<u>Quantum Optics Seminar</u>

Fock Space for Photons, Coherent States, Squeezed States

<u>Quantization of the Electro-Magnetic field</u>:

If we represent the vector potential $\vec{A}$ as a sum of plain waves so that

$$(1)\ \vec{A}(\vec{r},t)=\sqrt{\frac{8\pi c^2}{L^3}}\sum_{\vec{k},\alpha} q^{\alpha}{}_{\vec{k}}(t)e^{i\vec{k}\vec{r}}$$

with L the size of the space (which can be taken infinity at will), then

$$(2)\ \vec{E}(\vec{r},t)=\sqrt{\frac{8\pi c^2}{L^3}}\sum_{\vec{k},\alpha} \dot{q}^{\alpha}{}_{\vec{k}}(t)e^{i\vec{k}\vec{r}}\ ;\ (3)\ \vec{B}(\vec{r},t)=\sqrt{\frac{8\pi c^2}{L^3}}\sum_{\vec{k},\alpha} \vec{k}\times q^{\alpha}{}_{\vec{k}}(t)e^{i\vec{k}\vec{r}}$$

Then the Hamiltonian of the field is (using orthonormality of the functions and the relation $\omega_{\vec{k}}=ck$)

$$(4)\ H=\frac{1}{8\pi}\int\left(\left|\vec{E}\right|^2+\left|\vec{B}\right|^2\right)d\vec{r}=\sum_{\vec{k},\alpha}\left|\dot{q}^{\alpha}{}_{\vec{k}}\right|^2+\omega^2\left|q^{\alpha}{}_{\vec{k}}\right|^2$$

We define $p^{\alpha}{}_{\vec{k}}\equiv\dot{q}^{\alpha}{}_{\vec{k}}$ and this is the canonic-momentum for the coordinate $q^{\alpha}{}_{\vec{k}}$.

Now we will quantize the field, making $q^{\alpha}{}_{\vec{k}}$ and $p^{\alpha}{}_{\vec{k}}$ hermitian operators satisfying the commutation relation:

$$(5)\ \left[\hat{q}^{\alpha}{}_{\vec{k}},\hat{p}^{\alpha'}{}_{\vec{k}'}\right]=i\hbar\delta_{\vec{k},\vec{k}'}\delta_{\alpha,\alpha'}$$

Defining:

$$(6)\ \hat{q}^{\alpha}{}_{k}=\sqrt{\frac{\hbar}{\omega_{\vec{k}}}}\vec{E}_{\vec{k}}{}^{\alpha}\hat{C}_{\vec{k},\alpha}e^{-i\omega_{\vec{k}}t}$$

$$(7)\ \hat{p}^{\alpha}{}_{k}=i\omega_{\vec{k}}\sqrt{\frac{\hbar}{\omega_{\vec{k}}}}\vec{E}_{\vec{k}}{}^{\alpha}\hat{C}_{\vec{k},\alpha}e^{i\omega_{\vec{k}}t}$$

Which satisfy (using (5),(6),(7))

$$(8)\left[\hat{C}^{\alpha}{}_{\vec{k}},\hat{C}^{+\alpha'}{}_{\vec{k}'}\right]=\delta_{\vec{k},\vec{k}'}\delta_{\alpha,\alpha'}$$

$$(9)\ \left[\hat{C}^{\alpha}{}_{\vec{k}},\hat{C}^{\alpha'}{}_{\vec{k}'}\right]=0=\left[\hat{C}^{+\alpha}{}_{\vec{k}},\hat{C}^{+\alpha'}{}_{\vec{k}'}\right]$$

The electric and magnetic fields become:

$$(10)\ \vec{E}(\vec{r},t)=\frac{i}{c}\sum_{\vec{k},\alpha}\omega_{\vec{k}}\left(\hat{C}_{\vec{k},\alpha}\vec{U}_{\vec{k},\alpha}(t)-H.C.\right)$$

$$(11)\ \vec{B}(\vec{r},t)=i\sum_{\vec{k},\alpha}\vec{k}\times\left(\hat{C}_{\vec{k},\alpha}\vec{U}_{\vec{k},\alpha}(t)-H.C.\right)$$

With

$$(12)\ \vec{U}_{\vec{k},\alpha}(t) = \sqrt{\frac{2\pi\hbar c^2}{\omega_{\vec{k}} L^3}}\vec{E}_{\vec{k}} \exp(i\vec{k}\vec{r} - \omega_{\vec{k}} t)$$

And most importantly - the Hamiltonian becomes:

$$(13)\ \ H = \frac{1}{2}\sum_{\vec{k},\alpha} \hat{p}^{\alpha}{}_{\vec{k}}\hat{p}^{\alpha}{}_{\vec{k}}{}^{+} + \omega_{\vec{k}}\hat{q}^{\alpha}{}_{\vec{k}}\hat{q}^{\alpha}{}_{\vec{k}}{}^{+} = \sum_{\vec{k},\alpha}\hbar\omega_{\vec{k}}\left(\hat{C}^{\alpha}{}_{\vec{k}}{}^{+}\hat{C}^{\alpha}{}_{\vec{k}} + \frac{1}{2}\right)$$

This Hamiltonian is the same as the one for harmonic oscillator.
(From now on, for convenience, we will suppress the polarization index).

<u>Fock States and Fock Space</u>:

We will define a "Number" operator for each $\vec{k}$ thus:

$$(14)\ \ \hat{N}_{\vec{k}} = \hat{C}_{\vec{k}}{}^{+}\hat{C}_{\vec{k}}$$

$$(15)\ \ H = \sum_{\vec{k}}\hbar\omega_{\vec{k}}(\hat{N}_{\vec{k}} + \tfrac{1}{2})$$

With the commutation relations:

$$(16)\ \ \left[\hat{N}_{\vec{k}}, \hat{C}_{\vec{k}}\right] = -\hat{C}_{\vec{k}}$$

$$(17)\ \ \left[\hat{N}_{\vec{k}}, \hat{C}^{+}{}_{\vec{k}}\right] = \hat{C}^{+}{}_{\vec{k}}$$

Now we see, from (16) and (17) that if $\left|n_{\vec{k}}\right\rangle$ is an eigenstate of $\hat{N}_{\vec{k}}$ with an eigenvalue $n_{\vec{k}}$, then

$$(18)\ \ \hat{N}_{\vec{k}}\hat{C}_{\vec{k}}\left|n_{\vec{k}}\right\rangle = \left(\left[\hat{N}_{\vec{k}}, \hat{C}_{\vec{k}}\right] + \hat{C}_{\vec{k}}\hat{N}_{\vec{k}}\right)\left|n_{\vec{k}}\right\rangle = \left(-\hat{C}_{\vec{k}} + n_{\vec{k}}\hat{C}_{\vec{k}}\right)\left|n_{\vec{k}}\right\rangle = \left(n_{\vec{k}} - 1\right)\hat{C}_{\vec{k}}\left|n_{\vec{k}}\right\rangle$$

and similarly

$$(19)\ \ \hat{N}_{\vec{k}}\hat{C}^{+}{}_{\vec{k}}\left|n_{\vec{k}}\right\rangle = \left(n_{\vec{k}} + 1\right)\hat{C}^{+}{}_{\vec{k}}\left|n_{\vec{k}}\right\rangle$$

So $\hat{C}_{\vec{k}}\left|n_{\vec{k}}\right\rangle$ is also an eigenstate of $\hat{N}_{\vec{k}}$, this time with eigenvalue $n_{\vec{k}} - 1$, and $\hat{C}^{+}{}_{\vec{k}}\left|n_{\vec{k}}\right\rangle$ is an eigenstate with an eigenvalue $n_{\vec{k}} + 1$. We will name $\hat{C}_{\vec{k}}$ an "annihilation operator" and $\hat{C}^{+}{}_{\vec{k}}$ a "creation" operator.
A question arises – can we lower and raise the eigenvalue as low and as high as we'd like?
We go on to note that $\hat{N}_{\vec{k}}$ has only non-negative eigenvalues. Since for every eigenstate $\left|n_{\vec{k}}\right\rangle$

$$(20)\ \ n_{\vec{k}} = \left\langle n_{\vec{k}}\left|\hat{N}_{\vec{k}}\right|n_{\vec{k}}\right\rangle = \left\langle n_{\vec{k}}\left|\hat{C}_{\vec{k}}{}^{+}\hat{C}_{\vec{k}}\right|n_{\vec{k}}\right\rangle = \left\|\hat{C}_{\vec{k}}\left|n_{\vec{k}}\right\rangle\right\|^2 \geq 0$$

But we have seen that if $\left|n_{\vec{k}}\right\rangle$ is an eigenstate then $\hat{C}_{\vec{k}}\left|n_{\vec{k}}\right\rangle$ is also an eigenstate. The only way to stop the lowering process is if at one point the eigenvalue will be zero. In order for that to happen, only integer eigenvalues are allowed (making the choice of $n_{\vec{k}}$ an appropriate notation).

So we may define a lowest-energy state and name it the "vacuum" state: $|0\rangle$, for which

$$(21)\ \hat{C}_{\vec{k}}|0\rangle = 0$$

A general eigenstate with an eigenvalue $n_{\vec{k}}$ one must operate with the creation operator $n_{\vec{k}}$ times on the vacuum.

We go on to find the normalization needed for such a construction. Using (20) we readily see that the norm of $\hat{C}_{\vec{k}}|n_{\vec{k}}\rangle$ is $n_{\vec{k}}$, so

$$(22)\ \hat{C}_{\vec{k}}|n_{\vec{k}}\rangle = \sqrt{n_{\vec{k}}}\,|n_{\vec{k}} - 1\rangle$$

and

$$(23)\ \hat{C}^{+}{}_{\vec{k}}|n_{\vec{k}}\rangle = \sqrt{n_{\vec{k}} + 1}\,|n_{\vec{k}} + 1\rangle$$

So we can now write a general state $|n_{\vec{k}}\rangle$ constructed from the vacuum state:

$$(24)\ |n_{\vec{k}}\rangle = \frac{\left(C^{+}{}_{\vec{k}}\right)^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}|0\rangle$$

And generalizing for more than a single wave-vector $\vec{k}$:

$$(25)\ |n_{\vec{k}_1}, n_{\vec{k}_2}, \ldots, n_{\vec{k}_m}\rangle = \frac{\left(C^{+}{}_{\vec{k}_1}\right)^{n_{\vec{k}_1}}}{\sqrt{n_{\vec{k}_1}!}} \frac{\left(C^{+}{}_{\vec{k}_2}\right)^{n_{\vec{k}_2}}}{\sqrt{n_{\vec{k}_2}!}} \cdots \frac{\left(C^{+}{}_{\vec{k}_m}\right)^{n_{\vec{k}_m}}}{\sqrt{n_{\vec{k}_m}!}}|0\rangle$$

Noting that going from $|n_{\vec{k}}\rangle$ to $|n_{\vec{k}} + 1\rangle$ we added $\hbar\omega_{\vec{k}}$ to the energy of the system, which is the energy of a single photon with wave-vector $\vec{k}$, we can see the state $|n_{\vec{k}}\rangle$ is a state with $n_{\vec{k}}$ photons each with energy $\hbar\omega_{\vec{k}}$. So the annihilation and creation operators do just what their name implies to photons, and the number operator counts the number of photons with certain frequency in a system.

These are **Fock states** of photons. A Fock state has a defined number of photons. Fock states are orthonormal (being non-degenerate eigenstates of an Hermitian operator)

$$(26)\ \langle n_{\vec{k}}|m_{\vec{k}'}\rangle = \delta_{\vec{k},\vec{k}'}\delta_{n,m}$$

and form a complete set:

$$(27)\ \sum_{n}|n_{\vec{k}}\rangle\langle n_{\vec{k}}| = 1_{\vec{k}}$$

With $1_{\vec{k}}$ being the unity operator for the sub-space of photons with wave-vector $\vec{k}$.

The Coherent State:

We turn on to see a strange phenomena – even though $\left|n_{\vec{k}}\right\rangle$ is an eigenstate of the Hamiltonian, the expectation values of the electric and magnetic field in that state are zero (using (10),(11)):

$$\text{(28)}\quad \begin{aligned} &\left\langle n_{\vec{k}}\left|\vec{E}\right|n_{\vec{k}}\right\rangle = \omega_{\vec{k}}\left\langle n_{\vec{k}}\left|\hat{C}_{\vec{k}}\vec{U}_{\vec{k}}(t) - H.C.\right|n_{\vec{k}}\right\rangle = \\ &= \omega_{\vec{k}}\left(\left\langle n_{\vec{k}}\left|\vec{U}_{\vec{k}}(t)\right|n_{\vec{k}}-1\right\rangle - \left\langle n_{\vec{k}}\left|\vec{U}^*_{\vec{k}}(t)\right|n_{\vec{k}}+1\right\rangle\right) = 0 \end{aligned}$$

And similarly for the magnetic field. So the Fock states cannot represent the classical states we witness in everyday life. We have to construct a state which will be a superposition of different Fock states in order to get the classical, or "coherent", state. We will first introduce this state, without clarifying why it is the desired state, and only later will show that this is the case. Please also note that from now one we will discuss a monochromatic light - i.e. we will work in the wave vector $\vec{k}$ sub-space of the Hilbert space. Thanks to the orthogonality of the different sub-spaces the results can be easily generalized to light with different wavelengths.

We introduce the Translation unitary operator:

$$\text{(29)}\quad D_{\vec{k}}(\alpha) = \exp\left(\alpha\hat{C}_{\vec{k}}^{+} - \alpha^*\hat{C}_{\vec{k}}\right)$$

With $\alpha \in C$. Operating with that operator on the vacuum we get

$$\text{(30)}\quad \begin{aligned} &\left|\alpha\right\rangle = \exp\left(\alpha\hat{C}_{\vec{k}}^{+} - \alpha^*\hat{C}_{\vec{k}}\right)\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{C}_{\vec{k}}^{+}} e^{\alpha^*\hat{C}_{\vec{k}}}\left|0\right\rangle = \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{C}_{\vec{k}}^{+}}\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n\left(\hat{C}_{\vec{k}}^{+}\right)^n}{n!}\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=0}^{\infty}\frac{\alpha^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\left|n_{\vec{k}}\right\rangle \end{aligned}$$

Where we used (24) and the Baker-Campbell-Hausdorff formula, which states that for two operators A and B, if $[A,[A,B]] = 0 = [B,[A,B]]$ then $e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B$.

A very useful property of this state is that it is an eigenstate of the annihilation operator with eigenvalue α

$$\text{(31)}\quad \begin{aligned} &\hat{C}_{\vec{k}}\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=0}^{\infty}\frac{\alpha^n\hat{C}_{\vec{k}}}{\sqrt{n_{\vec{k}}!}}\left|n_{\vec{k}}\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=1}^{\infty}\frac{\alpha^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\sqrt{n_{\vec{k}}}\left|n_{\vec{k}}-1\right\rangle = \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=1}^{\infty}\frac{\alpha^{n_{\vec{k}}-1}\alpha}{\sqrt{\left(n_{\vec{k}}-1\right)!}}\left|n_{\vec{k}}-1\right\rangle = \alpha e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=0}^{\infty}\frac{\alpha^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\left|n_{\vec{k}}\right\rangle \end{aligned}$$

Where we used (22).

Having shown that, it is easy to show that this state has the desired classical expectation value for the electric and magnetic field

$$\text{(32)}\quad \left\langle\alpha\left|\vec{E}\right|\alpha\right\rangle = \omega_{\vec{k}}\left\langle\alpha\left|\hat{C}_{\vec{k}}\vec{U}_{\vec{k}}(t) - H.C.\right|\alpha\right\rangle = \omega_{\vec{k}}\left(\vec{U}_{\vec{k}}(t)\alpha - \vec{U}^*_{\vec{k}}(t)\alpha^*\right)$$

With the last equality coming directly from (30). So this is indeed the desired classical coherent state which we looked for.

Another interesting aspect about the coherent state is that it's expectation value of the number of the photons (and hence – of the energy) is

$$(33)\ \langle\alpha|\hat{N}_{\vec{k}}|\alpha\rangle = \langle\alpha|\hat{C}_{\vec{k}}^{+}\hat{C}_{\vec{k}}|\alpha\rangle = |\alpha|^2$$

And the variance is

$$(34)\ \begin{aligned}&\langle\alpha|\hat{N}_{\vec{k}}^{2}|\alpha\rangle - \langle\alpha|\hat{N}_{\vec{k}}|\alpha\rangle^2 = \langle\alpha|\hat{C}_{\vec{k}}^{+}\hat{C}_{\vec{k}}\hat{C}_{\vec{k}}^{+}\hat{C}_{\vec{k}}|\alpha\rangle - \left(|\alpha|^2\right)^2 = \\ &= |\alpha|^2\left(\langle\alpha|\hat{C}_{\vec{k}}\hat{C}_{\vec{k}}^{+}|\alpha\rangle - |\alpha|^2\right) = |\alpha|^2\left(\langle\alpha|\hat{C}_{\vec{k}}^{+}\hat{C}_{\vec{k}}|\alpha\rangle + 1 - |\alpha|^2\right) = |\alpha|^2\end{aligned}$$

In which we used the commutation relations for the annihilation and creation operators (8).

This is not surprising once one realize that the coherent state is a Poisson distribution of Fock states with α as a parameter.

We end this section by noting that each of these properties can serve as the definition of the coherent state – being a Poisson distribution of Fock states with parameter α, being an eigenstate of the annihilation operator with eigenvalue α or being the result of operating on the vacuum state with the translation operator $D_{\vec{k}}(\alpha)$.

Coherent State Representation:

We have seen that the Fock states form a complete set, and can span each and every state of a system. What about the Coherent states? Do they form a complete set?

We'll observe that

$$(35)\ \begin{aligned}&\int d^2\alpha|\alpha\rangle\langle\alpha| = \int e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\vec{k}}=0}^{\infty}\frac{\alpha^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}|n_{\vec{k}}\rangle e^{-\frac{|\alpha|^2}{2}}\sum_{m_{\vec{k}}=0}^{\infty}\frac{\alpha^{*m_{\vec{k}}}}{\sqrt{m_{\vec{k}}!}}\langle m_{\vec{k}}|d^2\alpha = \\ &= \int e^{-|\alpha|^2}\sum_{n_{\vec{k}},m_{\vec{k}}=0}^{\infty}\frac{\alpha^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\frac{\alpha^{*m_{\vec{k}}}}{\sqrt{m_{\vec{k}}!}}|n_{\vec{k}}\rangle\langle m_{\vec{k}}|d^2\alpha = \pi\sum_{n_{\vec{k}}}|n_{\vec{k}}\rangle\langle n_{\vec{k}}|\end{aligned}$$

Where we used the mathematical formula

$$(36)\int d^2\alpha e^{-|\alpha|^2}\alpha^{*n}\alpha^m = \delta_{n.m}\pi n!$$

So due to the completeness of the Fock states, we see that every state can be span in terms of coherent states. However, the coherent states are not orthogonal

$$(37)\ \begin{aligned}&\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}\sum_{n_{\vec{k}},m_{\vec{k}}=0}^{\infty}\frac{\beta^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\frac{\alpha^{*m_{\vec{k}}}}{\sqrt{m_{\vec{k}}!}}\langle m_{\vec{k}}|n_{\vec{k}}\rangle = \\ &= e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}\sum_{n_{\vec{k}}=0}^{\infty}\frac{\beta^{n_{\vec{k}}}\alpha^{*n_{\vec{k}}}}{n_{\vec{k}}!} = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right)\end{aligned}$$

(So $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$ and as α and β differ, the state tend to be more "orthogonal"). Because the states are not orthogonal, they form an over-complete set, i.e. there is more than one way to span each set, since each coherent state itself can be spanned using other coherent states.

So how do one find the coherent state representation of a certain state or operator?

We would like to find the coherent state representation of the Thermal state. A thermal state is defined by a density operator

$$(38)\ \ \hat{\rho} = \frac{e^{-\beta\hat{H}}}{Tr(e^{-\beta\hat{H}})}$$

Which in Fock state representation is simply

$$(39)\ \ \hat{\rho} = \sum_{n_{\bar{k}}} \left(1 - e^{-\beta\hbar\omega_{\bar{k}}}\right) e^{-\beta\hbar\omega_{\bar{k}} n_{\bar{k}}} \left|n_{\bar{k}}\right\rangle\left\langle n_{\bar{k}}\right| = \sum_{n_{\bar{k}}} \frac{\langle n\rangle^{n_{\bar{k}}}}{\left(1+\langle n\rangle\right)^{n_{\bar{k}}+1}} \left|n_{\bar{k}}\right\rangle\left\langle n_{\bar{k}}\right|$$

Where we defined

$$(40)\ \ \langle n\rangle = \frac{1}{e^{-\beta\hbar\omega_{\bar{k}}} - 1}$$

We would like to find $P(\alpha,\alpha^*)$ so that:

$$(41)\ \ \hat{\rho} = \int P\left(\alpha,\alpha^*\right)\left|\alpha\right\rangle\left\langle\alpha\right| d^2\alpha$$

We first note that

$$(42)\ \ \left\langle -\beta\right|\hat{\rho}\left|\beta\right\rangle = \int P\left(\alpha,\alpha^*\right)\left\langle -\beta\,|\,\alpha\right\rangle\left\langle\alpha\,|\,\beta\right\rangle d^2\alpha = e^{-|\beta|^2}\int P\left(\alpha,\alpha^*\right) e^{-|\alpha|^2} e^{-\beta^*\alpha+\alpha^*\beta} d^2\alpha$$

In which we used (37) in order to calculate $\left\langle\beta\,|\,\alpha\right\rangle$. We'll write it down in the coordinates of α and β to find out that

$$(43)\ \ \left\langle -\beta\right|\hat{\rho}\left|\beta\right\rangle e^{|\beta|^2} = \int P\left(\alpha,\alpha^*\right) e^{-x_\alpha^2 - y_\alpha^2} e^{2i\left(y_\beta x_\alpha - x_\beta y_\alpha\right)} dx_\alpha dy_\alpha$$

Which is a Fourier transform in the coordinates of α. If we'll do the inverse transform, we can find $P(\alpha,\alpha^*)$

$$(44)\ \ P\left(\alpha,\alpha^*\right) = \frac{e^{|\alpha|^2}}{\pi^2}\int \left\langle -\beta\right|\hat{\rho}\left|\beta\right\rangle e^{|\beta|^2} e^{-\beta\alpha^*+\alpha\beta^*} d^2\beta$$

Now we'll apply this general formula to case of a thermal state:

$$(45)\ \ \begin{aligned} \left\langle -\beta\right|\hat{\rho}\left|\beta\right\rangle &= \sum_{n_{\bar{k}}} \frac{\langle n\rangle^{n_{\bar{k}}}}{\left(1+\langle n\rangle\right)^{n_{\bar{k}}+1}} \left\langle -\beta\,|\,n_{\bar{k}}\right\rangle\left\langle n_{\bar{k}}\,|\,\beta\right\rangle = \\ &= e^{-|\beta|^2} \frac{1}{1+\langle n\rangle} \sum_{n_{\bar{k}}} \left(\frac{\langle n\rangle}{\left(1+\langle n\rangle\right)}\right)^{n_{\bar{k}}} \frac{\left(-|\beta|^2\right)^{n_{\bar{k}}}}{n_{\bar{k}}!} = \frac{e^{-|\beta|^2}}{1+\langle n\rangle} e^{-|\beta|^2\frac{\langle n\rangle}{1+\langle n\rangle}} \end{aligned}$$

In which we used the Fock state representation of the coherent state (30) and the orthogonality of Fock states (26).

So

$$
\begin{aligned}
P(\alpha,\alpha^*) &= \frac{e^{|\alpha|^2}}{\pi^2\left(1+\langle n\rangle\right)}\int e^{-|\beta|^2\frac{\langle n\rangle}{1+\langle n\rangle}}e^{-\beta\alpha^*+\alpha\beta^*}d^2\beta = \\
&= \frac{e^{|\alpha|^2}}{\pi^2\left(1+\langle n\rangle\right)}\int \exp\left(-\beta\beta^*\frac{\langle n\rangle}{1+\langle n\rangle}-\beta\alpha^*+\alpha\beta^*+\frac{1+\langle n\rangle}{\langle n\rangle}\left(\alpha\alpha^*-\alpha\alpha^*\right)\right)d^2\beta = \\
&= \frac{e^{|\alpha|^2}e^{-|\alpha|^2\frac{1+\langle n\rangle}{\langle n\rangle}}}{\pi^2\left(1+\langle n\rangle\right)}\int \exp\left(\left(\sqrt{\frac{1+\langle n\rangle}{\langle n\rangle}}\alpha^*+\sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}}\beta^*\right)\left(\sqrt{\frac{1+\langle n\rangle}{\langle n\rangle}}\alpha-\sqrt{\frac{\langle n\rangle}{1+\langle n\rangle}}\beta\right)\right)d^2\beta = \\
&= e^{\frac{-|\alpha|^2}{\langle n\rangle}}\frac{1}{\pi\langle n\rangle}
\end{aligned}
\tag{46}
$$

Which is a Gaussian distribution. So the Thermal State is a Guassian distribution of Coherent States.

Squeezed States:

We now turn on to observe an interesting property of the coherent state. We'll define two hermitian operators that are analogue to the location and momentum operators of the quantum harmonic oscillator:

$$
\begin{aligned}
\hat{Q}' &= \hat{C}^+ + \hat{C} \\
\hat{P}' &= i\left(\hat{C}^+ - \hat{C}\right)
\end{aligned}
\tag{47}
$$

Note that now we have suppressed all indices. This is to be understood as "space" and "momentum" operators for specific wave-vector $\vec{k}$ and polarization r.

The commutation relations of these operators are

$$\left[\hat{Q}',\hat{P}'\right]=2i \tag{48}$$

Hence, the uncertainty principle for their observables is

$$\left(\left\langle \Delta\hat{Q}'^2\right\rangle\left\langle \Delta\hat{P}'^2\right\rangle\right)\geq 1 \tag{49}$$

Where by definition

$$\left\langle \Delta\hat{A}^2\right\rangle \equiv \left\langle \hat{A}^2\right\rangle - \left\langle \hat{A}^2\right\rangle \tag{50}$$

For a coherent state

$$\langle\alpha|\hat{Q}'|\alpha\rangle = \langle\alpha|\hat{C}^+ + \hat{C}|\alpha\rangle = \alpha+\alpha^* \tag{51}$$

$$
\begin{aligned}
\langle\alpha|\hat{Q}'^2|\alpha\rangle &= \langle\alpha|\hat{C}^{+2}+\hat{C}^2+\hat{C}\hat{C}^+ + \hat{C}^+\hat{C}|\alpha\rangle = \\
&= \langle\alpha|\hat{C}^{+2}+\hat{C}^2+\left[\hat{C},\hat{C}^+\right]+2\hat{C}^+\hat{C}|\alpha\rangle = \alpha^2+\alpha^{*2}+2|\alpha|^2+1
\end{aligned}
\tag{52}
$$

$$\left\langle \Delta\hat{Q}'^2\right\rangle = \left\langle \hat{Q}'^2\right\rangle - \left\langle \hat{Q}'\right\rangle^2 = 1 \tag{53}$$

In which we used the fact that a coherent state is an eigenstate of the annihilation operator (31) and the commutation relation of the operators (8).

Similarly, for the "momentum" operator

$$(54)\ \langle \alpha|\hat{P}'|\alpha\rangle = i\langle \alpha|\hat{C}^{+} - \hat{C}|\alpha\rangle = i\alpha - i\alpha^{*}$$

$$(54)\ \langle \alpha|\hat{P}'^{2}|\alpha\rangle = -\langle \alpha|\hat{C}^{+2} + \hat{C}^{2} - \left(\hat{C}\hat{C}^{+} + \hat{C}^{+}\hat{C}\right)|\alpha\rangle = -\alpha^{2} - \alpha^{*2} + 2|\alpha|^{2} + 1$$

$$(55)\ \left\langle \Delta\hat{P}'^{2}\right\rangle = \left\langle \hat{P}'^{2}\right\rangle - \left\langle \hat{P}'\right\rangle^{2} = 1$$

Hence

$$(56)\ \left(\left\langle \Delta\hat{Q}'^{2}\right\rangle\left\langle \Delta\hat{P}'^{2}\right\rangle\right)_{C.S.} = 1$$

So a coherent state has a minimum uncertainty in these variables.

We would like to have a more generalized "space" and "momentum" operators, so for a rotation in an angle β will get

$$(57)\ \begin{aligned} \hat{Q} &= \hat{C}^{+}e^{i\beta} + \hat{C}e^{-i\beta} \\ \hat{P} &= \hat{C}^{+}e^{i\left(\beta+\frac{\pi}{2}\right)} + \hat{C}e^{-i\left(\beta+\frac{\pi}{2}\right)} \end{aligned}$$

Which doesn't change the results (48)-(56).

We will now build states in which the uncertainty in $\hat{Q}$ is smaller than 1, while the uncertainty in $\hat{P}$ is larger than 1. These states will be appropriately named "Squeezed States".

We first introduce a unitary "Squeezing" Operator, with a parameter $z \in C$

$$(58)\ \hat{S}(z) = \exp\left(\frac{1}{2}z^{*}C^{2} - \frac{1}{2}zC^{+2}\right)$$

A useful operator for the upcoming calculations will be the "squeezed annihilation" operator defined

$$(59)\ \hat{A}(z) = \hat{S}(z)\hat{C}\hat{S}^{+}(z) = \hat{C} + z\hat{C} + \frac{|z|^{2}\hat{C}}{2!} + ... = \hat{C}\cosh r + \hat{C}^{+}e^{i\theta}\sinh r$$

With $z = re^{i\theta}$. We'll define $\mu = \cosh r; \nu = e^{i\theta}\sinh r$ and then

$$(60)\ \begin{aligned} \hat{A}(z) &= \hat{C}\mu + \hat{C}^{+}\nu \\ \hat{A}^{+}(z) &= \hat{C}\nu^{*} + \hat{C}^{+}\mu^{*} \end{aligned}$$

And since $|\mu|^{2} - |\nu|^{2} = 1$ then

$$(61)\ \left[\hat{A}^{+}(z), \hat{A}(z)\right] = 1$$

We'll define a Squeezed State with parameter z as the result of operating with the Squeezing operator (58) on a coherent state.

$$(62)\ \hat{S}(z)|\alpha\rangle = |(z,\alpha)\rangle$$

And then

$$(63)\ \hat{A}(z)|(z,\alpha)\rangle = \hat{A}(z)\hat{S}(z)|\alpha\rangle = \hat{S}(z)\hat{C}\hat{S}^{+}(z)\hat{S}(z)|\alpha\rangle = \alpha|(z,\alpha)\rangle$$

So $|(z,\alpha)\rangle$ is an eigenstate of the "squeezed annihilation" operator with eigenvalue α.

We can now finally turn on to see what is the uncertainty in the "space" operator $\hat{Q}$ for the squeezed state:

$$\begin{aligned} &\langle(z,\alpha)|\hat{Q}|(z,\alpha)\rangle = \langle(z,\alpha)|\hat{C}^{+}e^{i\beta} + \hat{C}e^{-i\beta}|(z,\alpha)\rangle = \\ (64)\quad &= \langle(z,\alpha)|\left(\mu\hat{A}^{+} - \nu^{*}\hat{A}\right)e^{i\beta} + \left(\mu^{*}\hat{A} - \nu\hat{A}^{+}\right)e^{-i\beta}|(z,\alpha)\rangle = \\ &= \left(\mu\alpha^{*} - \nu^{*}\alpha\right)e^{i\beta} + \left(\mu^{*}\alpha - \nu\alpha^{*}\right)e^{-i\beta} \end{aligned}$$

$$\begin{aligned} &\langle(z,\alpha)|\hat{Q}^{2}|(z,\alpha)\rangle = \langle(z,\alpha)|\left(\mu\hat{A}^{+} - \nu^{*}\hat{A}\right)^{2}e^{2i\beta} + \left(\mu^{*}\hat{A} - \nu\hat{A}^{+}\right)^{2}e^{-2i\beta} + \\ &+ 2\left(\mu\hat{A}^{+} - \nu^{*}\hat{A}\right)\left(\mu^{*}\hat{A} - \nu\hat{A}^{+}\right) + 1|(z,\alpha)\rangle = \\ (65)\quad &= \left(\mu^{2}\alpha^{*2} + \nu^{*2}\alpha^{2} - \mu\nu^{*}\left(2|\alpha|^{2} + 1\right)\right)e^{2i\beta} + \left(\mu^{*2}\alpha^{2} + \nu^{2}\alpha^{*2} - \mu^{*}\nu\left(2|\alpha|^{2} + 1\right)\right)e^{-2i\beta} \\ &+ 2\left(|\mu|^{2}|\alpha|^{2} + |\nu|^{2}|\alpha|^{2} + |\nu|^{2} - \nu^{*}\mu^{*}\alpha^{2} - \nu\mu\alpha^{*2}\right) + 1 \end{aligned}$$

Therefore

$$\begin{aligned} (66)\quad &\left\langle\Delta\hat{Q}^{2}\right\rangle = \left\langle\hat{Q}^{2}\right\rangle - \left\langle\hat{Q}\right\rangle^{2} = -\mu\nu^{*}e^{2i\beta} - \mu^{*}\nu e^{-2i\beta} + 2|\nu|^{2} + 1 = \\ &= \cosh(2r) - \sinh(2r)\cos(\theta - 2\beta) \end{aligned}$$

Where again $z = re^{i\theta}$

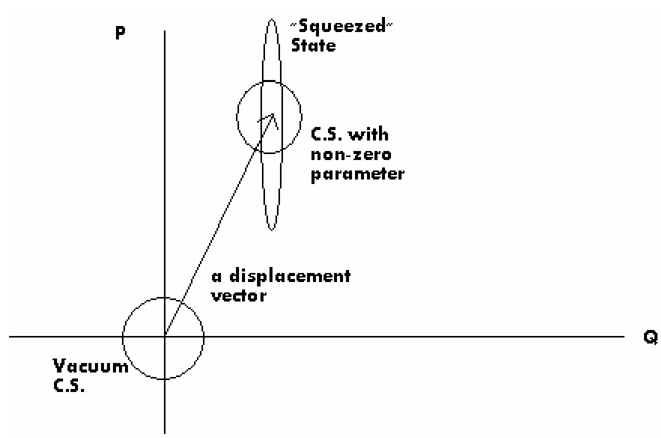
If we choose $\beta = \dfrac{\theta}{2}$ we will get the minimal possible value, which is

$$(67)\quad \left\langle\Delta\hat{Q}^{2}\right\rangle = e^{-2r}$$

This value can be as small as we'd like since we are free to choose r (a parameter of the squeezing operator). Since the uncertainty principle naturally holds, then with no need to calculate we get

$$(68)\quad \left\langle\Delta\hat{P}^{2}\right\rangle \geq e^{2r}$$

We can draw these results on a diagram describing the expectation values of $\hat{P}$ and $\hat{Q}$:



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