Quantum Optics

Wigner function, Quantum Tomography, Homodyne Detection

Wigner Function

Definiton:

$$W(x,p) = \int \frac{d\xi}{2\pi\hbar} e^{-ip\xi/\hbar} \langle x + \tfrac{1}{2}\xi | \hat{\rho} | x - \tfrac{1}{2}\xi \rangle$$

Using Coherent-Basis:

$$W(\alpha) = \int \frac{d^2\lambda}{\pi^2} e^{\lambda^* \alpha - \lambda \alpha^*} \mathrm{Tr}\Big(\mathrm{e}^{\lambda \mathrm{a}^\dagger - \lambda^* \mathrm{a}} \hat{\rho} \Big)$$

Marginals:

$$\langle x | \hat{\rho} | x \rangle = \int dp W(x,p)$$

$$\langle p | \hat{\rho} | p \rangle = \int dx W(x,p)$$

Joined Trace:

$$\mathrm{Tr}\Big(\hat{\rho}_1 \hat{\rho}_2 \Big) = 2\pi\hbar \int \mathrm{dxdp W_1(x,p) W_2(x,p)}$$

Quantum Mechanics in Phase Space

Wigner-Weyl Symbol

$$A(x,p) = \int \frac{d\xi}{2\pi\hbar} e^{-ip\xi/\hbar} \langle x + \tfrac{1}{2}\xi | \hat{A} | x - \tfrac{1}{2}\xi \rangle$$

Expectation Values:

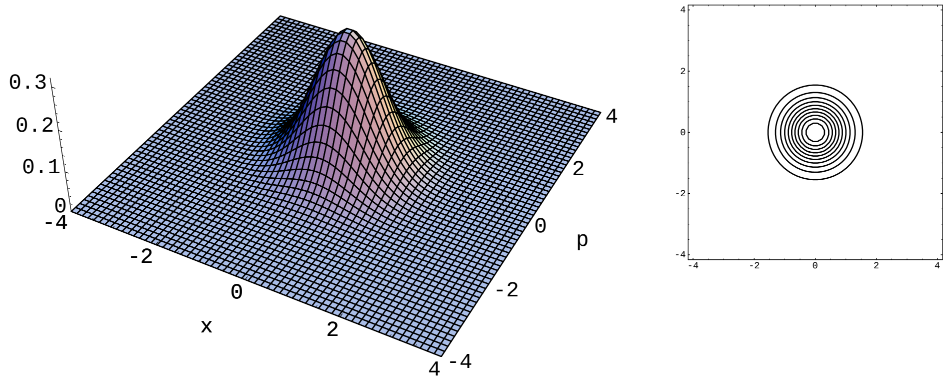
$$\mathrm{Tr}\Big(\hat{\rho} \hat{\mathrm{A}} \Big) = 2\pi\hbar \int \mathrm{dxdp W(x,p) A(x,p)}$$

Dynamics:

$$\Big[\frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - \frac{\partial U(x)}{\partial x} \frac{\partial}{\partial p} \Big] W(x,p) = \sum_{n=1}^{\infty} \frac{(-1)^n (\hbar/2)^{2n}}{(2n+1)!} \frac{\partial^{2n+1} U(x)}{\partial x^{2n+1}} \frac{\partial^{2n+1} W(x,p)}{\partial p^{2n+1}}$$

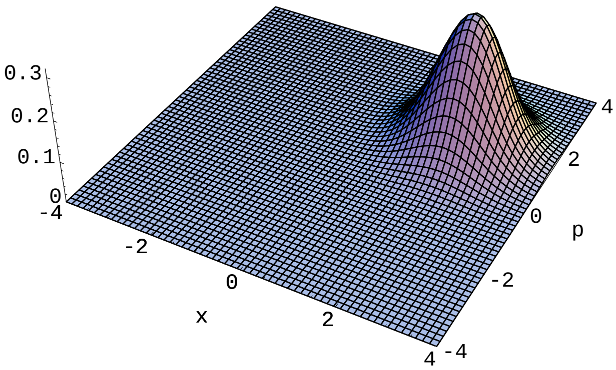
Vaccum State: $\hat{\rho} = |0\rangle\langle 0|$

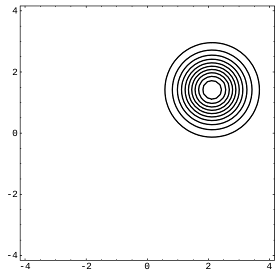
$$W(x,p) = \frac{1}{\pi\hbar} \exp\Big\{ - \Big[(\kappa x)^2 + (p/\hbar\kappa)^2 \Big] \Big\}$$



Coherent State: $\hat{\rho} = |\alpha\rangle\langle\alpha|$

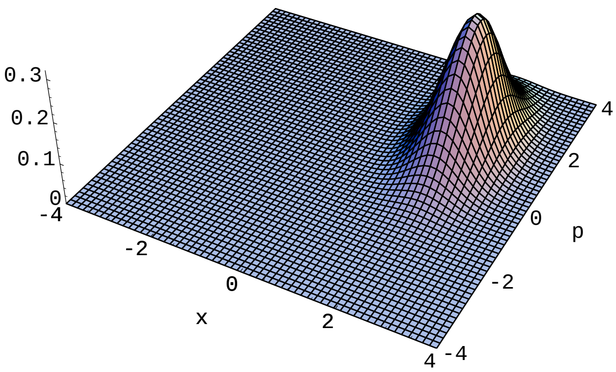
$$W(x,p) = \frac{1}{\pi\hbar}\exp\Big\{ - \Big[(\kappa x - \sqrt{(}2)\mathrm{Re}\alpha)^2 + (p/\hbar\kappa - \sqrt{(}2)\mathrm{Im}\alpha)^2\Big]\Big\}$$

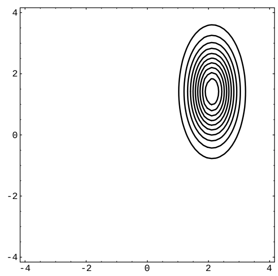




Squeezed State: $\hat{\rho} = |\alpha, s\rangle\langle\alpha, s|$

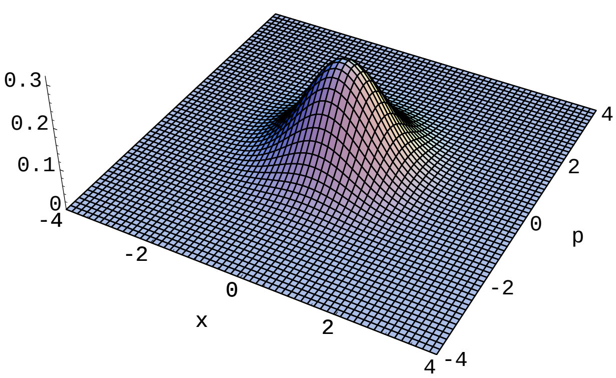
$$W(x,p) = \frac{1}{\pi\hbar}\exp\Big\{ - \Big[s(\kappa x - \sqrt{(}2)\mathrm{Re}\alpha)^2 + s^{-1}(p/\hbar\kappa - \sqrt{(}2)\mathrm{Im}\alpha)^2\Big]\Big\}$$

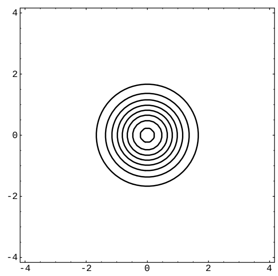




Thermal State: $\hat{\rho} = (1 - e^{-\hbar\omega/k_BT})\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT}|n\rangle\langle n|$

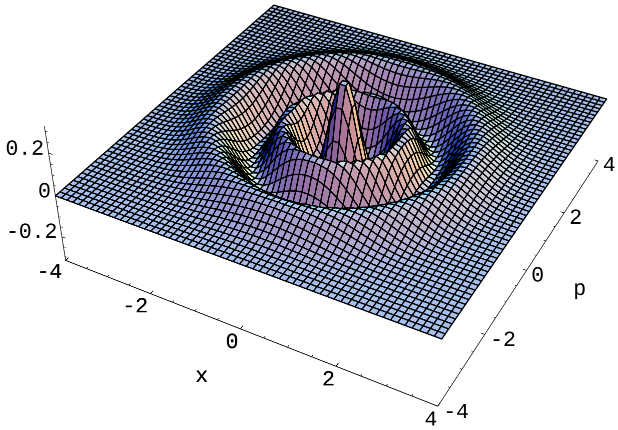
$$W(x,p) = \frac{\tanh(\hbar\omega/2k_BT)}{\pi\hbar}\exp\Big\{ - \tanh(\hbar\omega/2k_BT)\omega\Big[(\kappa x)^2 + (p/\hbar\kappa)^2\Big]\Big\}$$

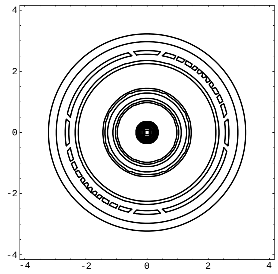




Fock State: $\hat{\rho} = |n\rangle\langle n|$

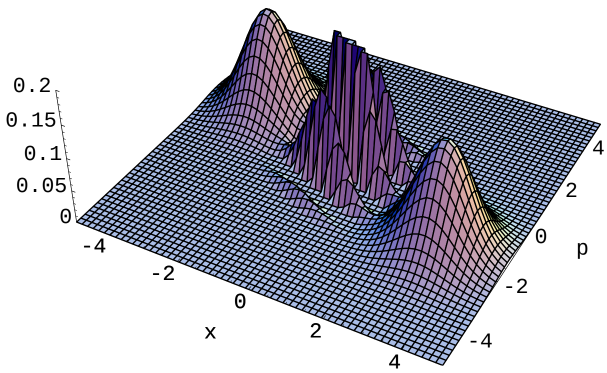
$$W(x,p) = \frac{(-1)^n}{\pi\hbar} \exp\Big\{ -\Big[(\kappa x)^2 + (p/\hbar\kappa)^2\Big]\Big\} L_n\Big\{2\Big[(\kappa x)^2 + (p/\hbar\kappa)^2\Big]\Big\}$$

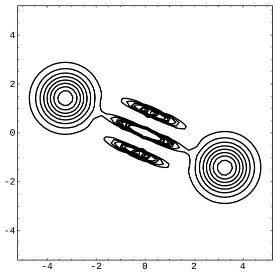




Schrödinger Cat State: $\hat{\rho} = N^2[|\alpha\rangle + |\beta\rangle][\langle\alpha| + \langle\beta|]$ $\qquad N^2 = 2\Big[1 + e^{-(|\alpha|^2+|\beta|^2)/2}\cosh\alpha^*\beta\Big]$

$$\begin{aligned} W(x,p) =& \frac{N^2}{\pi\hbar} \exp\Big\{ -\Big[(\kappa x - \sqrt(2)\mathrm{Re}\alpha)^2 + (p/\hbar\kappa - \sqrt(2)\mathrm{Im}\alpha)^2\Big]\Big\} + \frac{N^2}{\pi\hbar} \exp\Big\{ -\Big[(\kappa x - \sqrt(2)\mathrm{Re}\beta)^2 + (p/\hbar\kappa - \sqrt(2)\mathrm{Im}\beta)^2\Big]\Big\} \\ &+ \frac{2N^2}{\pi\hbar} \exp\Big\{ -\Big[(\kappa x - \mathrm{Re}(\alpha+\beta)/\sqrt(2))^2 + (p/\hbar\kappa - \mathrm{Im}(\alpha+\beta)/\sqrt(2))^2\Big]\Big\} \\ &\times \cos\Big[\sqrt{2}\mathrm{Im}(\alpha-\beta)\kappa x - \sqrt{2}\mathrm{Re}(\alpha-\beta)p/\hbar\kappa + \mathrm{Im}\alpha^*\beta\Big] \end{aligned}$$





Localised States:

$$\hat{\rho} = |x_0\rangle\langle x_0| \quad W(x,p) = \frac{1}{2\pi\hbar}\delta(x - x_0)$$

$$\hat{\rho} = |p_0\rangle\langle p_0| \quad W(x,p) = \frac{1}{2\pi\hbar}\delta(p - p_0)$$

$$\hat{\rho} = |X_\theta\rangle\langle X_\theta| \quad W(x,p) = \frac{1}{2\pi\hbar}\delta\big[X_\theta - (\cos\theta\kappa x + \sin\theta(\hbar\kappa)^{-1}p)\big]$$

Quantum Tomography

The Quadrature operator:

$$\hat{X}_\theta = \cos\theta\kappa\hat{x} + \sin\theta(\hbar\kappa)^{-1}\hat{p} = \frac{1}{\sqrt{2}}\Big[e^{-i\theta}a + e^{i\theta}a^\dagger\Big]$$

With conjugate momentum $\hat{P}_\theta = \hat{X}_{\theta+\pi/2}$. Here $\kappa = (m\omega/\hbar)^{1/2}$. The overlap with the x coordinate:

$$\langle x|X_\theta\rangle = \Big(\frac{\kappa}{2\pi\sin\theta}\Big)^{1/2}\exp\Big[-i\frac{(X_\theta - \cos\theta\kappa x)^2}{2\sin\theta\cos\theta}\Big]$$

The marginal in the direction n_θ is a Radon Transform

$$R(X_\theta,\theta) = \langle X_\theta|\hat{\rho}|X_\theta\rangle = \int dxdp\delta\big[X_\theta - (\cos\theta\kappa x + \sin\theta(\hbar\kappa)^{-1}p)\big]W(x,p)$$

And can be inverted by

$$W(x,p) = \frac{1}{4\pi^2\hbar}\int_{-\infty}^{\infty}dt|t|\int_{-\pi/2}^{\pi/2}d\theta\int_{-\infty}^{\infty}dX_\theta e^{it(X_\theta-\cos\theta\kappa x-\sin\theta(\hbar\kappa)^{-1}p)}R(X_\theta,\theta)$$

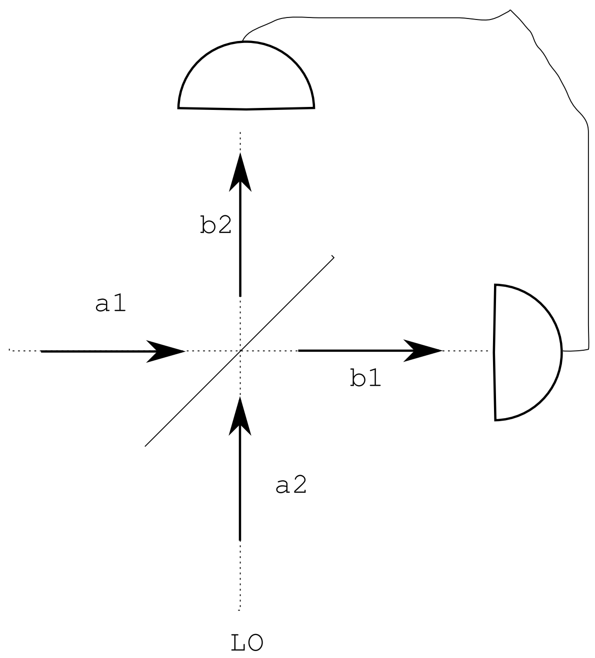
By knowing the wigner function one can get the density matrix by inverse Fourier transform

$$\hat{\rho} = \int dx\int dx'\int dp e^{ip(x-x')/\hbar}W\Big(\frac{x+x'}{2},p\Big)|x\rangle\langle x'|$$

Homodyne Detection

50%/50% Beam Splitter:

$$\begin{pmatrix}b_1\\b_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

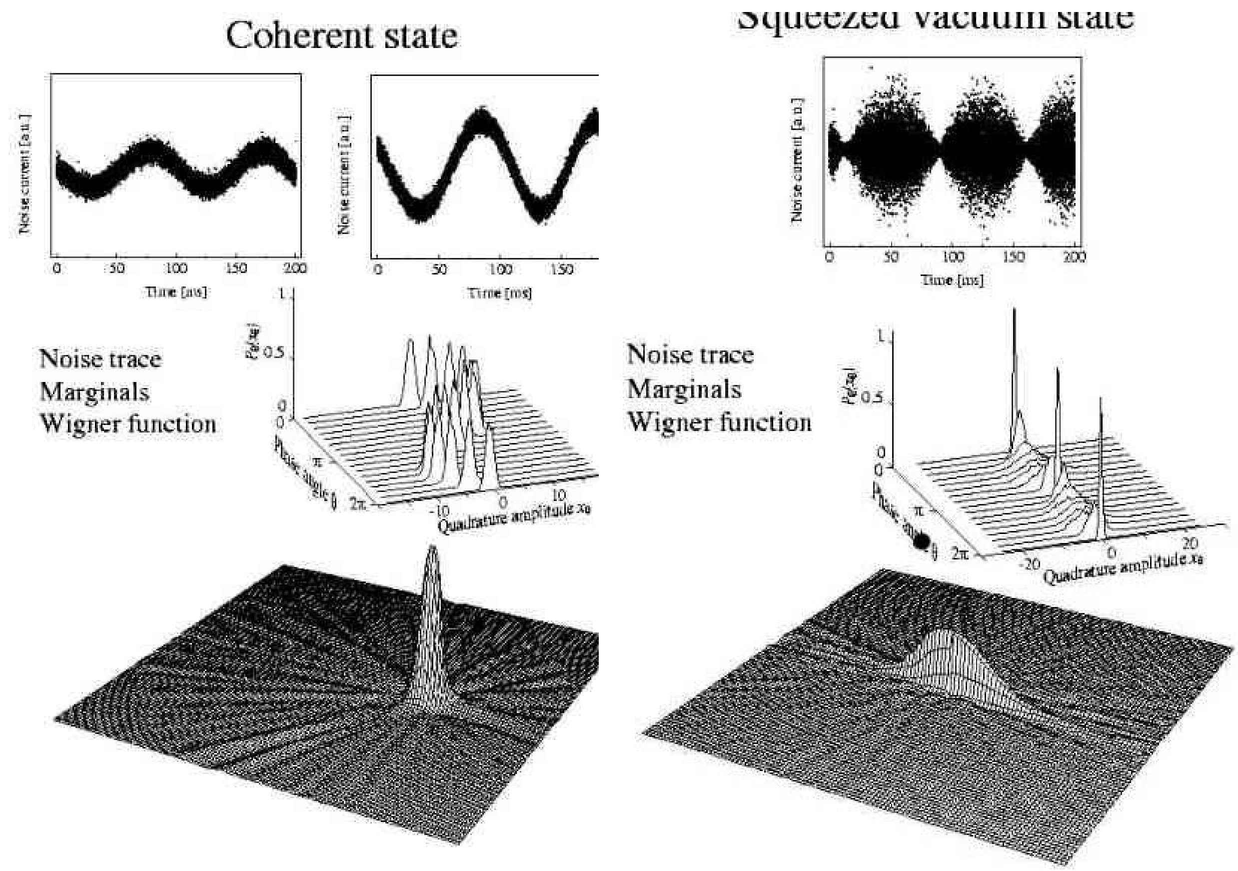


The difference of photon counting

$$N_1 - N_2 = b_1^\dagger b_1 - b_2^\dagger b_2 = a_2^\dagger a_1 + a_1^\dagger a_2$$

which for an classical input $a_2 \to |\alpha|e^{i\theta}$ is

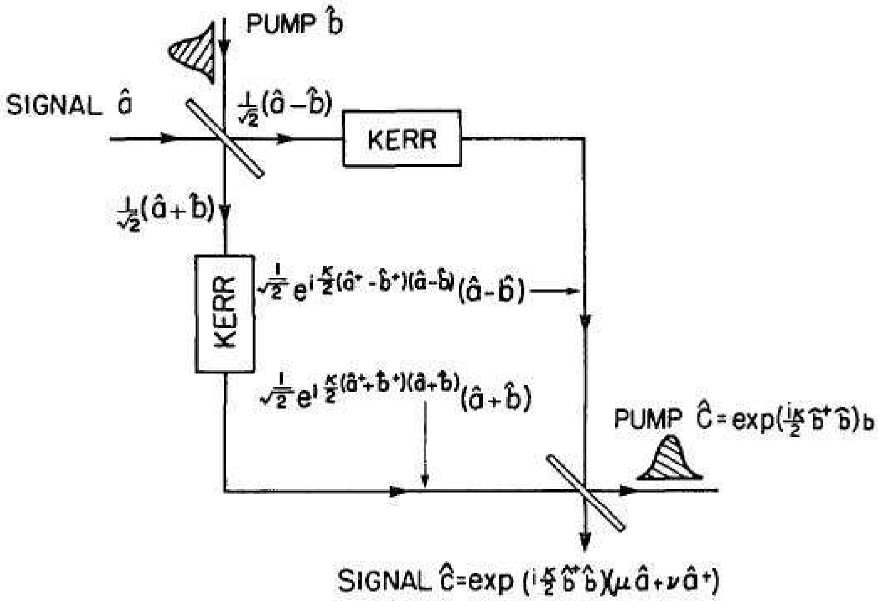
$$N_1 - N_2 = |\alpha|\Big[e^{-i\theta}a_1 + e^{i\theta}a_1^\dagger\Big] = \sqrt{2\bar{n}_{LO}}\hat{X}_\theta$$



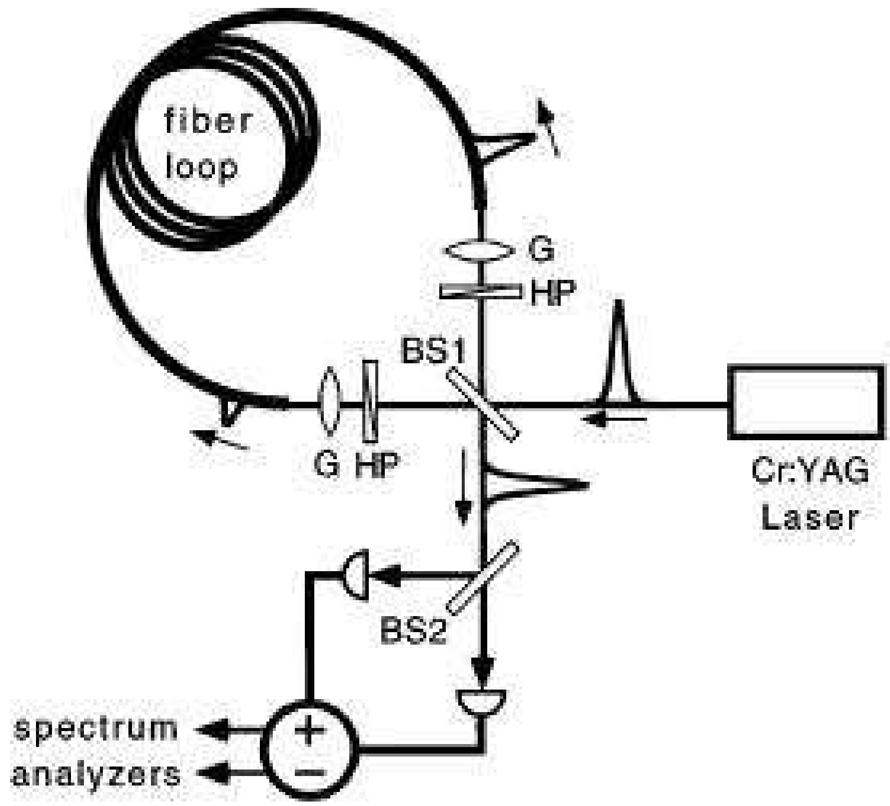
Images taken from work of J. Mlynek et al., Nature **387**, 471 (1997)

Experiments

Sagnac Interferometer:



G. Leuchs et. al., *Photon-Number Squeezed Solitons from an Asymmetric Fiber-Optic Sagnac Interferometer*, Phys. Rev. Lett. **81**, 2446 (1998)



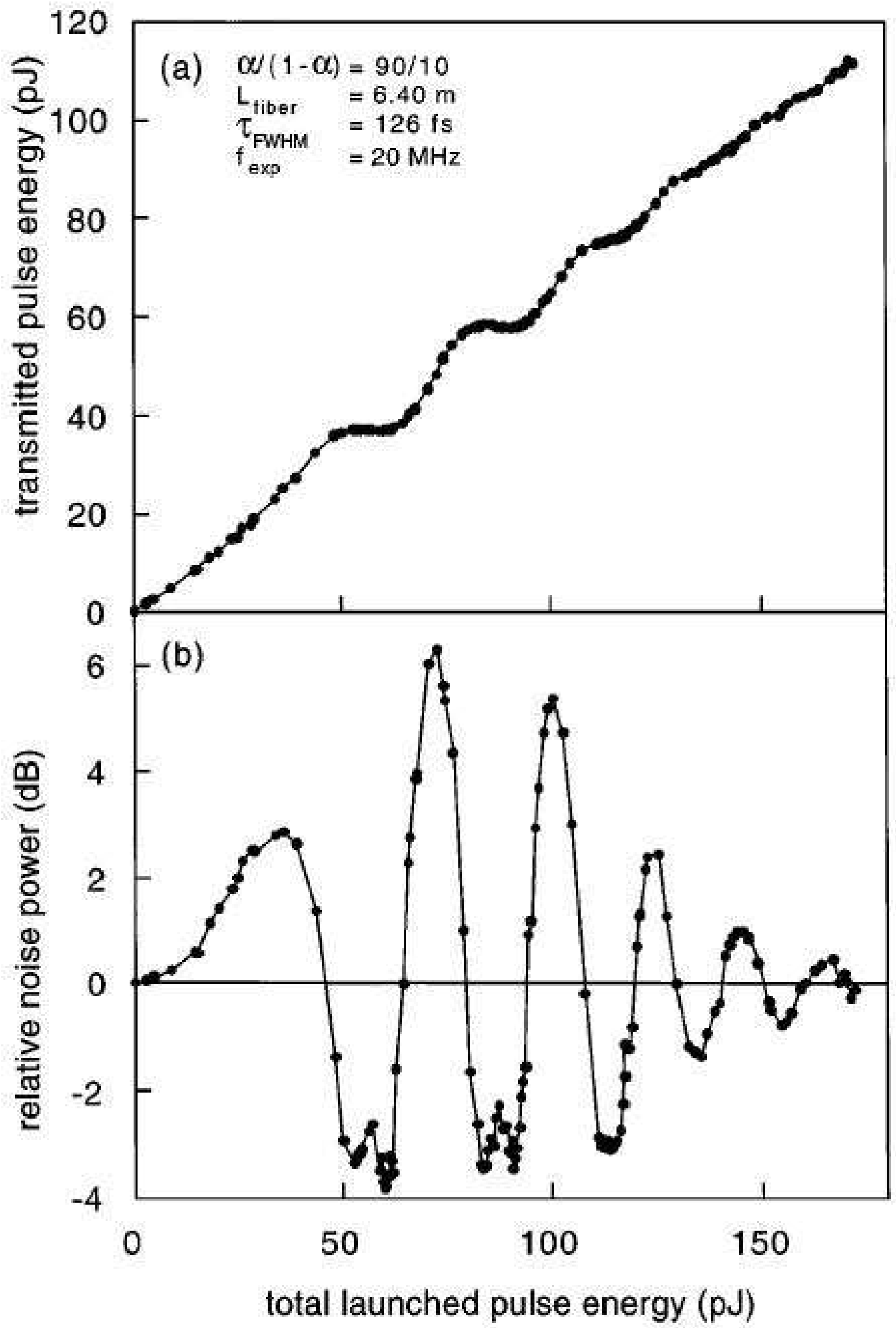
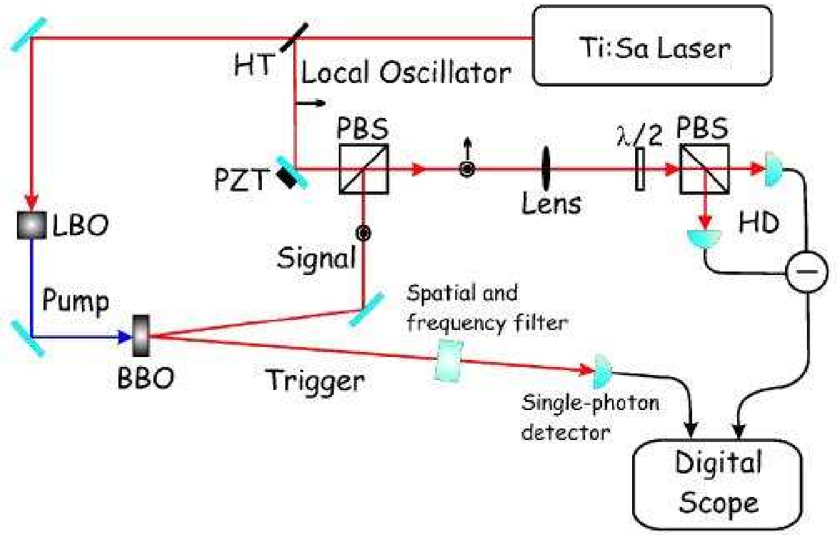


FIG. 2. Nonlinear energy-transfer function and squeezing from a 90:10 asymmetric Sagnac loop, plotted versus the launched pulse energy. (a) The transmitted output pulse energy shows an optical limiting effect at input energies of 53 pJ and 83 pJ. (b) Photocurrent noise power relative to shot noise (0 dB). The quantum fluctuations are reduced below shot noise at input energies where optical limiting occurs.

A. Zavatta, S. Viciani, M. Bellini, *Tomographic reconstruction of the single-photon Fock state by high-frequency homodyne detection*, Phys. Rev. A. **70**, 053821 (2004)



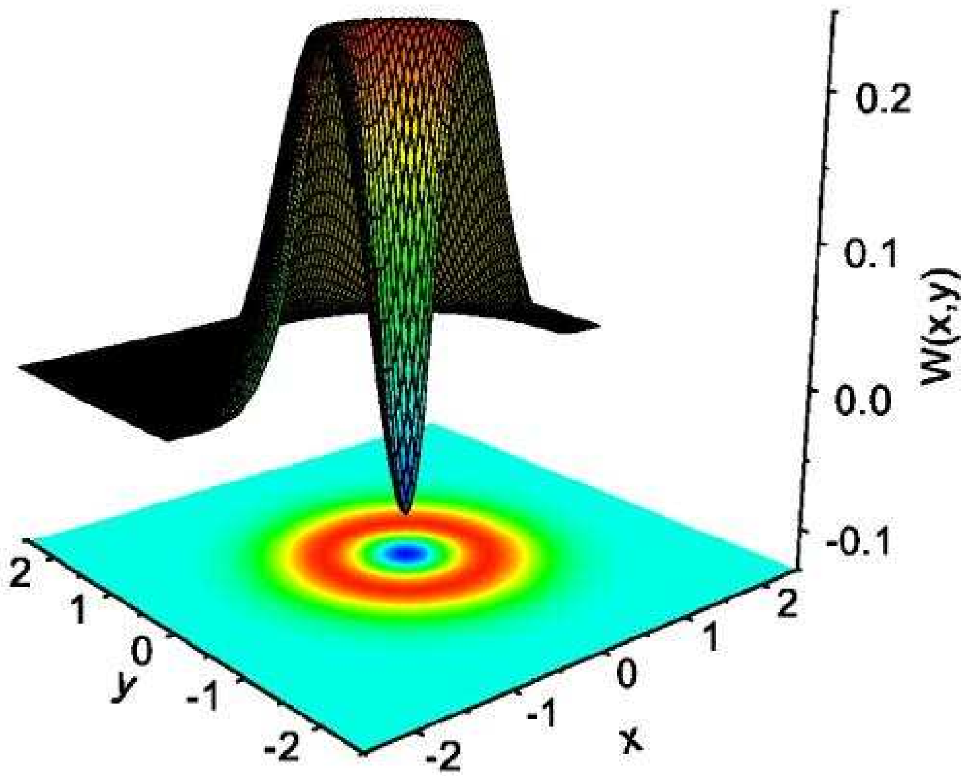


FIG. 4. (Color online) Wigner function of the single-photon Fock state as obtained from the reconstructed density-matrix elements. The negativity of the distribution, a clear proof of the non-classical character of the state, is evident around the origin of the shot-noise normalized quadrature axes.

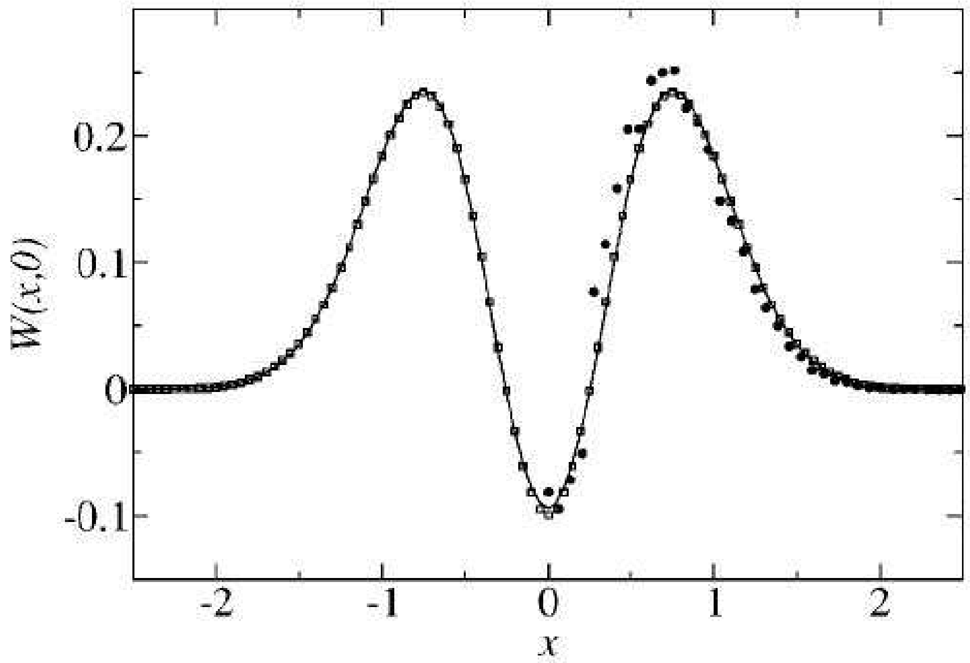


FIG. 5. $W(x,0)$ section of the reconstructed Wigner function as a function of the normalized quadrature x. Filled circles, Abel transform; empty squares, section of Fig. 4; solid line, section corresponding to the mixed state of Eq. (13).

Bibliography

Wigner Function

Wigner function and other quasi-distributions are found in every decent book on quantum optics:

M. O. Scully, M. S. Zubairy, Quantum Optics

D. F. Walls, G. J. Milburn, Quantum Optics

W. P. Schleich, Quantum Optics in Phase Space

They also contain description of homodyne detection, and the latter describes the tomographic reconstruction of the Wigner function.

Short lecture notes on wigner function by Stefan Keppeler: http://www.matfys.lth.se/stefan.keppeler/

Another review: A. M. Nassimi, *Quantum Mechanics in Phase Space*, quant-ph/0706.0237

The original paper by Wigner (for the sake of some history):
E. Wigner, *On the Quantum Correction For Thermodynamic Equilibrium*, Phys. Rev., **40**, 749 (1932)

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Gentle introductions:

http://qis.ucalgary.ca/quantech/wigner.html

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