Quantum Cloning

We shall discuss three major points:

- The Problem with Perfect Cloning
- The necessity of Universal Cloning
- The limit of Optimal Cloning

(Naïve) No Cloning Theorem

- Assuming the copying machine is unaffected by the process, and exists only as a unitary operation –

$$U\left|\psi\right\rangle\left|e\right\rangle=\left|\psi\psi\right\rangle \qquad U\left|\phi\right\rangle\left|e\right\rangle=\left|\phi\phi\right\rangle$$

due to the unitarity

$$\left\langle\psi\,|\,\phi\right\rangle=\left\langle\psi\,|U^{\dagger}U\,|\,\phi\right\rangle=\left\langle\psi\psi\,|\,\phi\phi\right\rangle=\left\langle\psi\,|\,\phi\right\rangle^{2}$$

(NO) Cloning Theorem $|A_0\rangle \otimes |\psi\rangle \overset{?}{\rightarrow} |A_f\rangle \otimes |\psi\psi\rangle$

$$|A_0\rangle \otimes \left[\alpha|V\rangle + \beta|H\rangle\right] \rightarrow |A_0\rangle \otimes \alpha|V\rangle + |A_0\rangle \otimes \beta|H\rangle =$$

$$= \alpha|A_V\rangle \otimes |VV\rangle + \beta|A_H\rangle \otimes |HH\rangle \overset{*}{=}$$

$$|A_f\rangle \otimes \left[\alpha|VV\rangle + \beta|HH\rangle\right]$$

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* In order to receive a pure state $|A_f\rangle$ must be independent of $|\psi\rangle$: $|A_H\rangle \overset{!}{=} |A_V\rangle \square |A_f\rangle$

$|A_0\rangle \otimes |H\rangle \rightarrow |A_H\rangle \otimes |HH\rangle$ $\quad$ $|A_0\rangle \otimes |V\rangle \rightarrow |A_V\rangle \otimes |VV\rangle$

(NO) Cloning Theorem $|A_0\rangle \otimes |\psi\rangle \overset{?}{\rightarrow} |A_f\rangle \otimes |\psi\psi\rangle$

$$|A_0\rangle \otimes \left[\alpha|V\rangle + \beta|H\rangle\right] \rightarrow |A_0\rangle \otimes \alpha|V\rangle + |A_0\rangle \otimes \beta|H\rangle =$$

$$= \alpha|A_V\rangle \otimes |VV\rangle + \beta|A_H\rangle \otimes |HH\rangle \overset{*}{=}$$

$$|A_f\rangle \otimes \left[\alpha|VV\rangle + \beta|HH\rangle\right]$$

$$\text{but } |\psi\psi\rangle = \frac{1}{\sqrt{2}}(\alpha V^\dagger + \beta H^\dagger)^2|0\rangle =$$

$$\frac{1}{\sqrt{2}}(\alpha^2|VV\rangle + \alpha\beta\left[|HV\rangle + |VH\rangle\right] + \beta^2|HH\rangle$$

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$|A_0\rangle \otimes |H\rangle \rightarrow |A_H\rangle \otimes |HH\rangle \qquad |A_0\rangle \otimes |V\rangle \rightarrow |A_V\rangle \otimes |VV\rangle$

The Tool Box

Fidelity $F(\rho_i, \rho_f) \equiv tr(\sqrt{\rho_i}\rho_f\sqrt{\rho_i})$

Von Neumann Entropy $S \equiv -tr(\rho \ln \rho)$

Hilbert Schmidt norm $|A|^2 \equiv tr(A^\dagger A) \Rightarrow |\rho_o - \rho_i|^2 = tr(\rho_o - \rho_i)^2$

The Grinder

$$|\psi\rangle = \sin(\theta/2)|V\rangle + e^{i\phi}\cos(\theta/2)|H\rangle$$

$$\rho_i = \sin^2(\theta/2)|V\rangle\langle V| + \cos^2(\theta/2)|H\rangle\langle H| + \frac{1}{2}\sin\theta\left(|H\rangle\langle V| + |V\rangle\langle H|\right)$$

$\rho_o = tr_i(\rho_{oo})$ since ρ_i is a pure state $F = \langle\psi|\rho_o|\psi\rangle$

$$\bar{F} = \int_{\substack{Poincare\\Sphere}} d\Omega \cdot F$$

Several Schemes for Cloning

- Measurement – Choosing a specific axis

$$\rho_o = |H\rangle\langle H| \quad \bar{F} = \frac{1}{2}$$

- Simple QCM

$$\rho_o = \sin^2(\theta/2)|V\rangle\langle V| + \cos^2(\theta/2)|H\rangle\langle H| \quad \bar{F} = \frac{2}{3}$$

Universality – can we do better?

Although the Simple QCM copied eigenstates perfectly, the average fidelity was poor. Consider the following Cloning scheme:

$$|0\rangle_a|Q\rangle_x \rightarrow |0\rangle_a|0\rangle_b|Q_0\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|Y_0\rangle_x$$

$$|1\rangle_a|Q\rangle_x \rightarrow |1\rangle_a|1\rangle_b|Q_1\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|Y_1\rangle_x$$

Under the constraints:

$${}_x\langle Q_1|Y_0\rangle_x = {}_x\langle Q_0|Y_1\rangle_x \neq 0 = \xi$$

$${}_x\langle Q_i|Q_i\rangle_x + 2\,{}_x\langle Y_i|Y_i\rangle_x = 1, \quad i = 0,1$$

$${}_x\langle Q_i|Y_i\rangle_x = {}_x\langle Y\ |Y_1\rangle_x = {}_x\langle Y\ |Y_0\rangle_x = 0$$

$$\rho_o = \left[(1-\xi)\sin^2(\theta/2) + \xi\cos^2(\theta/2)\right]|H\rangle\langle H|$$
$$+\left[(1-\xi)\cos^2(\theta/2) + \xi\sin^2(\theta/2)\right]|V\rangle\langle V|$$
$$+\left[(1/2-\xi)\sin(\theta)\right]\left(|H\rangle\langle V| + |V\rangle\langle H|\right)$$

Demanding $\frac{\partial F}{\partial \theta} = 0$ gives $\xi = \frac{1}{6}$, and further calculation gives

$$F = \frac{5}{6}$$

Optimal Quantum Cloning Machines

A $U_{1,M}$ QCM acting on an arbitrary pure state gives

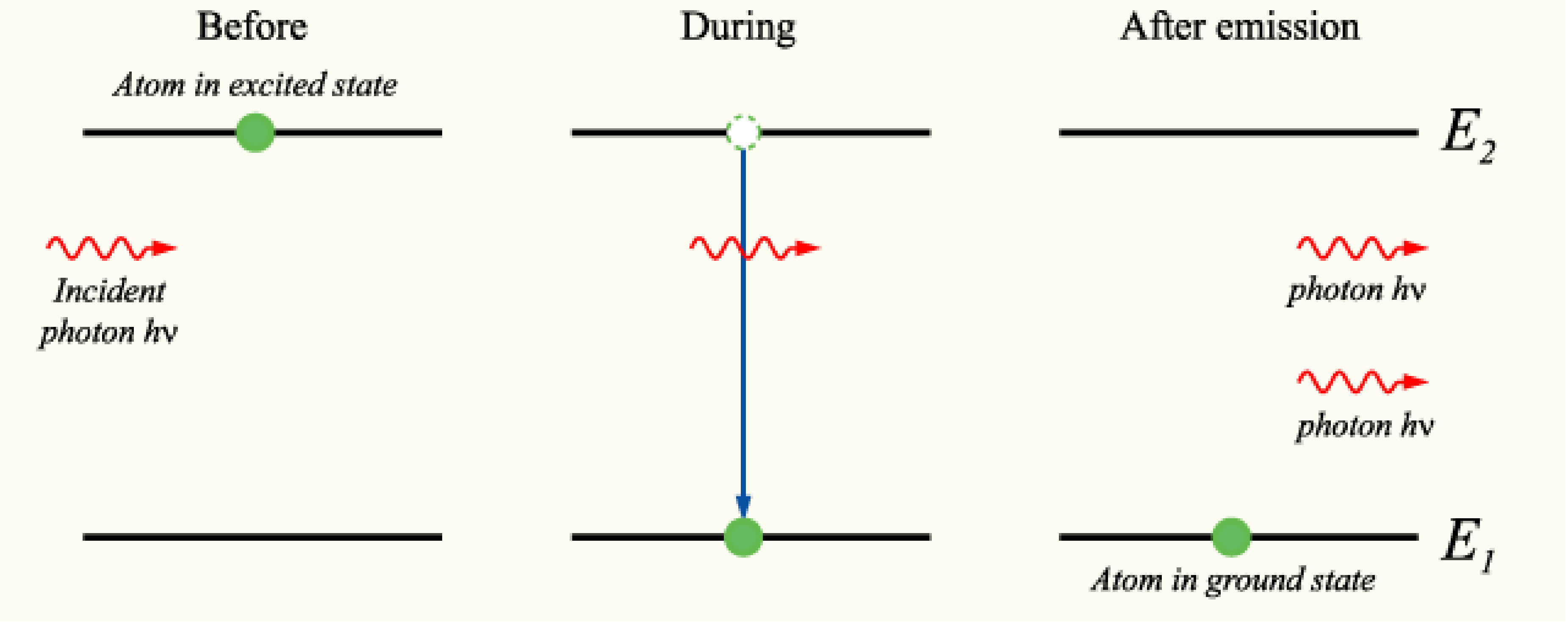
$$\left|\psi\right\rangle \otimes \left|S_0\right\rangle \rightarrow \sum_{j=0}^{M-1} \sqrt{P_N(j)} \left|(M-j)\psi, j\psi^{\perp}\right\rangle \otimes \left|S_j(\psi)\right\rangle$$

$P_N(j)$ (probability of having j errors in N clones) = $\dfrac{2(N-j)}{N(N+1)}$

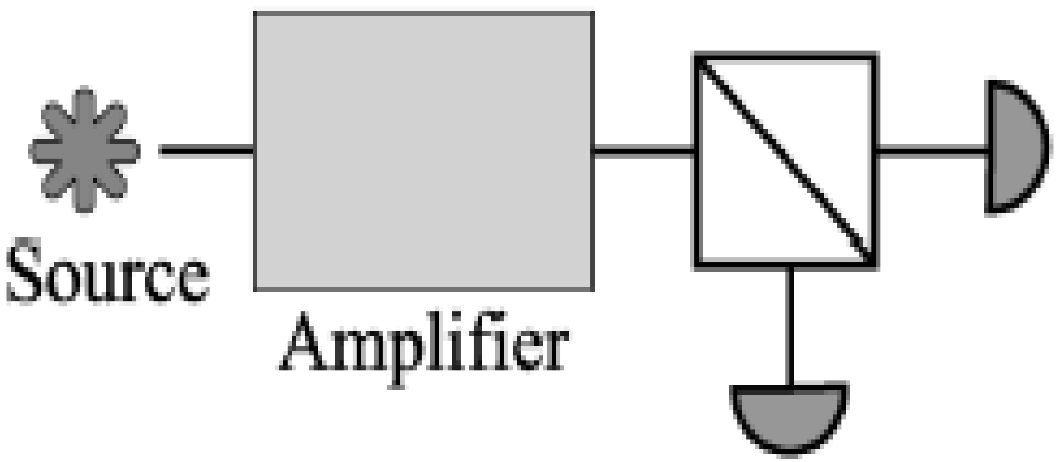
$\left|S_j\right\rangle$ The state of the cloning system

$$F = \sum_{j=0}^{M-1} P_{M-1}(j) = \sum_{j=0}^{M-1} P_M(j) \binom{M-1}{j} \Big/ \binom{M}{j} = \frac{2}{3} + \frac{1}{3M}$$

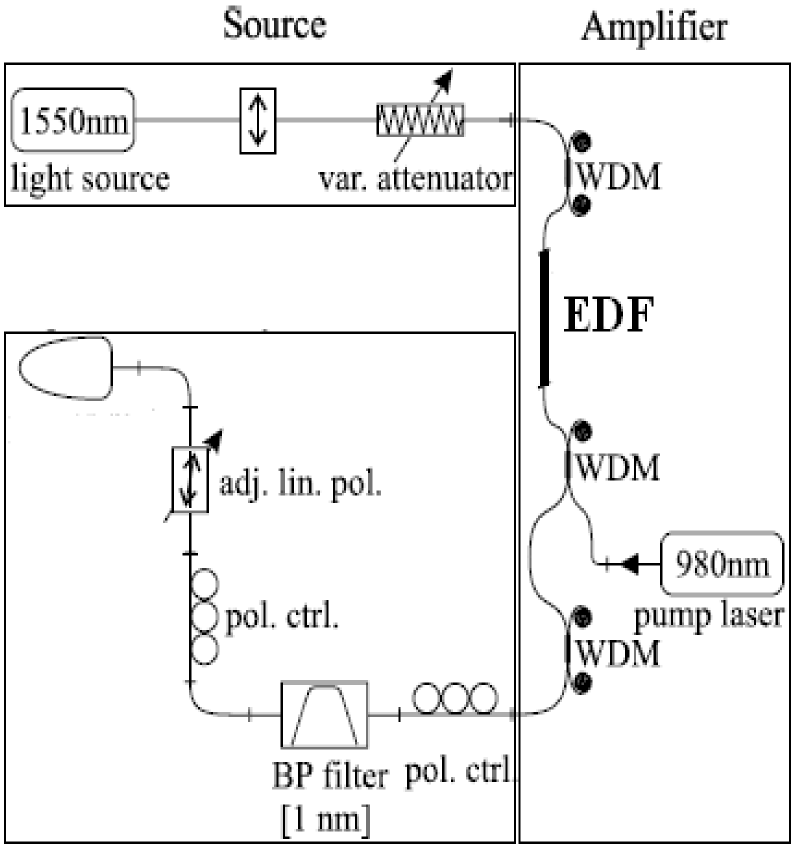
Stimulated Emission

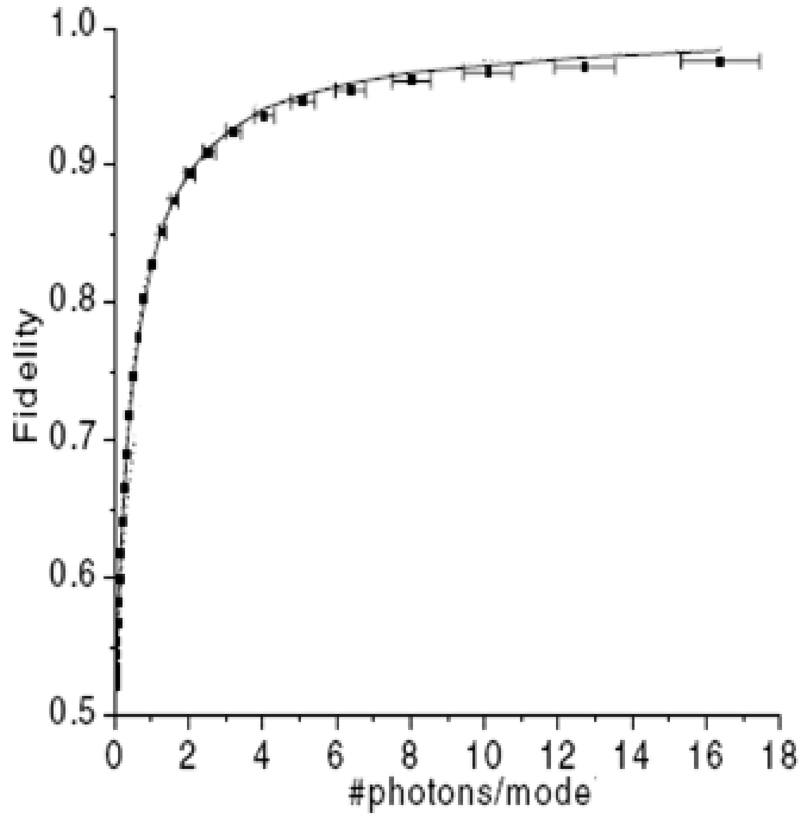


The General Apparatus



Quantum Cloning with an Optical Fiber Amplifier

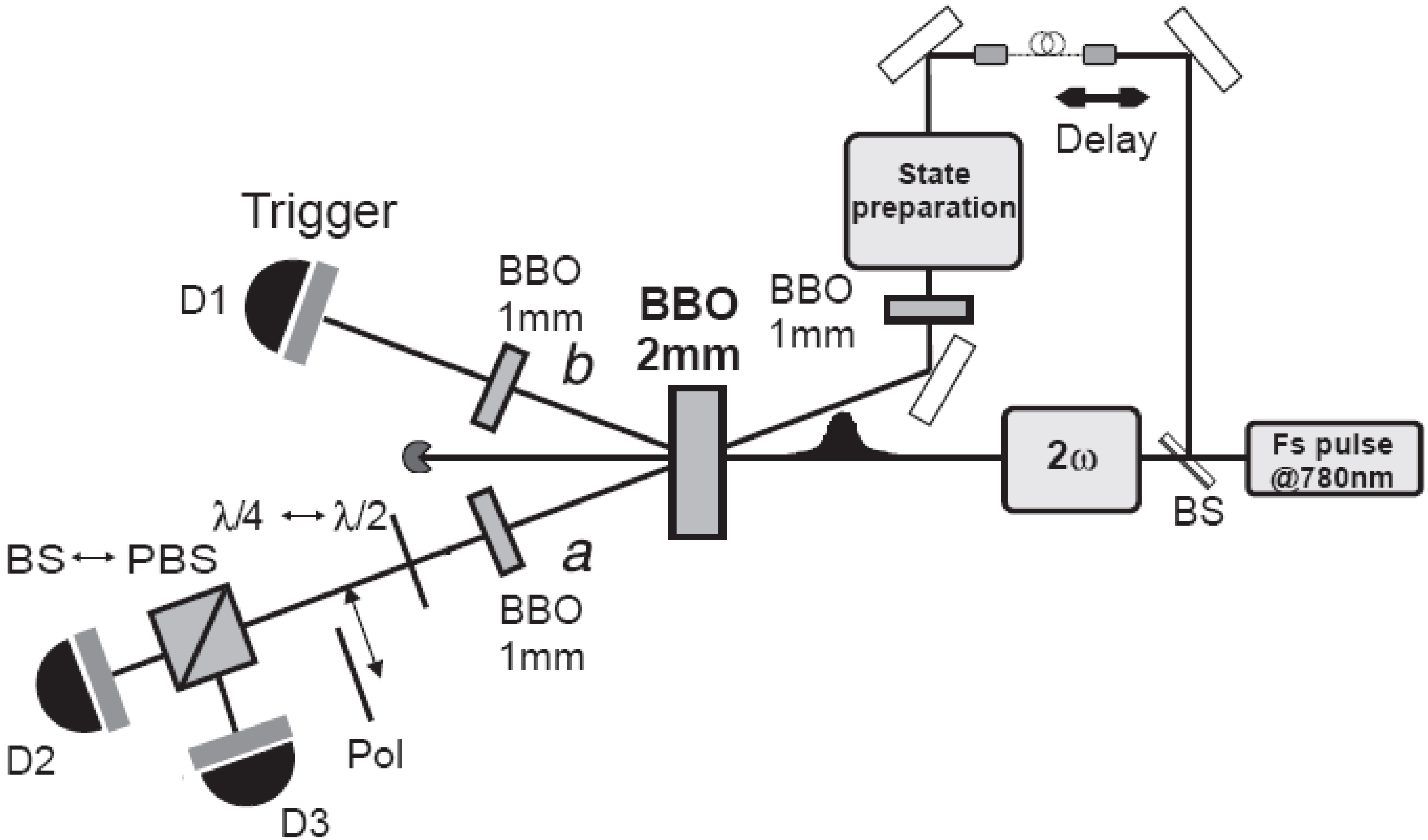




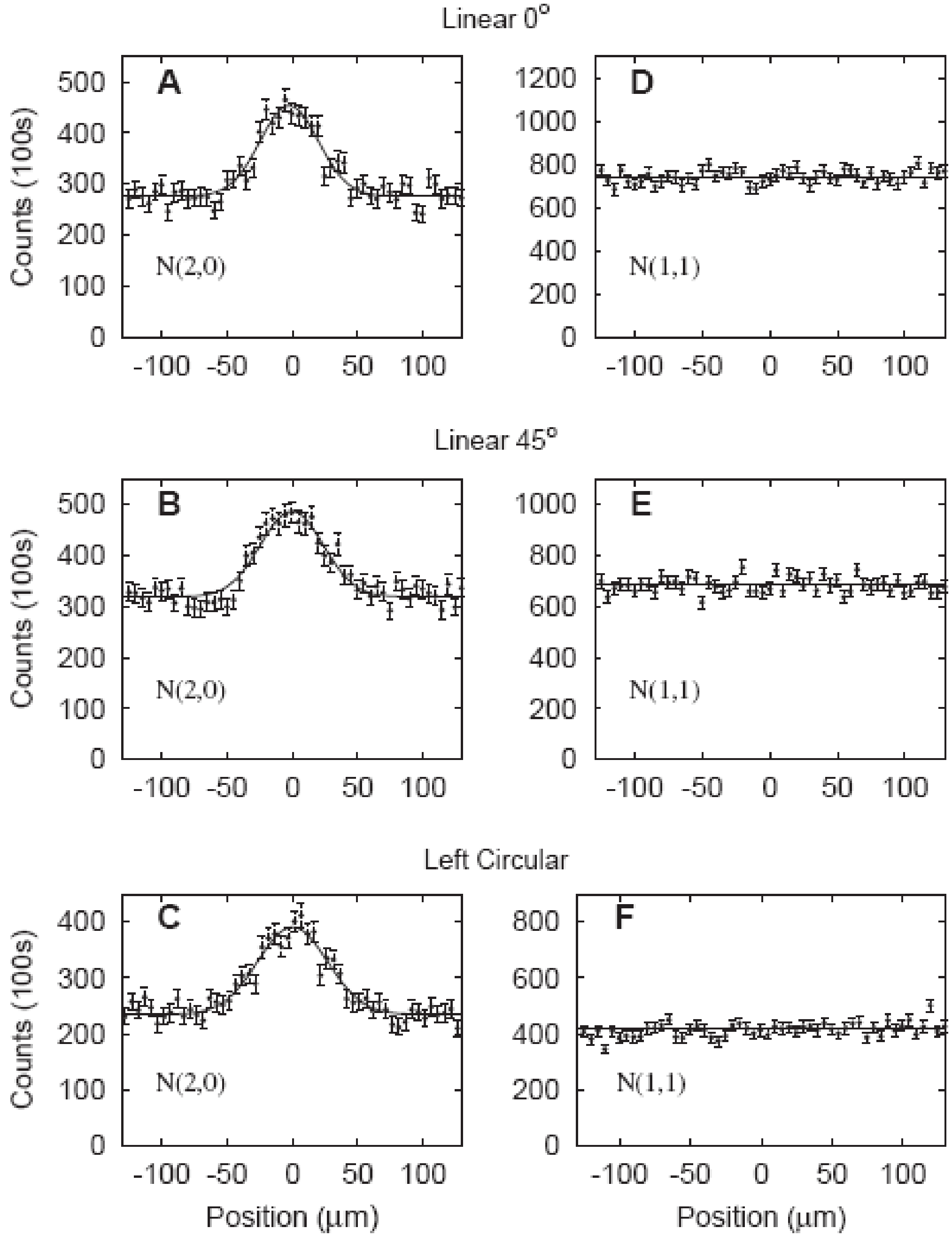
WDM – Wavelength Division Multiplexer

EDF - Erbium Doped Fiber

Optimal Quantum Cloning on a Beam Splitter

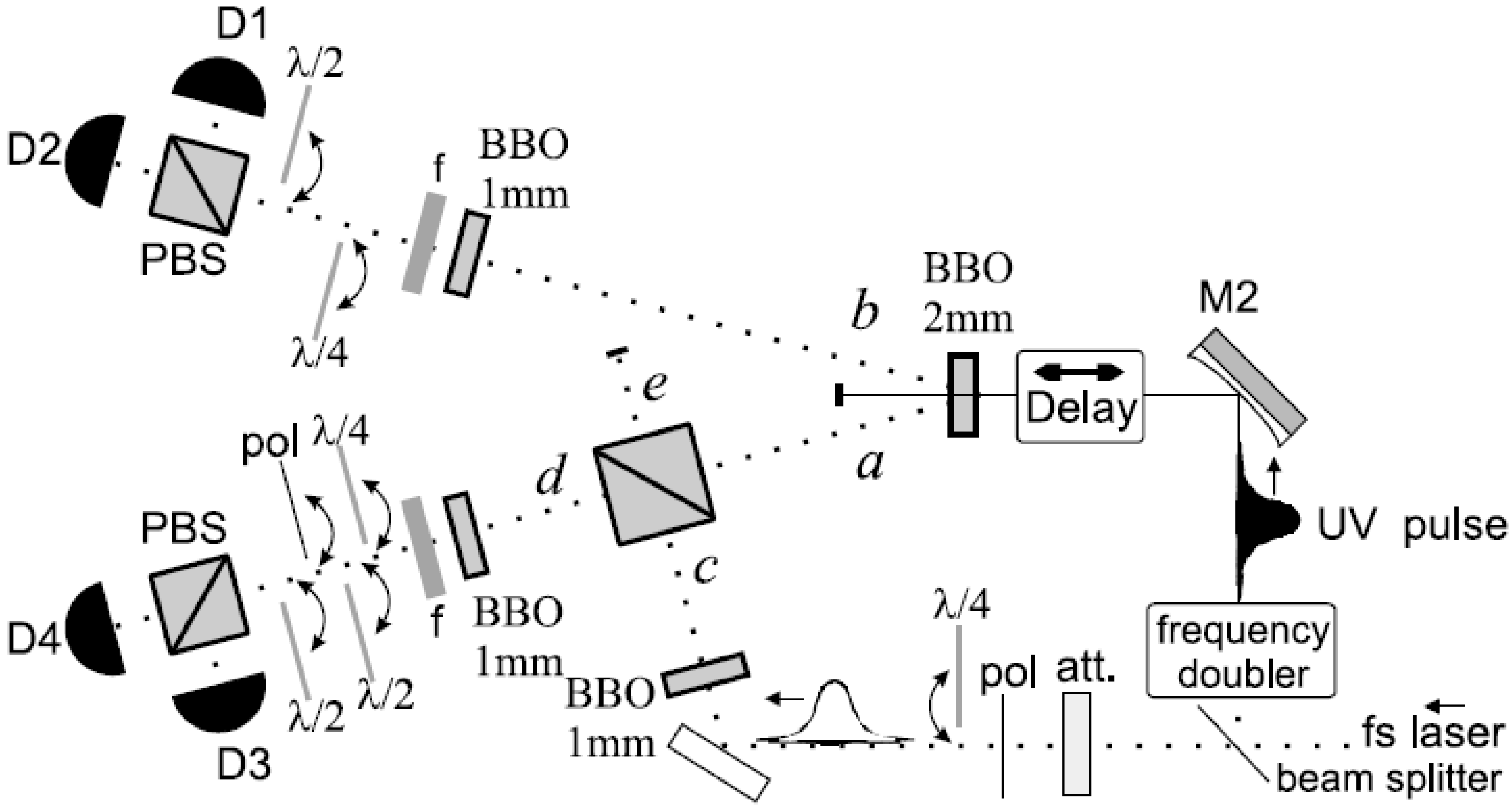


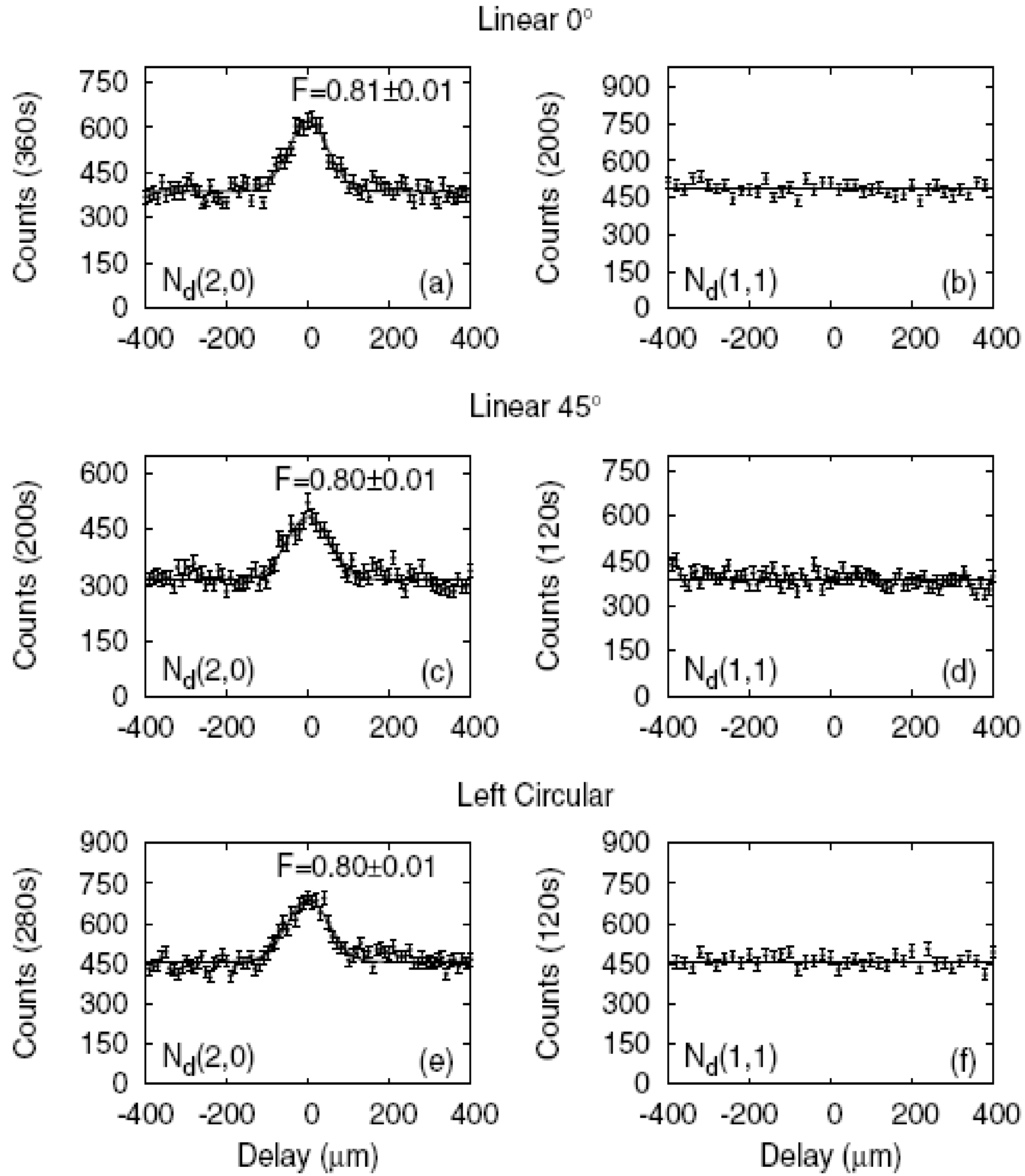
$$-i\kappa t\,(a^{\dagger}{}_{v}b^{\dagger}{}_{h} - a^{\dagger}{}_{h}b^{\dagger}{}_{v})a^{\dagger}{}_{v}|0\rangle = -i\kappa t\,(2|2,0\rangle a|0,1\rangle b - |1,1\rangle_a|1,0\rangle_b)$$



Linear 0°
A
N(2,0)
D
N(1,1)
Linear 45°
B
N(2,0)
E
N(1,1)
Left Circular
C
N(2,0)
F
N(1,1)
Counts (100s)
Position (μm)

Optimal Quantum Cloning on a Beam Splitter and NOT




Linear 0°
F=0.81±0.01
$N_d(2,0)$
(a)
$N_d(1,1)$
(b)
Counts (360s)
Counts (200s)
Linear 45°
F=0.80±0.01
$N_d(2,0)$
(c)
$N_d(1,1)$
(d)
Counts (200s)
Counts (120s)
Left Circular
F=0.80±0.01
$N_d(2,0)$
(e)
$N_d(1,1)$
(f)
Counts (280s)
Counts (120s)
Delay (μm)
Delay (μm)

- Wooters, Zurek – (Nature Vol. 299 1982)

  A Single Quantum cannot be cloned
- Buzek, Hillery – (PRA 54 3 1996)

  Quantum copying: Beyond the no-cloning theorem
- Gisin, Massar – (PRL 79 11 1997)

  Optimal Quantum Cloning Machines
- Simon, Weihs, and Zeilinger – (PRL 84 13 2000)

  Optimal Quantum Cloning via Stimulated Emission
- Bruss, Ekert,and Macchiavello – (PRL 81 12 1998)

  Optimal Universal Quantum Cloning and State Estimation
- Fasel, Gisin, Ribordy, Scarani, and Zbinden – (PRL 89 10 2002)

  Quantum Cloning with an Optical Fiber Amplifier
- Irvine, Linares,de Dood,and Bouwmeester – (PRL 92 4 2004)

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- Linares,Simon, Howell, Bouwmeester - *Science* ***296, 712 (2002);***

  Experimental Quantum Cloning of Single Photons