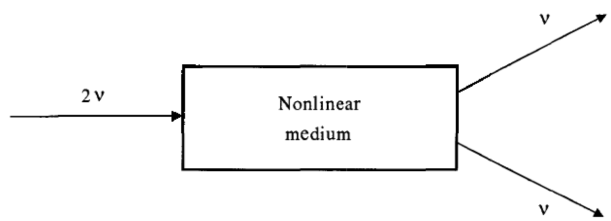

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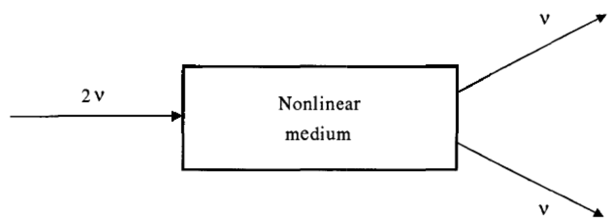

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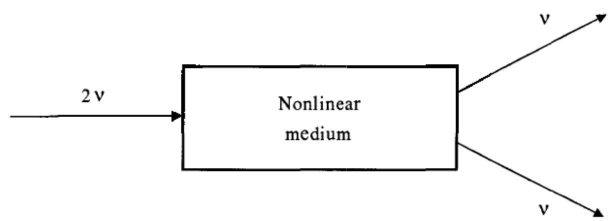

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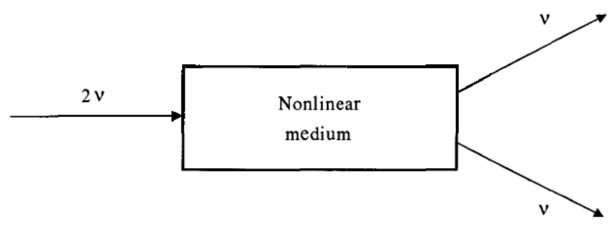

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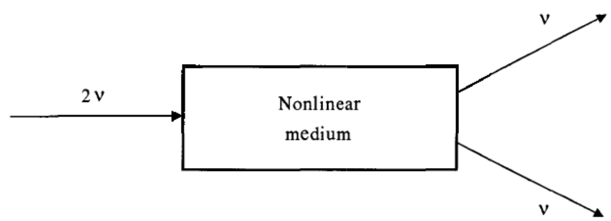

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Homodyne Detection

BALANCED HOMODYNE DETECTION

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BALANCED HOMODYNE DETECTION

- Two field are incident on a $50 : 50$ beam splitter

Homodyne Detection

BALANCED HOMODYNE DETECTION

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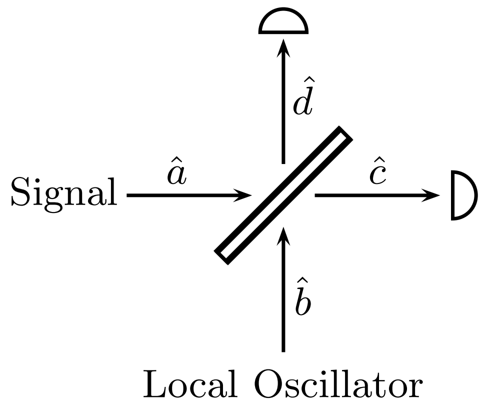
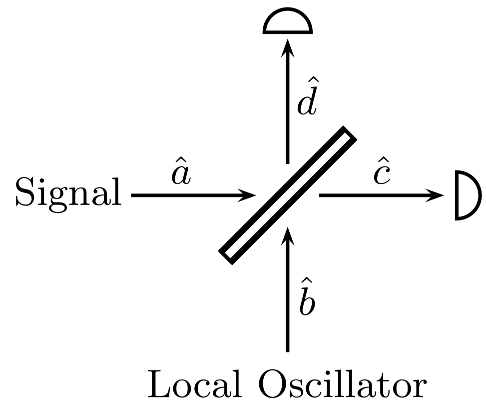


Homodyne Detection

BALANCED HOMODYNE DETECTION

- Two field are incident on a $50:50$ beam splitter
  - The local oscillator is in a large amplitude coherent state



Homodyne Detection

BALANCED HOMODYNE DETECTION

- Two field are incident on a $50:50$ beam splitter
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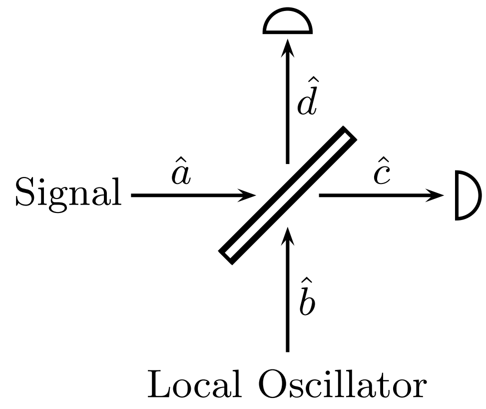
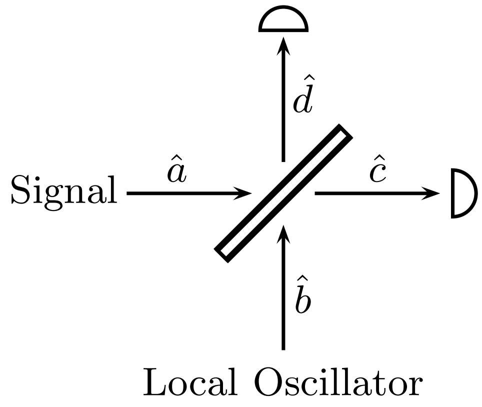


Homodyne Detection

BALANCED HOMODYNE DETECTION



- Two field are incident on a $50:50$ beam splitter
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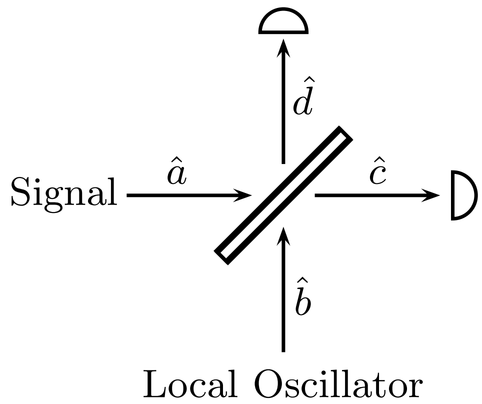
  - The input field is described by the operator $\hat{a}$
- The beam splitter:

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + i\hat{b})$$

$$\hat{d} = \frac{1}{\sqrt{2}}(i\hat{a} + \hat{b})$$

Homodyne Detection

BALANCED HOMODYNE DETECTION



- Two field are incident on a $50:50$ beam splitter
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- The beam splitter:

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  - The signals in the detectors are given by

$$\langle \hat{c}^\dagger \hat{c} \rangle = \frac{1}{2}(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + i\langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle)$$

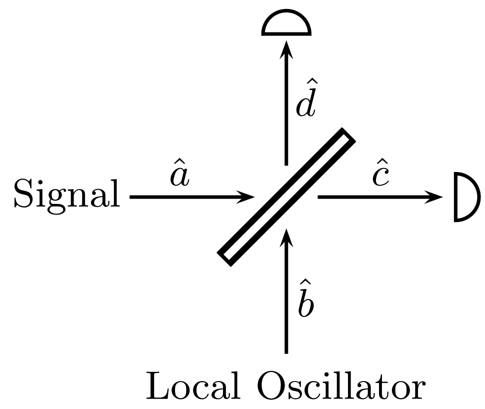
$$\langle \hat{d}^\dagger \hat{d} \rangle = \frac{1}{2}(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle - i\langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle)$$

Homodyne Detection

BALANCED HOMODYNE DETECTION

- In order to obtain the quadrature operator, we subtract the readings of the two detectors

$$\langle \hat{n}_{cd} \rangle = \langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{d}^\dagger \hat{d} \rangle = i \langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle$$



Homodyne Detection

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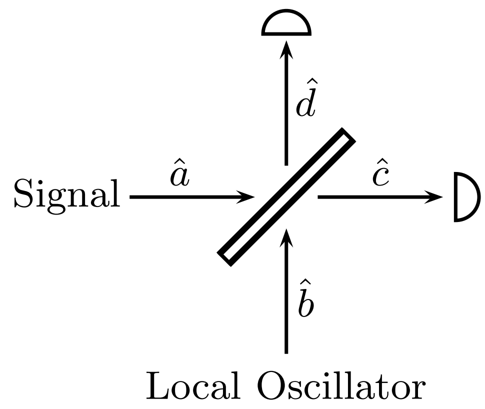
  - For a local oscillator exited to a large amplitude coherent state, $\hat{b} \longrightarrow |\beta| e^{i\theta}$

$$\langle \hat{n}_{cd} \rangle = i |\beta| \langle \hat{a}^\dagger e^{i\theta} - \hat{a} e^{-i\theta} \rangle$$



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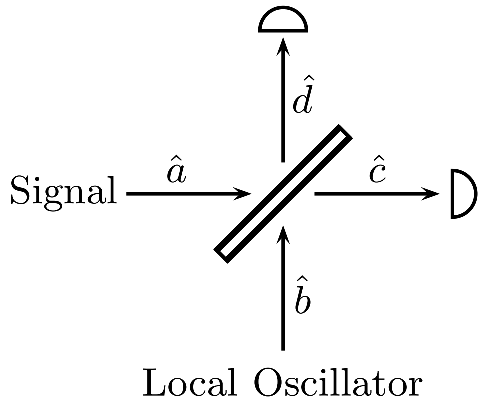

- For a local oscillator exited to a large amplitude coherent state, $\hat{b} \to |\beta| e^{i\theta}$

$$\langle \hat{n}_{cd} \rangle = i|\beta| \langle \hat{a}^\dagger e^{i\theta} - \hat{a} e^{-i\theta} \rangle = |\beta| \langle \hat{a}^\dagger e^{i(\theta + \frac{\pi}{2})} + \hat{a} e^{-i(\theta + \frac{\pi}{2})} \rangle$$

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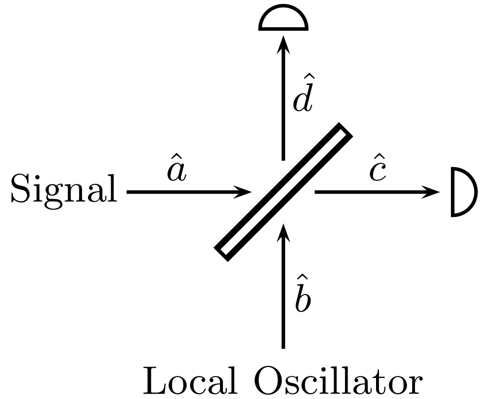

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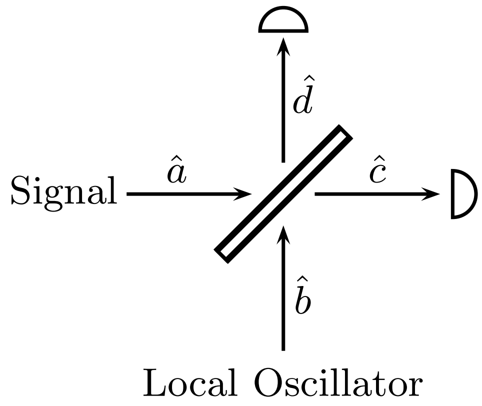
Homodyne Detection

BALANCED HOMODYNE DETECTION



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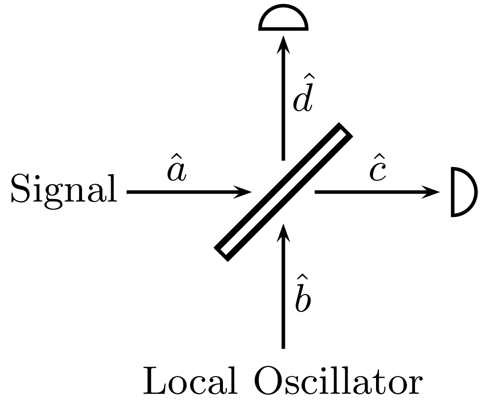
  - where $\hat{X}_\theta = \hat{a}^\dagger e^{i\theta} + \hat{a} e^{-i\theta}$

Homodyne Detection

BALANCED HOMODYNE DETECTION



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$$\langle \hat{n}_{cd} \rangle = \langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{d}^\dagger \hat{d} \rangle = i\langle \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \rangle$$

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$$\langle \hat{n}_{cd} \rangle = i|\beta|\langle \hat{a}^\dagger e^{i\theta} - \hat{a} e^{-i\theta} \rangle = |\beta|\langle \hat{a}^\dagger e^{i(\theta+\frac{\pi}{2})} + \hat{a} e^{-i(\theta+\frac{\pi}{2})} \rangle = |\beta|\langle \hat{X}_{\theta+\frac{\pi}{2}} \rangle$$

  - where $\hat{X}_\theta = \hat{a}^\dagger e^{i\theta} + \hat{a} e^{-i\theta}$

Thus, the subtraction of the detectors readings leads to the expectation value of the quadrature operator

Homodyne Detection - Experiments

HOMODYNE TOMOGRAPHY - Smithey, et al.

Homodyne Detection - Experiments

HOMODYNE TOMOGRAPHY - Smithey, et al.

Figure 1[1] The experimental setup is depicted in figure 1. A squeezed state is produced and it's Wigner function is reconstructed by homodyne tomography

[1] D. Smithey, M. Beck, M. Raymer and, A. Faridani *Phys. Rev. Lett.* ***70****, 1244 (1993).*

Homodyne Detection - Experiments

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Figure 1

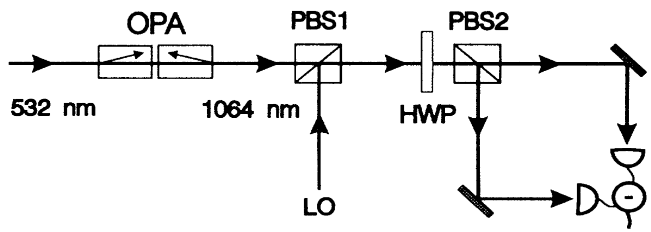

FIG. 2. Apparatus for balanced homodyne measurement of quadrature amplitude. The crystals are oriented at 45° with respect to the polarizer (PBS 1) axes in order to produce the squeezed field. Prisms (not shown) in front of each detector remove the 532 nm pump beam from the 1064 nm signal beam.

[1] D. Smithey, M. Beck, M. Raymer and, A. Faridani *Phys. Rev. Lett.* ***70****, 1244 (1993).*

Homodyne Detection - Experiments

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Figure 2
The probability distribution $P(X_\phi)$ as a function of the rotated quadrature X_ϕ for different phases ϕ.

Figure 2

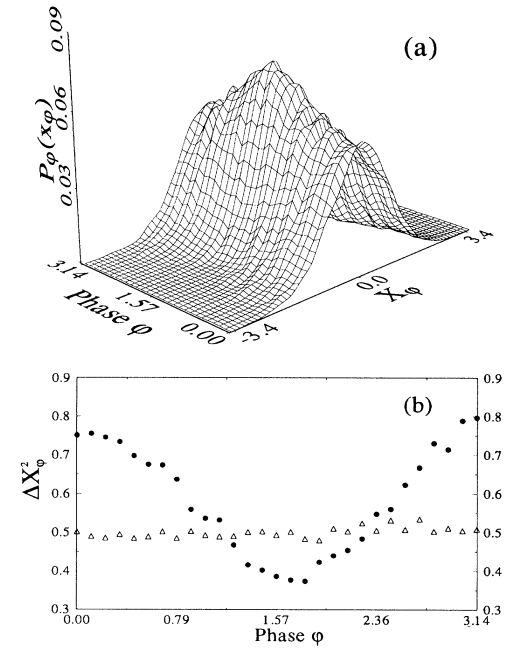

FIG. 3. (a) Measured quadrature-amplitude distributions at various values of local oscillator phase. Note that since these distributions are normalized, a decreasing width of a particular distribution is accompanied by an increase in its peak height. (b) Variances of quadrature amplitude vs LO phase: circles, squeezed state; triangles, vacuum state.

Homodyne Detection - Experiments

HOMODYNE TOMOGRAPHY - Smithey, et al.

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Figure 3
A plot of their results showing 3D plots and contours of the Wigner functions. The ellipse on the left side corresponds to a squeezed state and the circle on the right to the vacuum state.

Figure 3

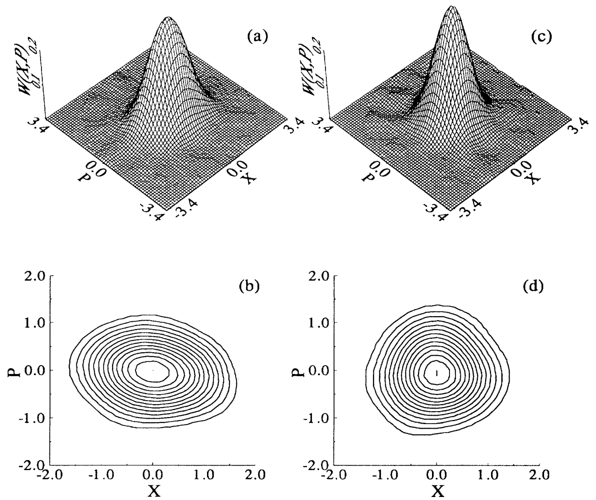

FIG. 1. Measured Wigner distributions for (a),(b) a squeezed state and (c),(d) a vacuum state, viewed in 3D and as contour plots, with equal numbers of constant-height contours. Squeezing of the noise distribution is clearly seen in (b).

Homodyne Detection - Experiments

HOMODYNE TOMOGRAPHY - Zavatta, et al.

Homodyne Detection - Experiments

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Homodyne Detection - Experiments

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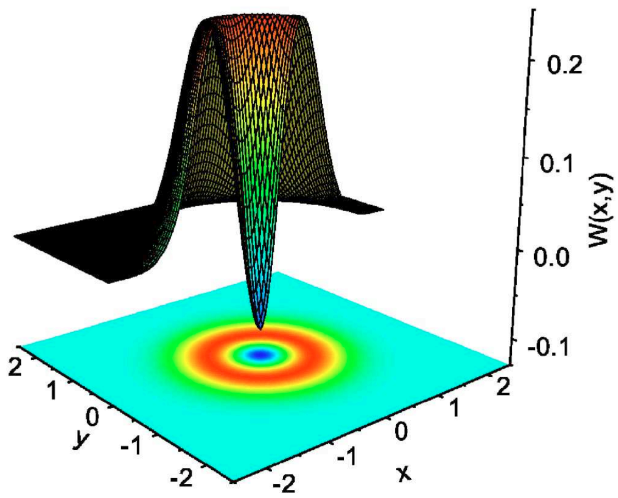


FIG. 4. (Color online) Wigner function of the single-photon Fock state as obtained from the reconstructed density-matrix elements. The negativity of the distribution, a clear proof of the nonclassical character of the state, is evident around the origin of the shot-noise normalized quadrature axes.

Homodyne Detection - Experiments

HOMODYNE TOMOGRAPHY - Zavatta, et al.

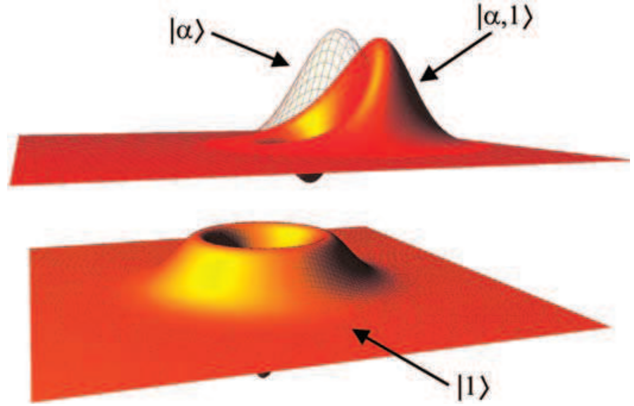
Figure 1 Zavatta, et al. have obtained the Wigner function of the single-photon Fock state using homodyne tomography.

Figure 2[3] In this paper, Zavatta, et al. report the creation and detection of a single-photon-added coherent state. Theoretical plots of the Wigner functions of the one-photon Fock state and of the single-photon-added coherent state.

[3] A. Zavatta, S. Viciani, M. Bellini *Science **306**, 660 (2004).*

Figure 2

Fig. 1. Theoretical Wigner functions for some of the quantum states of light discussed in the text. Upper surface, SPACS $|\alpha,1\rangle$; wire-frame surface, original unexcited coherent state $|\alpha\rangle$; lower surface, single-photon Fock state $|1\rangle$. The horizontal plane coordinates represent two orthogonal quadratures of the field. The single-photon Wigner function is centered at the origin of the phase space. A value of $|\alpha|^2 = 1$ is used.



Homodyne Detection - Experiments

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Figure 3[3] The experimental reconstructed Wigner functions

Figure 3

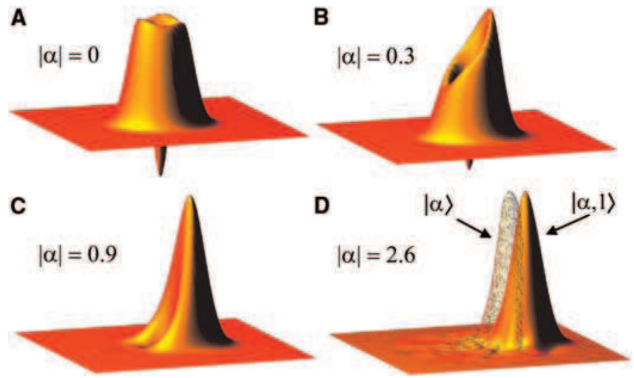

Fig. 3. Experimental Wigner functions for the SPACS. (**A**) Reconstructed Wigner function for the single-photon Fock state obtained without injection. (**B** to **D**) Same, but with an input coherent field of increasing amplitude. In (D), the reconstructed Wigner function for both the SPACS and the unexcited seed coherent state (wire-frame surface) are shown.

[3] A. Zavatta, S. Viciani, M. Bellini *Science* ***306****, 660 (2004).*

Squeezing - Experiments

PHOTON NUMBER SQUEEZING - Schmitt, et al.

Squeezing - Experiments

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Figure 1[4]

The experimental setup consisting of an asymmetric Sagnac interferometer. The nonlinearity of the fiber induces an intensity-dependent phase. The beam-splitter BS1 has splitting ratio of 90 : 10, in order to give different phase to each loop direction.

Figure 1

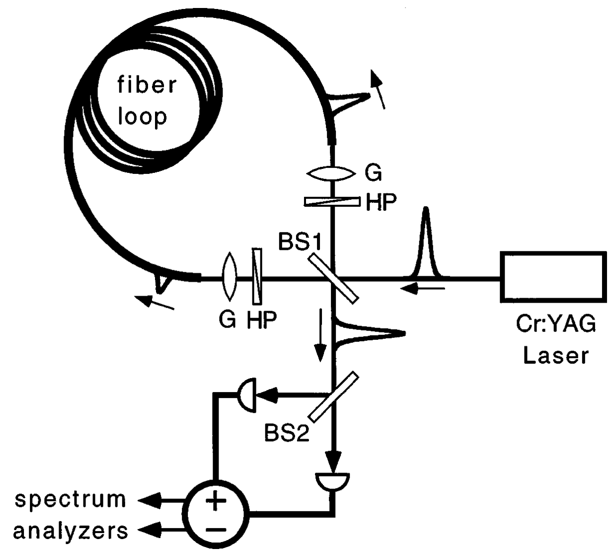

FIG. 1. Experimental setup for directly detectable squeezing from an asymmetric fiber-optic Sagnac interferometer. (BS) beam splitter, (HP) half-wave plate, (G) grin lens.

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[4]S. Schmitt, J. Ficker, M. Wolff, F. Koenig, A. Sizmann and G. Leuchs *Phys. Rev. Lett.* ***81**, 2446 (1998).*

Squeezing - Experiments

PHOTON NUMBER SQUEEZING - Schmitt, et al.

Figure 1[4]

The experimental setup consisting of an asymmetric Sagnac interferometer. The nonlinearity of the fiber induces an intensity-dependent phase. The beam-splitter BS1 has splitting ratio of $90 : 10$, in order to give different phase to each loop direction.

Figure 2

The measured intensity as a function of input intensity. There are oscillations in the transmitted intensity due to the interference. The correlation between the steps in the upper figure and the noise level in the lower figure is evident.

Figure 2

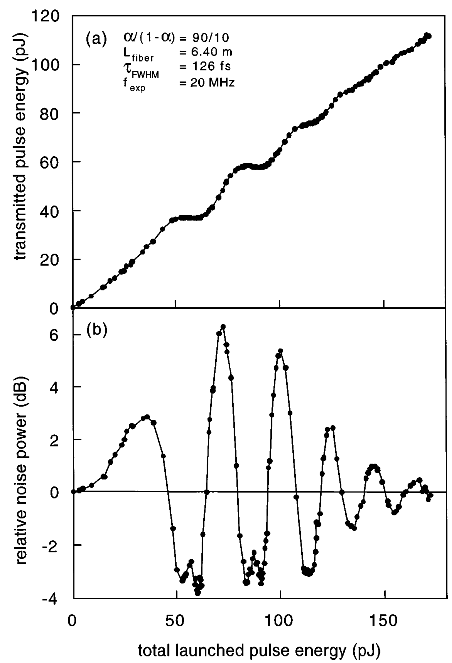

FIG. 2. Nonlinear energy-transfer function and squeezing from a 90:10 asymmetric Sagnac loop, plotted versus the launched pulse energy. (a) The transmitted output pulse energy shows an optical limiting effect at input energies of 53 pJ and 83 pJ. (b) Photocurrent noise power relative to shot noise (0 dB). The quantum fluctuations are reduced below shot noise at input energies where optical limiting occurs.