Path Entanglement

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Quantum Optics Seminar

March 2008

Lecture Outline

- Path entangled states.
- Generation of path entangled states.
- Characteristics of the entangled state:
  - Super Resolution
- Beating classical limits:
  - Super Phase sensitivity
  - Quantum Lithography

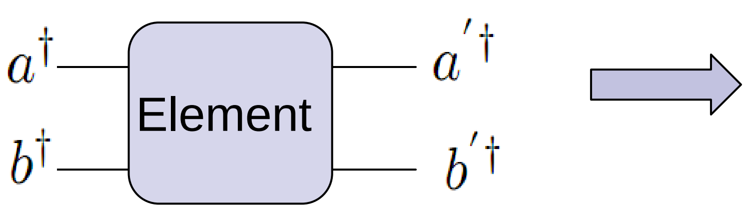
Entanglement

- Entangled state: $\frac{1}{\sqrt{2}}\left[|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right]$
- Optical mode - physical state (polarization, frequency, direction)
- Types of Entanglement:
  - Polarization (eg. down conversion) $\frac{1}{\sqrt{2}}\left[|HV\rangle + |VH\rangle\right]$
  - Frequency (not very useful)
  - Direction = Path Entanglement
- Path (number) entangled states: $\frac{1}{\sqrt{2}}\left[|P,Q\rangle_{AB} + e^{i\phi}|Q,P\rangle_{AB}\right]$
  - P photons in A $\longleftrightarrow$ Q photons in B
  - Most interesting : *NOON* states

Optical Quantum Systems

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$
$$a^\dagger b^\dagger |0\rangle = |1\rangle_a |1\rangle_b$$

- Input State: $(a^\dagger)^n (b^\dagger)^m |0\rangle$
- Output State: $\left(a'^\dagger\right)^{n'} \left(b'^\dagger\right)^{m'} |0\rangle$
- The total photon number is preserved, $n + m = n' + m'$.
- Operation of elements is described by a unitary matrix U.
- U operates on the modes' creation operators.
- U determines creation operators for output modes.



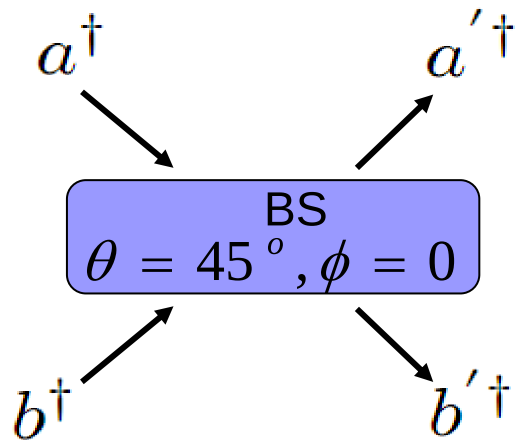
$$U \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \begin{pmatrix} a'^\dagger \\ b'^\dagger \end{pmatrix}$$

Beam Splitter

$$\boxed{\begin{array}{lcr} \cos^2\theta & \equiv & R \\ \sin^2\theta & \equiv & T \end{array}}$$

$$U(B_{\theta\phi}) = \begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}$$

50-50 beam splitter: $\theta = 45^o, \phi = 0$



$$U \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \begin{pmatrix} a'^\dagger \\ b'^\dagger \end{pmatrix}$$

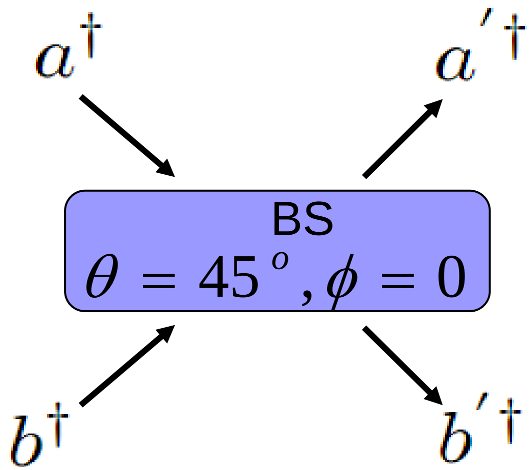
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a^\dagger - b^\dagger \\ a^\dagger + b^\dagger \end{pmatrix}$$

Beam Splitter – 1 Incident Photon

$$\boxed{\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a'^\dagger + b'^\dagger \\ b'^\dagger - a'^\dagger \end{pmatrix}}$$

$$|10\rangle_{ab} = \left(a^\dagger\right)^1 \left(b^\dagger\right)^0 |0\rangle \quad \xrightarrow{BS} \quad \frac{1}{\sqrt{2}} \left(a'^\dagger + b'^\dagger\right) |0\rangle$$

$$= \quad \frac{1}{\sqrt{2}} |10\rangle_{a'b'} + \frac{1}{\sqrt{2}} |01\rangle_{a'b'}$$



Photon appears at a'
Probability: 0.5

Photon appears at b'
Probability: 0.5

Beam Splitter – 2 Incident Photons

$$\boxed{\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a'^\dagger + b'^\dagger \\ b'^\dagger - a'^\dagger \end{pmatrix}}$$

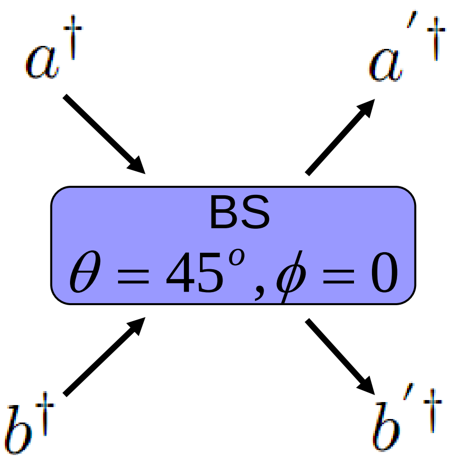
$$|20\rangle_{ab} = \frac{1}{\sqrt{2}} \left(a^\dagger\right)^2 \left(b^\dagger\right)^0 |0\rangle \quad \xrightarrow{BS} \quad \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \left(a'^\dagger + b'^\dagger\right)\right]^2 |0\rangle$$

$$= \quad \frac{1}{2}|20\rangle_{a'b'} + \frac{1}{2}|02\rangle_{a'b'} + \frac{1}{\sqrt{2}}|11\rangle_{a'b'}$$

2 photons appears at a'
Probability: 0.25

2 photon appears at b'
Probability: 0.25

1 photon appears at a'
1 photon appears at b'
Probability: 0. 5



Beam Splitter – Two Bunching Photons

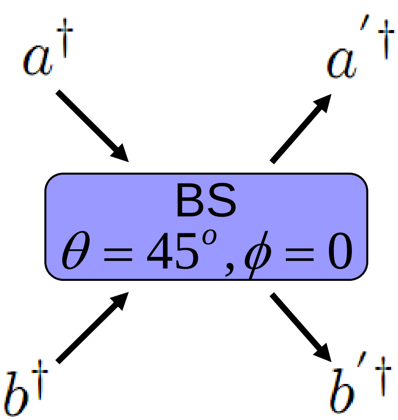
$$\boxed{\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a'^\dagger + b'^\dagger \\ b'^\dagger - a'^\dagger \end{pmatrix}}$$

$$|11\rangle_{ab} = \left(a^\dagger\right)^1 \left(b^\dagger\right)^1 |0\rangle \quad \xrightarrow{BS} \quad = \left[\frac{1}{\sqrt{2}}\left(a'^\dagger + b'^\dagger\right)\right]\left[\frac{1}{\sqrt{2}}\left(b'^\dagger - a^\dagger\right)\right]|0\rangle$$

$$= -\frac{1}{\sqrt{2}}|20\rangle_{a'b'} + \frac{1}{\sqrt{2}}|02\rangle_{a'b'}$$

2 photons appears at a'
Probability: 0.5

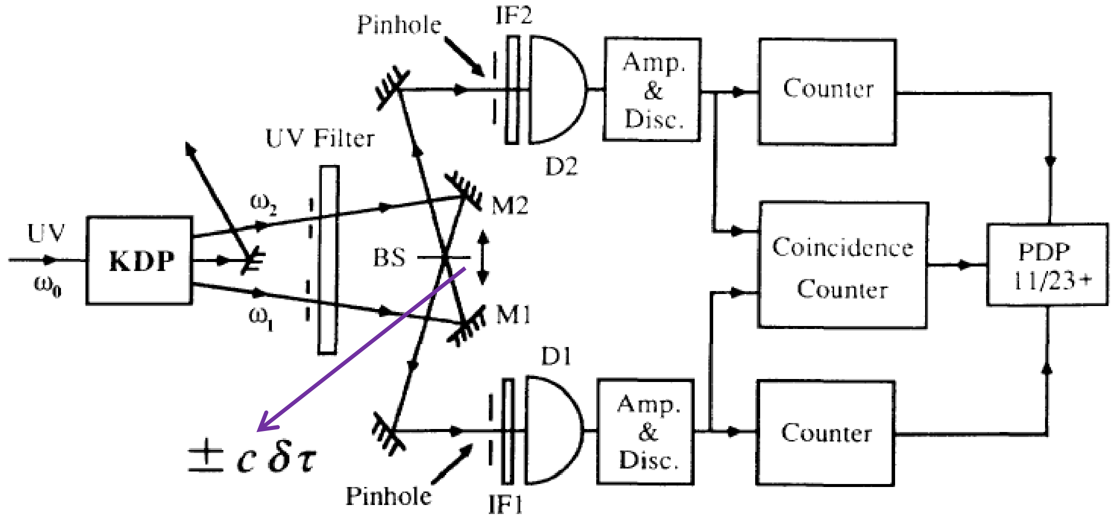
2 photon appears at b'
Probability: 0.5



The photons “bunch” (No 11 state) !!!

Two Photon Bunching - Hong Oh Mandel (HOM)

- Purpose: Measure time interval between two photons



- Output: $\psi_{out}\rangle = (R-T)\,|1_1,1_2\rangle + i(2RT)^{1/2}\,|2_1,0_2\rangle + i(2RT)^{1/2}\,|0_1,2_2\rangle$
- Coincidence in D1 and D2:

$$\boxed{P_{12}(\tau) = K\langle \hat{E}_1^{(-)}(t)\hat{E}_2^{(-)}(t+\tau)\hat{E}_2^{(+)}(t+\tau)\hat{E}_1^{(+)}(t)\rangle}$$

$$\hat{E}_1^{(+)}(t) = \sqrt{T}\hat{E}_{01}^{(+)}(t-\tau_1) + i\sqrt{R}\hat{E}_{02}^{(+)}(t-\tau_1+\delta\tau)$$

$$\hat{E}_2^{(+)}(t) = \sqrt{T}\hat{E}_{02}^{(+)}(t-\tau_1) + i\sqrt{R}\hat{E}_{01}^{(+)}(t-\tau_1-\delta\tau)$$

C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)

Two Photon Bunching - Hong Oh Mandel (HOM)

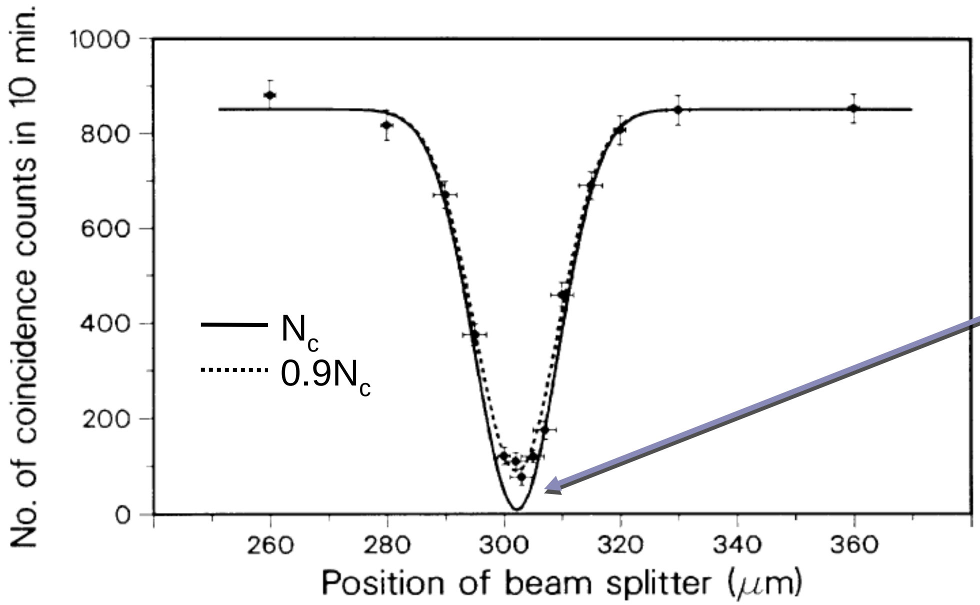
- Down converted photons are not monochromatic.
- $\Delta\omega$ is largely determined by band pass of filters.
- Coincidence rate is determined by distribution of frequencies.

$$N_c = C(T^2+R^2)\left[1 - \frac{2RT}{R^2+T^2} e^{-(\Delta\omega\,\delta\tau)^2}\right]$$

Gaussian

$$N_c = \begin{cases} C(R-T)^2 & \delta\tau \to 0 \\ C(T^2+R^2) & \delta\tau \to \infty \end{cases}$$

$$\frac{R}{T} = 0.95$$



bunching

Coherence Length

- A light beam is divided in an interferometer and reunited after Δt .
- Fringes are formed if $\Delta t \sim \dfrac{1}{\Delta v}$.
- Coherence length: $\Delta l = c\Delta t \sim \dfrac{c}{\Delta v}$

$$\boxed{\Delta l \sim \left(\frac{\bar{\lambda}}{\Delta\lambda}\right)\bar{\lambda}}$$

$$v = \frac{c}{\lambda}$$

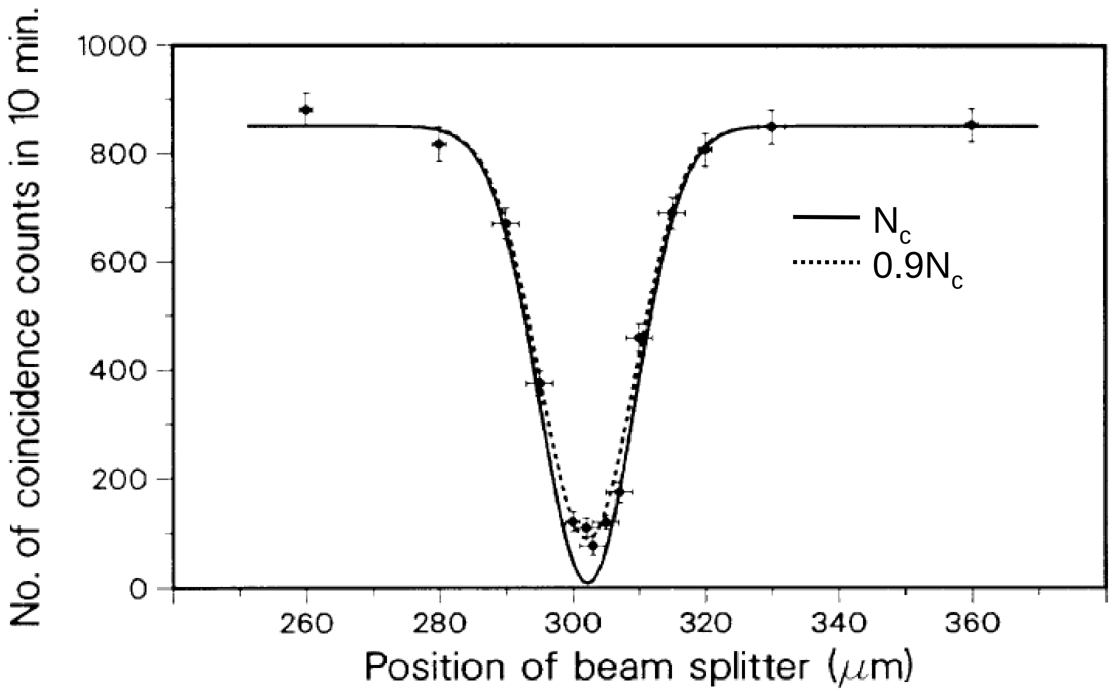
$$\Delta v \sim c\frac{\Delta\lambda}{\bar{\lambda}^2}$$

- Coherence time – time interval in which the phase of a EM wave is considered predictable.
- Δv ➔ different spatial periodicities.
- Increasing time delay ➔ maxima becomes out of step ➔ less well-defined fringe pattern.

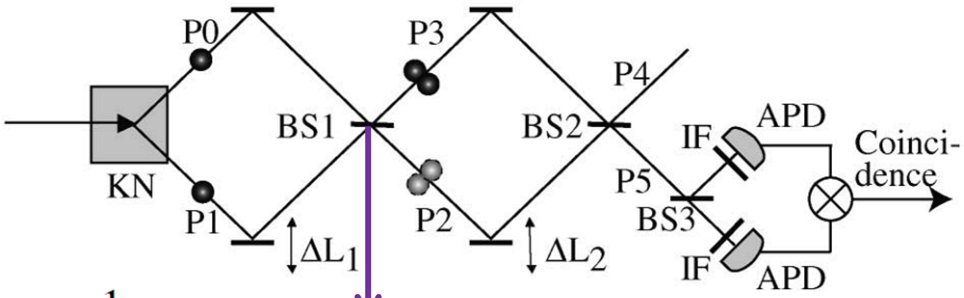
Two Photon Bunching - Hong Oh Mandel (HOM)

- Dip width provides measure of coherence time $\Delta t \approx 100 fs$
- Coherence time is consistent with filter band pass $\Delta t \sim \frac{1}{\Delta v}$

$$N_c = C(T^2 + R^2)\left[1 - \frac{2RT}{R^2 + T^2} e^{-(\Delta\omega\,\delta\tau)^2}\right] \quad N_c = \begin{cases} C(R-T)^2 & \delta\tau \to 0 \\ C(T^2 + R^2) & \delta\tau > \Delta t \end{cases}$$



Characteristics of Entangled Photon Pair



$$\frac{1}{\sqrt{2}}(|2\rangle_{P2}|0\rangle_{P3} + |0\rangle_{P2}|2\rangle_{P3})$$

- Counting rates at output ports:

$$R_i(|\psi_0, \psi_1\rangle) = \langle\psi_0, \psi_1|\hat{a}_i^\dagger \hat{a}_i|\psi_0, \psi_1\rangle$$

$$R_{ij}(|\psi_0, \psi_1\rangle) = \langle\psi_0, \psi_1|\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i|\psi_0, \psi_1\rangle$$

$$\begin{pmatrix}\hat{a}_4\\ \hat{a}_5\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}e^{i\phi_2} & 0\\ 0 & e^{i\phi_3}\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\hat{a}_0\\ \hat{a}_1\end{pmatrix}$$

- One photon interference:

$R_5(|0, 1\rangle) = \frac{1}{2}(1 - \cos\phi)$ ← oscillation period λ

$$\boxed{\phi \equiv \phi_2 - \phi_3 = 2\pi\Delta L_2/\lambda}$$

- Two-photon interference:

$R_{55}(|1, 1\rangle) = \frac{1}{2}(1 - \cos 2\phi)$

$R_{45}(|1, 1\rangle) = \frac{1}{2}(1 + \cos 2\phi)$

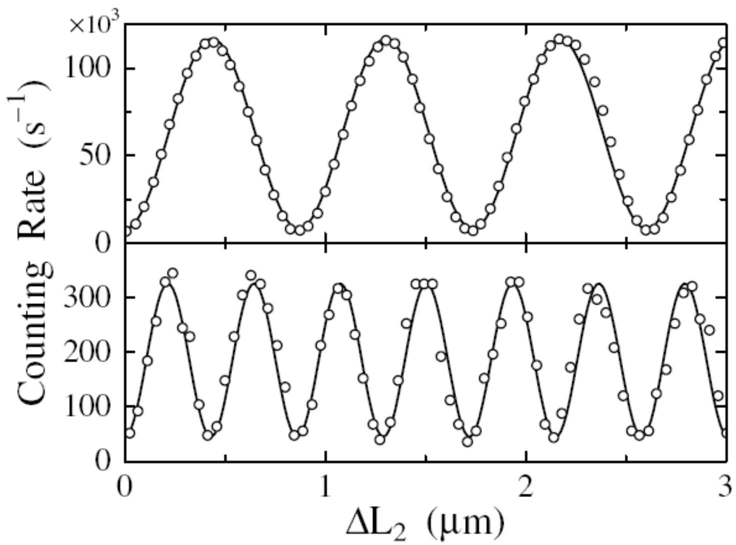
oscillation period λ/2

- Super Resolution

K. Edamatsu, R. Shimizu, and T. Itoh, Phys. Rev. Lett. **89**, 213601 (2002)

Characteristics of Entangled Photon Pair

- Reduced oscillation period in R_{55} : $R_{55}(|1,1\rangle) = \frac{1}{2}(1-\cos 2\phi)$
  - Photonic de Broglie wavelength λ/N (N=2) for biphoton state
  - Is not observed for a coherent state $R_{55}(|0,\alpha\rangle) = R_5^2 = \dfrac{|\alpha|^4}{4}(1-\cos\phi)^2$
  - Quantum nature of light
- Reduced oscillation period in R_{45} : $R_{45}(|1,1\rangle) = \frac{1}{2}(1+\cos 2\phi)$
  - Not a quantum effect (the two photons are split)
  - Can be observed for classical states: $R_{45}(|0,\alpha\rangle) = R_4 R_5 = \dfrac{|\alpha|^4}{8}(1-\cos 2\phi)$

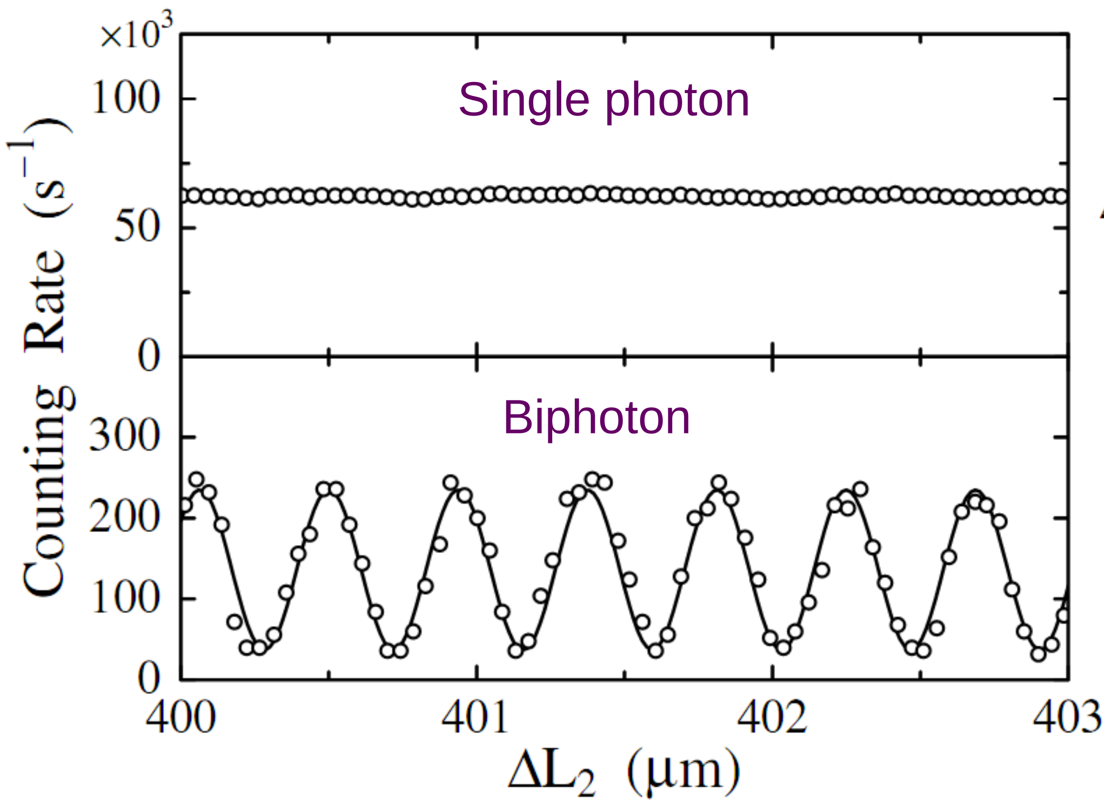


1 photon

2 photons

Coherence Length of Bi-photon

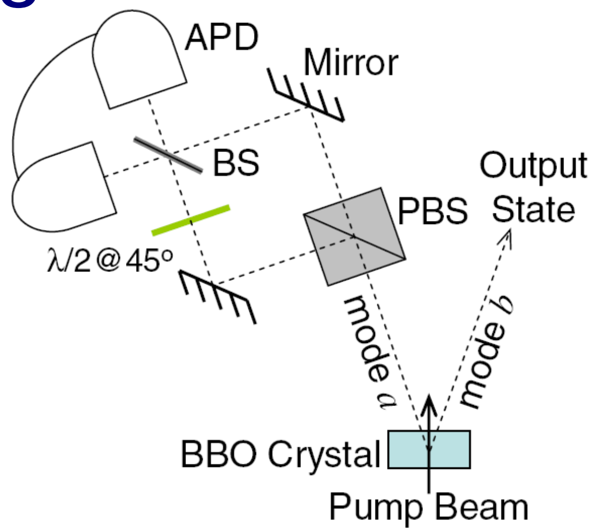
- Bi-photons have a much longer coherence length



$$\Delta l_{photon} = \frac{\lambda_c^2}{\Delta\lambda} \sim 70\mu m$$

$$\Delta l_{biphoton} = \frac{c}{\Delta\nu_{\mathrm{pump}}} \sim 400cm$$

Multi-photon Path Entanglement by Non-local Bunching



- Input state (PDC):

$$|\psi_2^-\rangle = \frac{1}{\sqrt{3}}(|2,0\rangle_a|0,2\rangle_b - |1,1\rangle_a|1,1\rangle_b + |\underset{\text{H}}{0},\underset{\text{V}}{2}\rangle_a|2,0\rangle_b)$$

- Measurement on mode a ➔ entangled state in mode b.

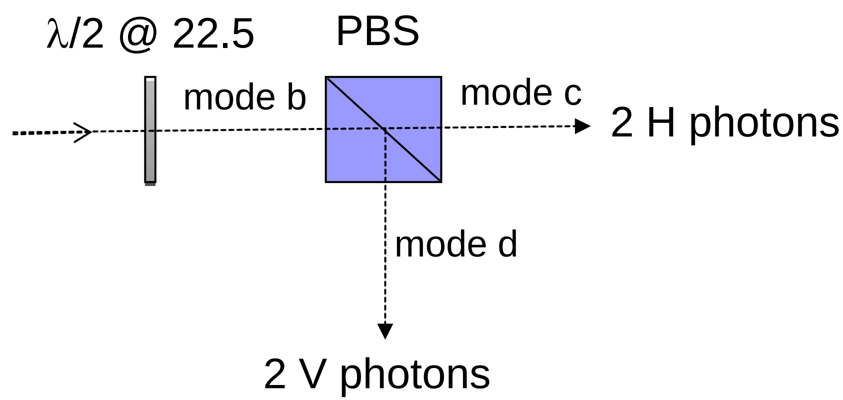
$$|\Psi_b\rangle = \frac{1}{\sqrt{2}}\left[|0,2\rangle_b + e^{i\Phi}|2,0\rangle_b\right]$$

- Coincidence counts arise from $|2,0\rangle_a|0,2\rangle_b$ and $|0,2\rangle_a|2,0\rangle_b$.

H. Eisneberg, J. Hodelin, G. Khoury, and D. Bouwmeester, Phys. Rev. Lett. **94**, 090502 (2005).

Multi-photon Path Entanglement by Nonlocal Bunching

- Photon number entanglement is between polarization modes.
- Beam splitter operation on two spatial modes $\longleftrightarrow$ Wave plate operation on two polarization modes.



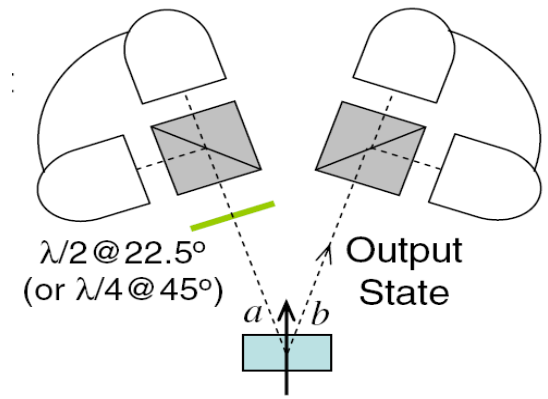
2 polarization modes

2 spatial modes

$$\frac{1}{\sqrt{2}}\left[\left|H^2\right\rangle_b + \left|V^2\right\rangle_b\right] \longrightarrow \frac{1}{\sqrt{2}}\left[\left|\left(\frac{H+V}{\sqrt{2}}\right)^2\right\rangle_b + \left|\left(\frac{H-V}{\sqrt{2}}\right)^2\right\rangle_b\right] \longrightarrow \left|0,2\right\rangle_{cd} + e^{i\Phi}\left|2,0\right\rangle_{cd}$$

Multi-photon Path Entanglement by Nonlocal Bunching

- Equivalent setup:



- $a_H^\dagger,\ a_V^\dagger$ modes bunch when rotated to 45º linear or right-left circular polarizations → No coincidence

$$|HV\rangle_a \Longrightarrow \left|\left(\frac{H+V}{\sqrt{2}}\right)\left(\frac{H-V}{\sqrt{2}}\right)\right\rangle_a = \left|\left(\frac{H^2-V^2}{2}\right)\right\rangle_a$$

- Going backwards:

$$a_h a_v \longrightarrow \begin{array}{lll} a_h^2 - a_v^2 & for & \frac{\lambda}{2} \\ a_h^2 + a_v^2 & for & \frac{\lambda}{4} \end{array}$$

$$\boxed{|\psi\rangle = (a_h^2 + e^{i\theta}a_v^2)|\psi_2^-\rangle = |0,0\rangle_a \otimes (|2,0\rangle_b + e^{i\theta}|0,2\rangle_b)}$$

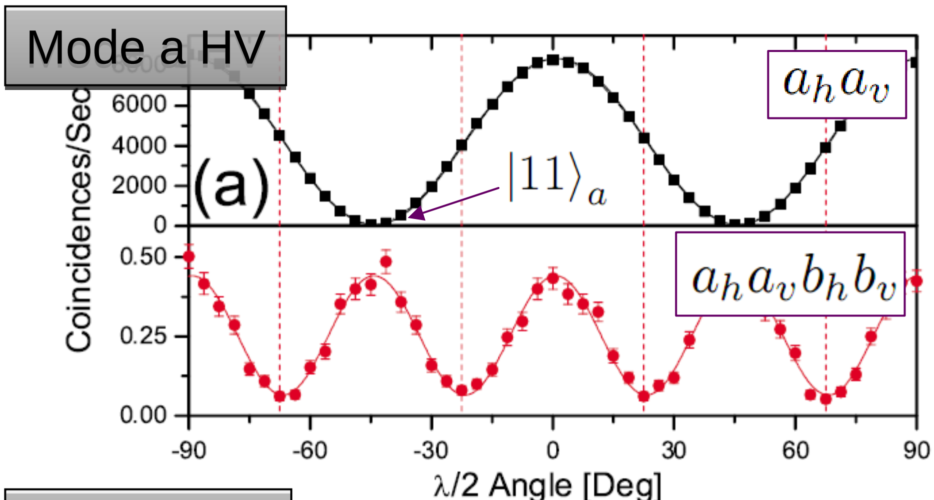
Multi-photon Path Entanglement by Nonlocal Bunching

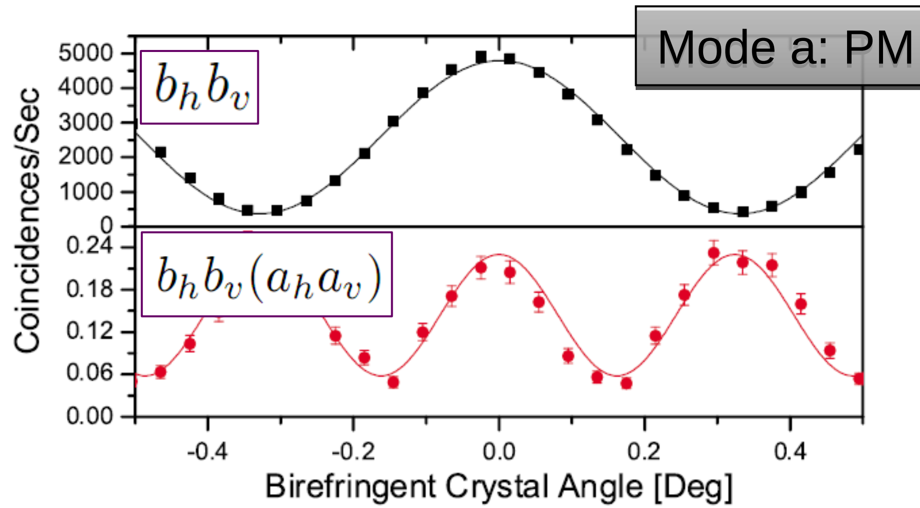
Path Entanglement

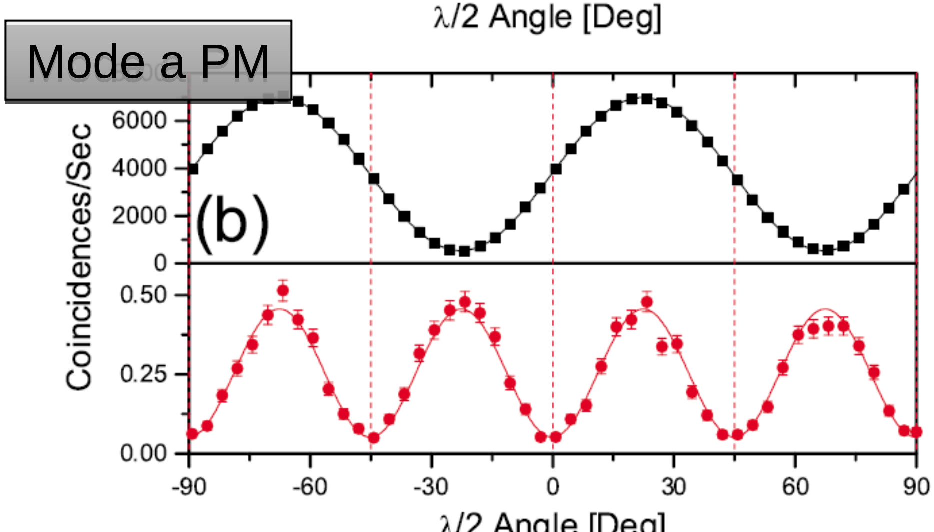
Mode b: HV→PM

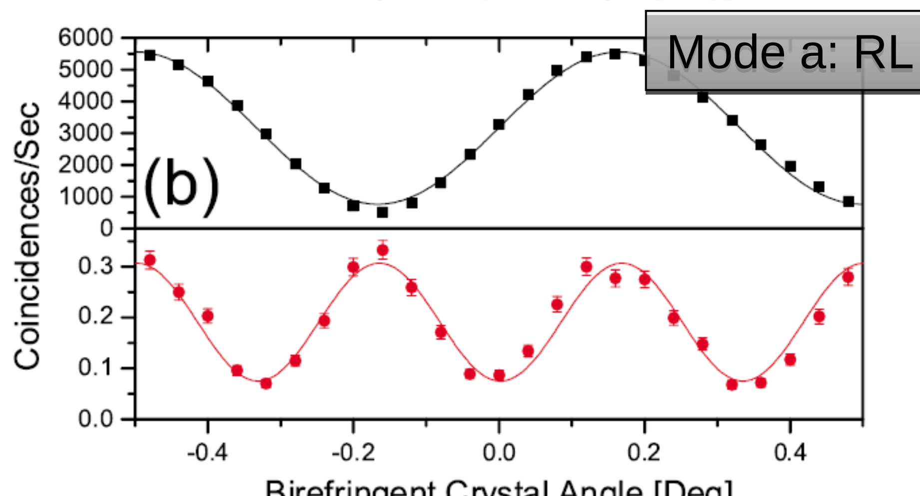
Coherence

Mode b: HV



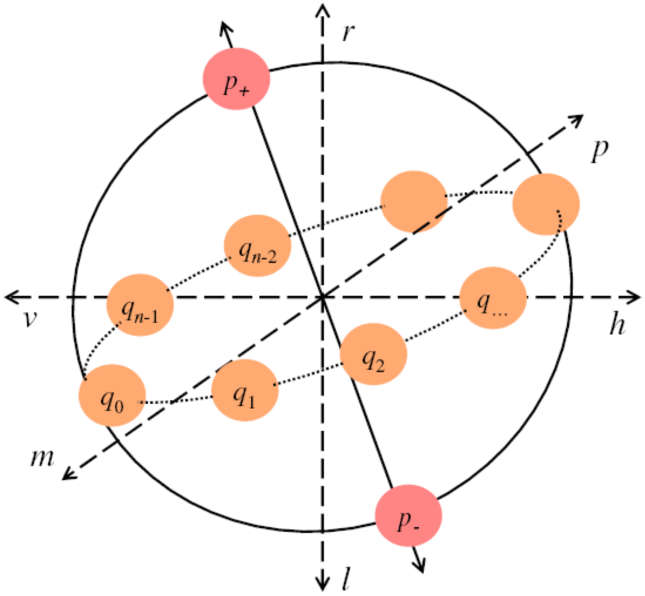






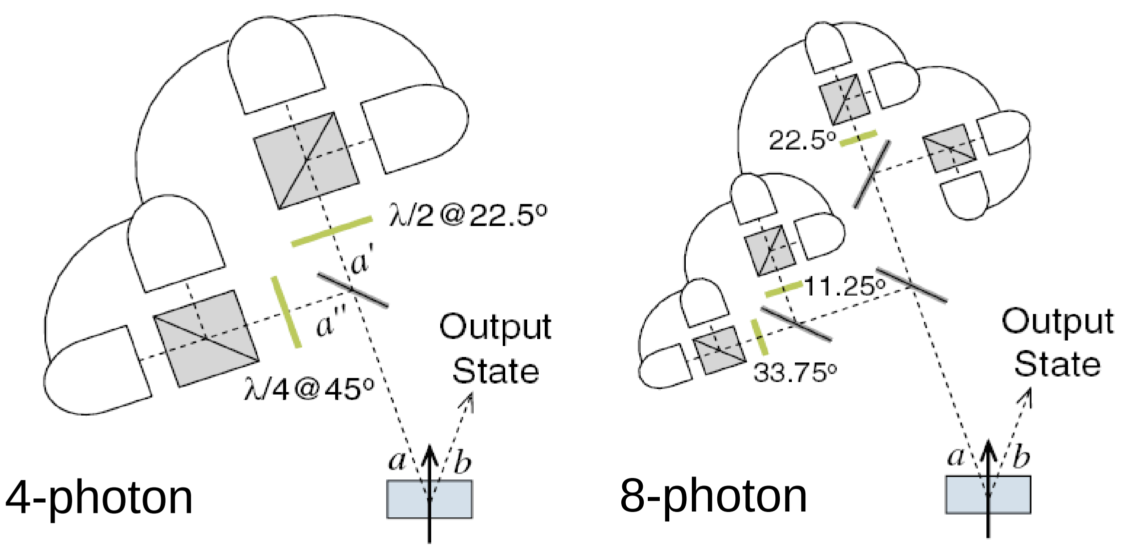
Multi-photon Path Entanglement by Nonlocal Bunching

- Expandable to N photons.
- Bunching in p basis requires N photons residing equidistant on the great circle perpendicular to p.

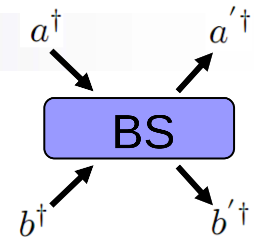


$$q_m = p_+ + e^{i[(2\pi m+\theta)/n]} p_-$$

$$\prod_{m=0}^{n-1} q_m = p_+^n - e^{i(n\pi+\theta)} p_-^n,$$



So far…



- Path entangled states: $\frac{1}{\sqrt{2}}\left[|P,Q\rangle_{AB} + e^{i\phi}|Q,P\rangle_{AB}\right]$
- Sources for *NOON* states:
  - Bunching (HOM): $|11\rangle \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}}\left[|20\rangle + |02\rangle\right]$
    - Non scalable $|22\rangle \xrightarrow{\text{BS}} \sqrt{\frac{3}{4}} \cdot \frac{|40\rangle + |04\rangle}{\sqrt{2}} + \frac{1}{\sqrt{4}}|22\rangle$
  - Non Local Bunching (measurement in a → bunching in b)
    - Heralded generation of entangled state
    - Expandable to large N
- Unique traits of bi-photons
  - Super Resolution – oscillations at twice the frequency of a single photon.

And Now…

- Super Phase Sensitivity
  - Enhancing measurement accuracy (beating the **S**tandard **Q**uantum **L**imit).
  - Reaching limits imposed by Heisenberg uncertainty laws.
- Super Resolution
  - X N faster oscillations
  - Beating the Rayleigh limit
    - Quantum Lithography

Super Phase Sensitivity

- Accuracy of phase measurement is limited:
  - Heisenberg uncertainty relation (fundamental)

number of photons phase

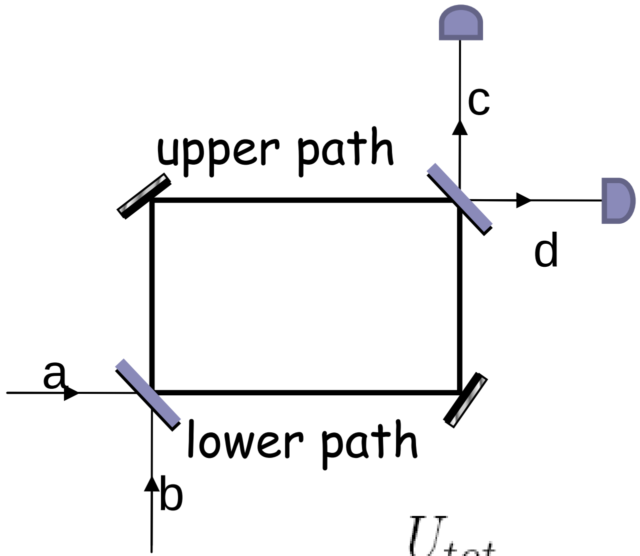
$$\boxed{\Delta N \Delta \Phi \geq 1}$$

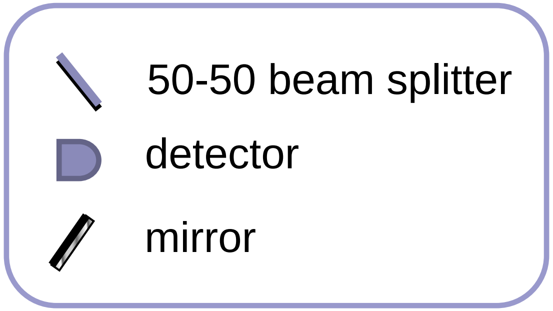
  - Other non fundamental:
    - Shot-noise – fluctuations due to discreteness of photons and Poisson statistics.
    - Standard Quantum Limit (SQL)

$$\boxed{\Delta \Phi = \frac{1}{\sqrt{N}}}$$

- Entangled Fock states → Heisenberg limit.

Sensitivity of Mach-Zehnder Interferometer



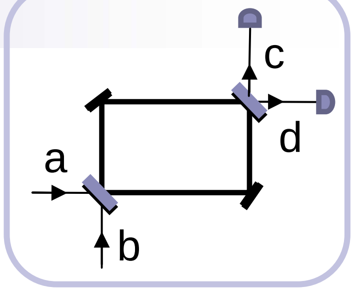


$$\begin{bmatrix} c^\dagger \\ d^\dagger \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} -i\left(e^{iu} - e^{iv}\right) & \left(e^{iu} + e^{iv}\right) \\ \left(e^{iu} + e^{iv}\right) & i\left(e^{iu} - e^{iv}\right) \end{bmatrix}}_{U_{tot}} \begin{bmatrix} a^\dagger \\ b^\dagger \end{bmatrix}$$

- The phase to be measured: $\Phi = \vec{k} \cdot l_{upper} - \vec{k} \cdot l_{lower}$

J. P. Dowling, Phys. Rev. A **57**, 4736 (1998)

Super Phase Sensitivity



- The Number of photons at the output ports:

$$\hat{N}_c = \hat{c}^\dagger \hat{c} = \hat{a}^\dagger \hat{a} \sin^2 \frac{\Phi}{2} + \hat{b}^\dagger \hat{b} \cos^2 \frac{\Phi}{2} + \frac{1}{2}\left(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}\right) \sin \Phi$$

$$\hat{N}_d = \hat{d}^\dagger \hat{d} = \hat{a}^\dagger \hat{a} \cos^2 \frac{\Phi}{2} + \hat{b}^\dagger \hat{b} \sin^2 \frac{\Phi}{2} - \frac{1}{2}\left(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}\right) \sin \Phi$$

- Let us define:

Conservation of particles

$$\hat{N} \equiv \hat{d}^\dagger \hat{d} + \hat{c}^\dagger \hat{c} = \hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a}$$

$$\hat{M} \equiv \hat{d}^\dagger \hat{d} - \hat{c}^\dagger \hat{c} = \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\right) \cos \Phi - \left(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}\right) \sin \Phi$$

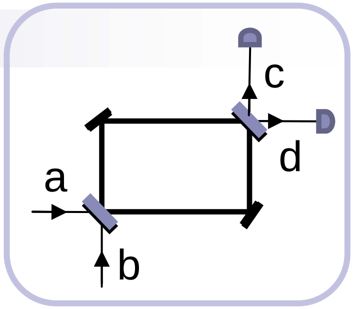
Phase information

- Quantum phase fluctuations:

$$\boxed{\Delta\Phi^2 = \frac{\Delta M^2}{\left|\frac{\partial\langle \hat{M}\rangle}{\partial\Phi}\right|^2}}$$

$$\Delta M^2 \equiv \left\langle \hat{M}^2 \right\rangle - \left\langle \hat{M} \right\rangle^2$$

Super Phase Sensitivity



- Defining: $\hat{M} \equiv \left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)\cos\Phi - \left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right)\sin\Phi$

Difference operator: $\hat{X} \equiv \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$

Exchange operator: $\hat{Y} \equiv \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}$

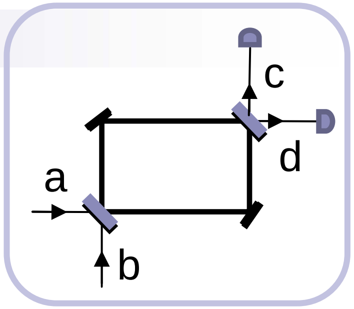
$$\Delta X^2 \equiv \left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle^2$$

$$\Delta Y^2 \equiv \left\langle \hat{Y}^2 \right\rangle - \left\langle \hat{Y} \right\rangle^2$$

- Phase Fluctuations:

$$\Delta\Phi^2 = \frac{\Delta X^2\cos^2\Phi - \left(\left\langle \hat{X}\hat{Y} \right\rangle - 2\left\langle \hat{X} \right\rangle\left\langle \hat{Y} \right\rangle + \left\langle \hat{Y}\hat{X} \right\rangle\right)\sin\Phi\cos\Phi + \Delta Y^2\sin^2\Phi}{\left|\left\langle \hat{X} \right\rangle\sin\Phi + \left\langle \hat{Y} \right\rangle\cos\Phi\right|^2}$$

One Input Port – Fock State Input



- N Fock-state particles are incident upon port A:

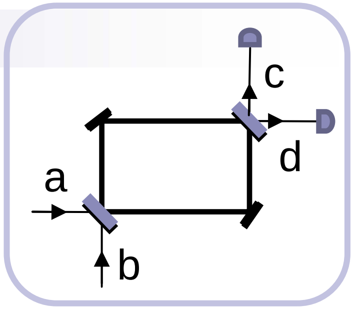
$$\boxed{|\Psi\rangle_I = |N\rangle_A |0\rangle_B}$$

- $$\begin{aligned}\left\langle \hat{X} \right\rangle_I &= {}_B\langle 0|_A\langle N|\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}|N\rangle_A|0\rangle_B \\ &= {}_A\langle N|\hat{a}^\dagger \hat{a}|N\rangle_A \; {}_B\langle 0|0\rangle_B - {}_A\langle N|N\rangle_A \; {}_B\langle 0|\hat{b}^\dagger \hat{b}|0\rangle_B \\ &= N \cdot 1 - 1 \cdot 0 = N\end{aligned}$$
- $\left\langle \hat{X}^2 \right\rangle_I = N^2$ $\left\langle \hat{Y}^2 \right\rangle_I = N$ $\left\langle \hat{Y} \right\rangle_I = 0$ $\left\langle \hat{X}\hat{Y} \right\rangle_I = \left\langle \hat{Y}\hat{X} \right\rangle_I = 0$
- $\Delta X^2 = 0, \Delta Y^2 = N$
- Phase uncertainty:

$$\Delta\,\Phi_I^2 = \frac{N \sin^2 \Phi}{|N \sin \Phi|^2} = \frac{1}{N} \longrightarrow \boxed{\Delta\,\Phi_I = \frac{1}{\sqrt{N}}}$$

Standard Quantum Limit

Two Input Ports – Fock State Input



- Fock-state particles are incident upon both ports

$$\boxed{|\Psi\rangle_{II} \equiv \frac{1}{\sqrt{2}}\{|N_+\rangle_A |N_-\rangle_B + |N_-\rangle_A |N_+\rangle_B\}}$$

$$N_\pm \equiv (N \pm 1)/2$$

- Minimum at Φ=0: $\Delta\Phi^2|_{\Phi=0} = \frac{\Delta X^2}{|\langle Y\rangle|}$

order of unity

- $\langle \hat{X}\rangle_{\mathrm{II}} = 0$ $\quad \langle \hat{X}^2\rangle_{\mathrm{II}} = (N_+ - N_-)^2 = 1$ $\quad \Delta X^2_{\mathrm{II}} = 1$ $\quad \langle \hat{Y}\rangle_{\mathrm{II}} = N_+$

- Phase uncertainty:

$$\boxed{\Delta\,\Phi_{II} = \frac{1}{N_+} = \frac{2}{N+1} \sim \frac{1}{N}}$$

Heisenberg Limit!

- Entangled Fock states → Heisenberg limit.

One Input Port - Coherent State Input

$$|\alpha|^2 = \bar{n}$$

- Input State: $|\Psi\rangle_{\alpha I} = |\alpha\rangle_A |0\rangle_B$
- Fluctuations in photon number: $\Delta X^2 = \bar{n}$

$$\Delta\Phi^2_{\alpha I} = \frac{1}{\bar{n}\sin\Phi}$$

- Phase uncertainty is dependent on choice of phase
- Minimal phase uncertainty:

$$\boxed{\Delta\Phi_{\alpha I}\big|_{\phi=\frac{\pi}{2}} = \frac{1}{\sqrt{\bar{n}}}}$$ Standard Quantum Limit

- So far, equivalent to Fock state input.

Two Input Ports - Coherent State Input

- Input State: $|\Psi\rangle_{CII} \equiv \frac{1}{\sqrt{2}} \{ |\alpha_+\rangle_A |\alpha_-\rangle_B + |\alpha_-\rangle_A |\alpha_+\rangle_B \}$ $\boxed{|\alpha|^2 = \bar{n}}$
- Coherent-state number fluctuations are of the same order as $\bar{n}$:

$$\Delta X^2 = O(\alpha^2) = O(\bar{n})$$

- Minimal phase uncertainty:

$$\boxed{\Delta \, \Phi_{CII}|_{\Phi=0} = O\left(\frac{1}{\sqrt{\bar{n}}}\right)}$$

Standard Quantum Limit

- Fluctuations in photon number →Standard Quantum Limit cannot be beaten with coherent states.

Path entangled states increase sensitivity

Super Resolution

- A *NOON* state can be made *to* oscillate N times faster than single photon:

Φ for every photon

NOON $\frac{1}{\sqrt{2}}\left[|N,0\rangle + |0,N\rangle\right] \xrightarrow{\text{Phase}} \frac{1}{\sqrt{2N!}}\left(a^{\dagger N} + e^{iN\phi} b^{\dagger N}\right)|0\rangle$

$$\xrightarrow{\text{BS}} \frac{1}{2^{\frac{N+1}{2}}} \cdot \frac{1}{\sqrt{N!}} \left(\left(a'^{\dagger} + b'^{\dagger}\right)^{N} + e^{iN\phi}\left(b'^{\dagger} - a'^{\dagger}\right)^{N}\right)|0\rangle$$

$$= \frac{1}{2^{\frac{N+1}{2}}} \cdot \left[\sum_{M=odd} \eta_{MN}\left[1 - e^{iN\phi}\right] + \sum_{M=even} \eta_{MN}\left[1 + e^{iN\phi}\right]\right]|M, N-M\rangle$$

$\sin\frac{N\phi}{2}$ $\qquad$ $\cos\frac{N\phi}{2}$

$$\boxed{P(|M, N-M\rangle) = \eta_{MN}\frac{1 \pm \cos(N\phi)}{2}}$$

Super Resolution & Super Sensitivity

$$\boxed{P(|M,N-M\rangle)=\eta_{MN}\frac{1\pm\cos(N\phi)}{2}}$$

- Phase uncertainty: $\Delta\Phi^2=\dfrac{\Delta A^2}{\left|\dfrac{\partial\langle\hat{A}\rangle}{\partial\Phi}\right|^2}$

Denominator is increased
by super resolution

- Super Resolution can improve phase sensitivity!
- Super Resolution $\not\rightarrow$ Super Phase Sensitivity

$$\Delta\,\Phi = \frac{1}{\sqrt{N}}$$

Requirement for Beating the SQL

- Fringe visibility $\mathfrak{V}$:
  - Sharpness of interference fringes
  - A measure of coherence.

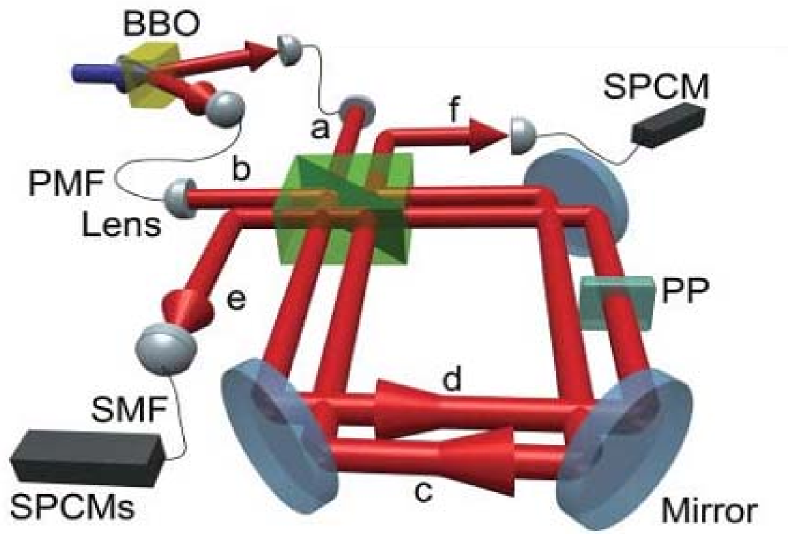
$$\mathfrak{V} = \frac{\langle I\rangle_{max} - \langle I\rangle_{min}}{\langle I\rangle_{max} + \langle I\rangle_{min}}$$

- Experimentally fringes show: $\eta\frac{1 - \mathfrak{V}\cos(N\Phi)}{2}$
- The phase uncertainty: $\Delta\Phi^2 = \frac{\Delta A^2}{\left|\frac{\partial\langle\hat{A}\rangle}{\partial\Phi}\right|^2} \geq \frac{1}{[\mathfrak{V}\eta N]^2}$
- SQL is beat if $\mathfrak{V} \geq \mathfrak{V}_{threshold}$

$$\boxed{\mathfrak{V}_{threshold} = \frac{1}{\sqrt{\eta N}}}$$

K. J. Resch, K. Pregnell, R. Prevedel, A. Gilchrist, G. Pryde, J. O'Brien, and A. White, Phys. Rev. Lett. **98**, 223601 (2007).

Beating the SQL with Four-Entangled Photons



- $|22\rangle_{ab} \xrightarrow{\text{BSI}} \sqrt{3/4}\left(|40\rangle_{cd} + |04\rangle_{cd}\right)/\sqrt{2} + {}^{1}/_{\sqrt{4}}|22\rangle_{cd}$

  BSII

  $|13\rangle_{ef} \qquad |31\rangle_{ef}$

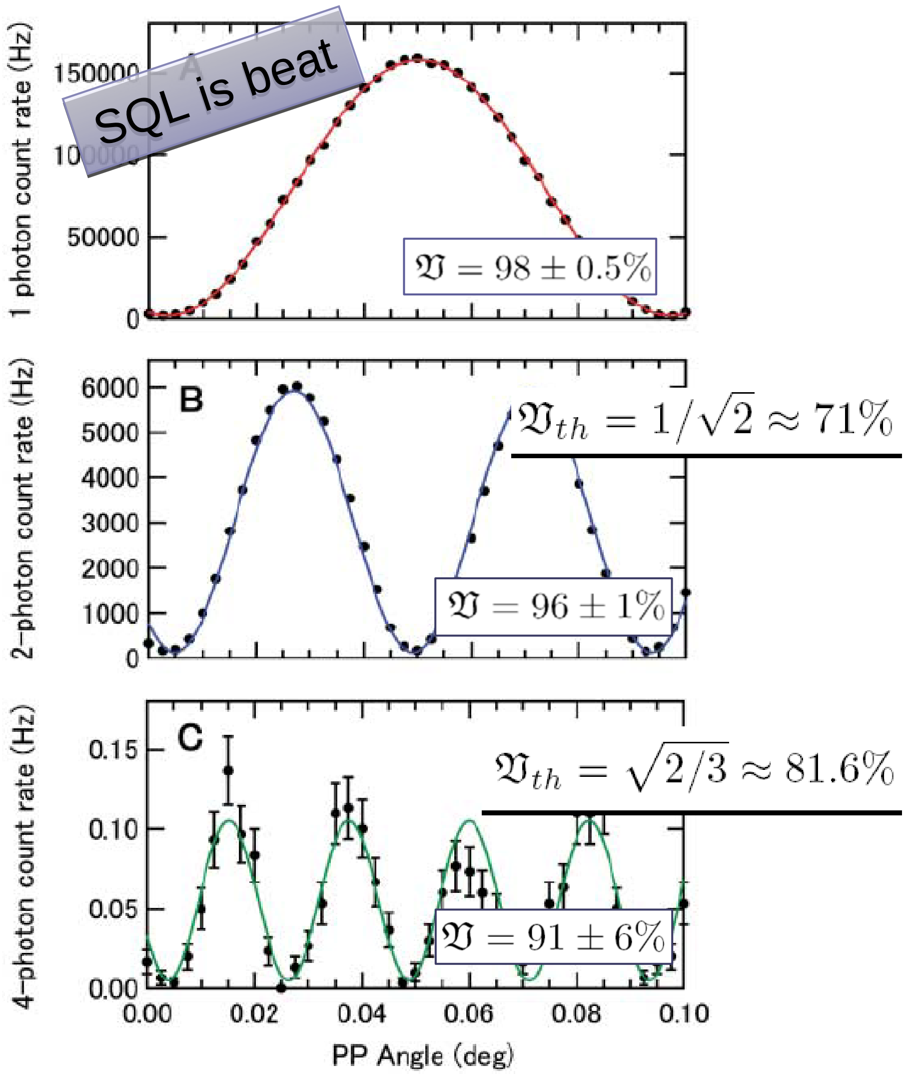
- Probability for four-photon path entangled state:

$$P_{3ef} = {}^{3}/_{8}\ (1 - \cos 4\phi)/2$$

T. Nagata, R. Okamokt, O. J.L., K. Sasaki, and T. S., Science **316**, 726 (2007).

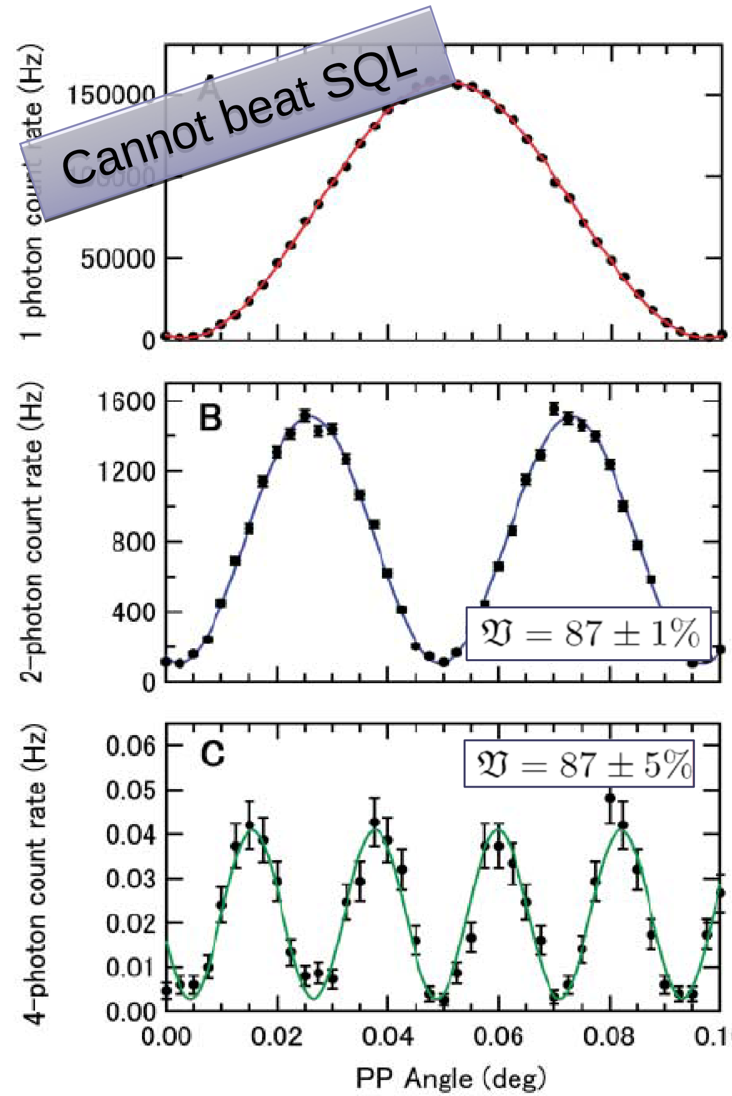
Input: $|22\rangle_{ab}$

$\boxed{\mathfrak{V} \geq \mathfrak{V}_{th}}$



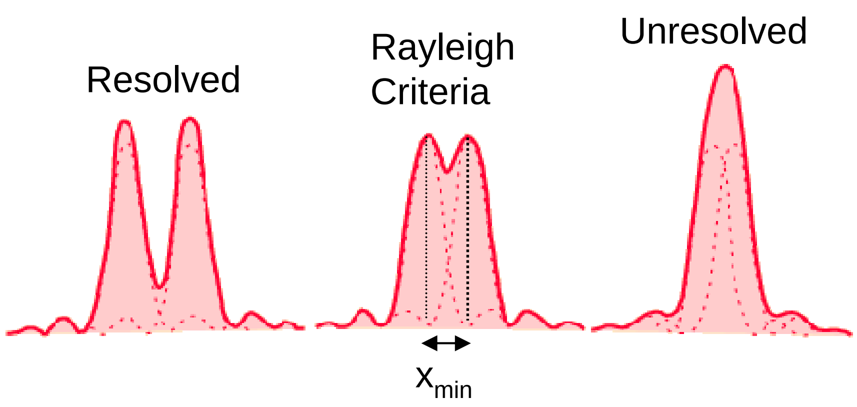
Input: $|1111\rangle_{a^t a^{t'} b^t b^{t'}}$

$\boxed{\mathfrak{V}_{th} = \sqrt{2}}$



Quantum Interferometric Optical Lithography

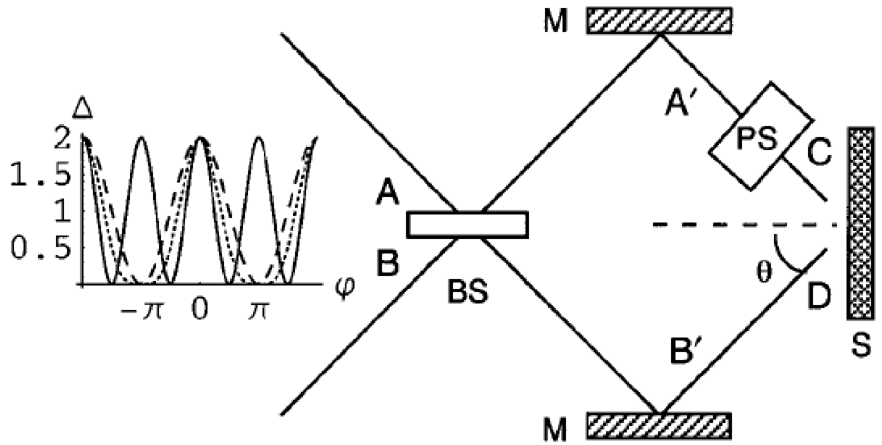
- Rayleigh Criteria for Minimum Resolvable Detail
  - Light with two monochromatic components.
  - Two displaced maxima in the interference pattern.



- Best resolution when principal maximum of one component coincides with the first minimum of the other.

Quantum Interferometric Optical Lithography

- Classical Lithography:



$$\phi = k \cdot x$$

- Exposure dose at the substrate $\Delta(x) = 1 + \cos(2\phi)$

$$\phi^{\min} = \frac{\pi}{2} \Longrightarrow \boxed{x_{\min} = \frac{\lambda}{2}}$$

- Number of elements writable on a surface ~ $1/x_{min}^2$.
- Super Resolution enables a resolution N times better (λ/N).

A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams, and J. P. Dowling, Phys. Rev. Lett. **85**, 2733 (2000)

Quantum Interferometric Optical Lithography

- Can be used to better approximate functions.

$$|\psi_{N,P}(\varphi)\rangle = (e^{iP\varphi}|N-P\rangle_A|P\rangle_B + e^{i(N-P)\varphi}|P\rangle_A|N-P\rangle)$$

A Basis is formed:

$$|\Psi_{N,P,P'}\rangle = \alpha_{N,P}|\psi_{N,P}\rangle + \beta_{N,P'}|\psi_{N,P'}\rangle$$

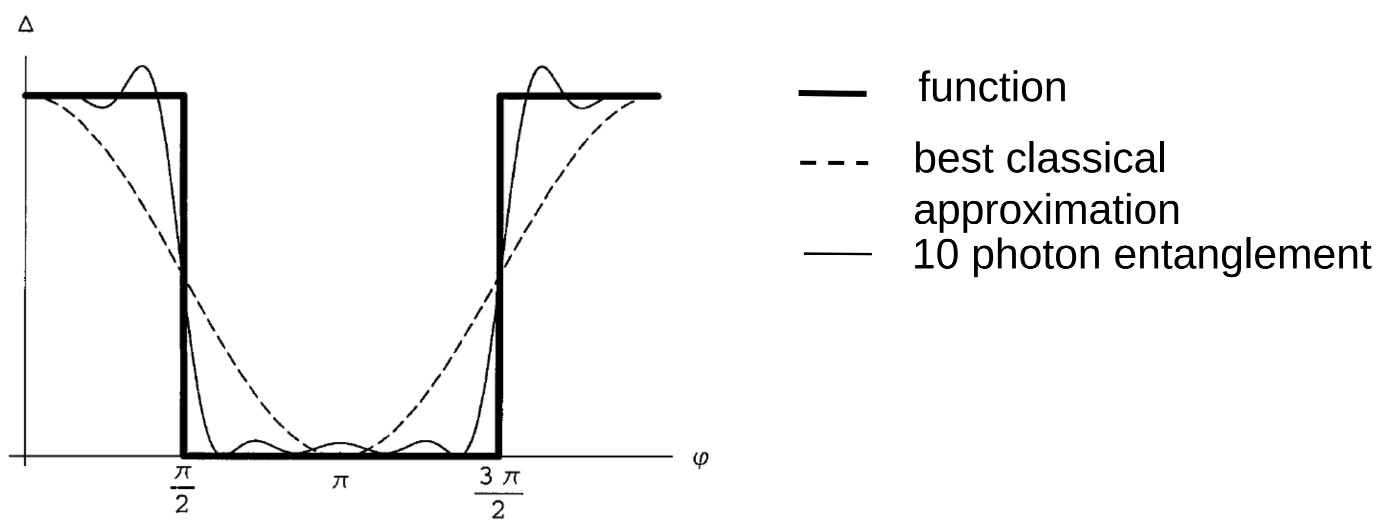
An arbitrary function can be expanded in terms of

$$\Delta_N^{P,P'} \equiv \langle\Psi_{N,P,P'}|\hat{\delta}_N|\Psi_{N,P,P'}\rangle$$

$$\hat{\delta} \equiv (\hat{e}^\dagger)^N(\hat{e})^N/N!$$

$P \in \{0,1,\ldots,5\}$

$N = 10$



Conclusion

- Creation of Path Entangled States (Bunching/Non-local Bunching)
- Characteristics of N-entangled photons:
  - Super Phase sensitivity – Beating the SQL $\Delta\Phi = \frac{1}{\sqrt{N}}$
  - Super Resolution
    - Can enhance phase sensitivity
    - Quantum Lithography

Heisenberg Limit: $\Delta n \Delta \Phi \geq 1$

- Dual nature of light:
  - Particles - number of photons, n
  - Waves – phase, Φ
- Electromagnetic field quantization ➔

  Annihilation & creation operators: $C_k^{\alpha}, C_k^{\dagger\alpha}$ where $\left[\hat{C_k^{\alpha}}, \hat{C_{k'}^{\beta}}^{\dagger}\right] = \delta_{kk'}\delta_{\alpha\beta}$
- A single mode phase can be defined as (Dirac):

$$\boxed{\hat{C} = e^{i\hat{\Phi}}\sqrt{\hat{N}}}$$

  - $\hat{N}$ - Number operator with eigenstates $|n\rangle$
  - $\sqrt{\hat{N}} = \sum_{n=0}^{\infty} \sqrt{n}\,|n\rangle\langle n|$
  - $\hat{N} = \sqrt{\hat{N}}\sqrt{\hat{N}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \sqrt{n}\,|n\rangle\langle n|m\rangle|m\rangle = \sum_{n=0}^{\infty} n\,|n\rangle\langle n|$

Heisenberg Limit: $\Delta n \Delta \Phi \geq 1$

- $\hat{C}^\dagger \hat{C}$ is the Number operator (If $\hat{\Phi}$ is hermitian):

$$\hat{C}^\dagger \hat{C} = \sqrt{\hat{N}} e^{-i\hat{\Phi}} e^{i\hat{\Phi}} \sqrt{\hat{N}} = \hat{N}$$

- The requirement $\left[\hat{C}, \hat{C}^\dagger\right] = 1$ constitutes $\left[\hat{\Phi}, \hat{N}\right]$

$$\begin{aligned} \left[\hat{C}, \hat{C}^\dagger\right] &= 1 \\ e^{i\hat{\Phi}} \hat{N} - \hat{N} e^{i\hat{\Phi}} &= e^{i\hat{\Phi}} \\ \sum_m \frac{i}{m!} \left[\hat{\Phi^m} \hat{N} - \hat{N} \hat{\Phi^m}\right] &= e^{i\hat{\Phi}} \end{aligned}$$

$$\boxed{\exp(i\hat{\Phi}) = \sum_m \frac{i\hat{\Phi}^m}{m!}}$$

  - If $\left[\hat{\Phi}, \hat{N}\right] = -i$ ➔ $\hat{\Phi}^m \hat{N} - \hat{N} \hat{\Phi}^m = -im\hat{\Phi}^{m-1}$ ➔ $\left[\hat{C}, \hat{C}^\dagger\right] = 1$
- Using $\left[\hat{A}, \hat{B}\right] = c$ ➔ $\Delta \hat{A} \Delta \hat{B} \geq |c|$

$$\boxed{\Delta n \Delta \Phi \geq 1}$$