The Lock-In Amplifier: Noise Reduction and Phase Sensitive Detection

The two essential reasons for using a Lock-In amplifier in a scientific experiment are its ability to **“reduce noise"**, i.e. to **improve the Signal-to-Noise ratio** of the signal to be measured, and to do **phase sensitive detection**, the latter being mainly deployed in technical applications, as e.g. in phase locked loops and control circuits.

Fundamentally, the Lock-In is **sensitive to ac signals**. So, whatever you do, you have to care for an ac signal at the input of the Lock-In. How this can be done in an experiment, you will learn in **Part Three : Scientific Applications** (for a short introduction, however, see below).

Furthermore, it needs a **“reference"**, i.e. a **periodic ac voltage**, with the signal frequency, fed into the reference input. Most Lock-Ins have a **built in reference generator** you can work with. In this case you need not apply an external reference signal, of course.

To see how our Project Lock-In works, have a look at the Schematic I. You will find there the internal electronic components of the instrument and their interconnections. The most essential parts to make the Lock-In work as such, are a **multiplier** circuit and a low pass circuit, or **integrator**, which together form a **“PhaseSensitive Detector"**, or PSD. For reasons you will immediately understand, a **phase shifter** is needed as well in the overall circuit.

Since the experiment is being "modulated" by a sine function with frequency f_o, the signal from the experiment has the same frequency, or, at least, a strong component with frequency f_o.

Under these conditions, the **output** of the Lock-In will be a **dc voltage**

$$U_{out} = \frac{ab}{2} \cdot \cos \Delta \qquad (1)$$

where a and b are the **rms amplitudes** of the signal and the reference ac input voltages, respectively.

In general, the reference amplitude is **set internally to unity** so that the **output voltage** is equal to the **input signal amplitude times the cosine of the phase difference**, Δ, between the “signal" and the “reference" inputs.

For the **origin** of this equation, please refer to the sections Phase Sensitive Detection and The Correlation Function, respectively.

For **maximum output**, you have to **adjust the phase difference to 0º(or 180º)**, a task not very convenient to perform with very weak, noisy input signals. There are, however, standard methods to do so experimentally, as we will show you in Part Four. Alternatively, you may take the **Lock-In Analyzer** (see also Part Four, for reference) with which you can get rid of any manual phase settings.

With eq. (1) almost everything is told about **phase sensitive detection** : When you have two signals with known amplitudes, you can easily calculate the phase difference between these from the output voltage. This feature has found many interesting applications in technical applications. In Part Three we will go a bit more into some details of psd.

The next, and for some people : the **most important and most useful**, feature of the Lock-In is its ability to **“reduce noise”**, or to **improve the Signal-to-Noise ratio of noisy signals up to 60 dB, and more.** The theory for this will be presented in Part Two, together with the opportunity to do yourself **measurements** which will strongly support this theory ...

To **explore** this noise reduction feature, set the **Phase Angle** for **maximum output** level and **add noise** to your input signal by setting the Signal-to-Noise to a “very noisy" value, say **0 dB**, **or less**. As you can watch on the “input" screen, the original sine becomes more and more “chopped" by the noise. The output is still noisy, around its dc “mean value" = the rms value of the input signal, as set by the “Signal Level" slider, times the “Gain" setting on the front panel.

Note : For reasons of easier visualization, the **rms value** of an ac input signal (measuring signal and reference) has been set, throughout our Project, equal to the **amplitude** of this signal. So, “rms" value and “amplitude" are **synonymous throughout our Project**.

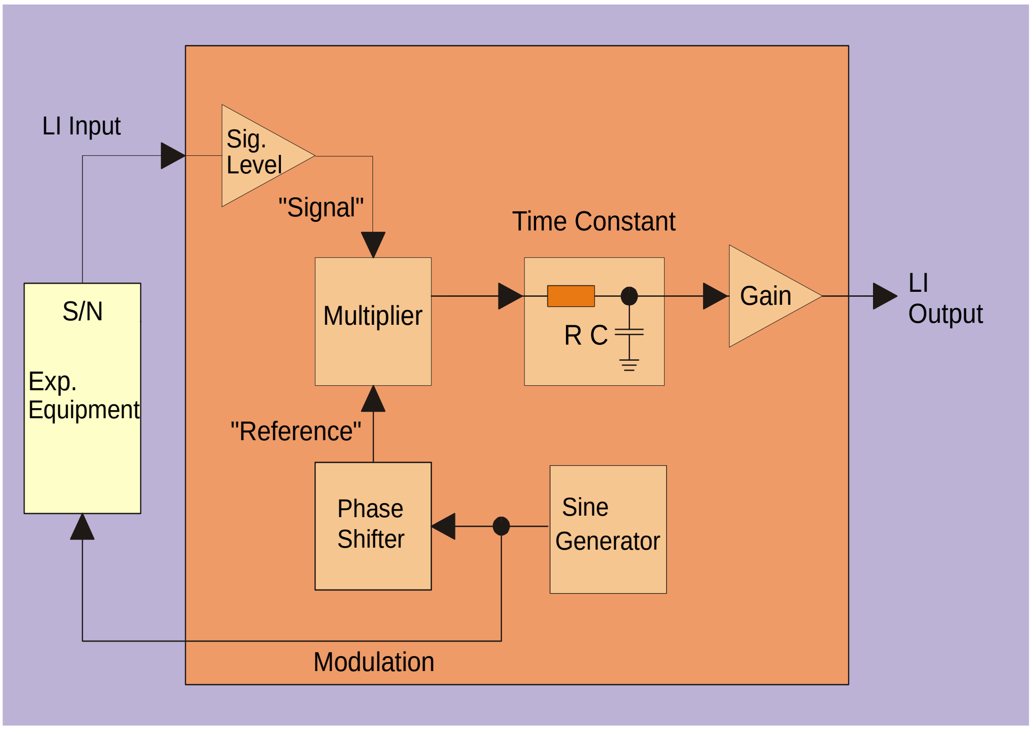
By **raising the time constant TC**, you will notice that the output noise becomes more and more **lower**.

To perform **quantitative measurements** of the dependence of the noise rms value on TC, you will need the **password** mentioned in the Introduction. But also without this password, you can **qualitatively** explore this behavior very nicely, without going into too much detail.

As will be pointed out in more detail in Part Three, under realistic conditions of a physical experiment in which a Lock-In is employed, the **System under Investigation** is typically **stimulated –** or modulated - by some kind of periodic signal. As a consequence, as we will see, the interesting **response** of the system will contain the **same frequency** (and possibly some **harmonics** if the system's behavior is **nonlinear**). But, due to the**"travelling time"** of the signals within the system, the modulated **output signal from the system** - which is our **input signal into the PSD** - is generally **delayed** with respect to the stimulating ac signal, which itself may be used as the **reference input** into the PSD.

In this case, the two signals are **automatically "locked" to one another**, the Lock-In amplifier further processing these two "locked" signals.

Schematic I



The **principal components** of the Lock-In are :

(1) a **multiplier**,
(2) an RC circuit, low pass filter, or **integrator**,
(3) a **phase shifting circuit**.

In most cases, like in the present one, a **sine generator** as the reference source is built in.

The **multiplier** in fact „multiplies“ both, the input and the reference signals. The resulting output signal is integrated, by an RC circuit, or, preferably, an **OpAmp integrator** over a time set by the time constant, TC.

As you know from your basic electronics course, this RC circuit generally serves as a **low pass filter**. The combination multiplier unit plus RC circuit, however, acts completely different, as you will see during the course of your investigations.

In our experiment, the RC circuit is "responsible" for the upper integration limit, the Time Constant **RC = nT**.

The **Phase Shifter** circuit shifts the phase of the "reference" with respect to the "signal", has been added (Remember: The phase difference Δ has to be adjusted to 0° or 180°, respectively, to get maximum output).

Gaussian or Normal Distribution

Electronic noise is a **continuous random process**, the **amplitude probability density** function of which is described by the well known **Gaussian or „normal“ distribution**

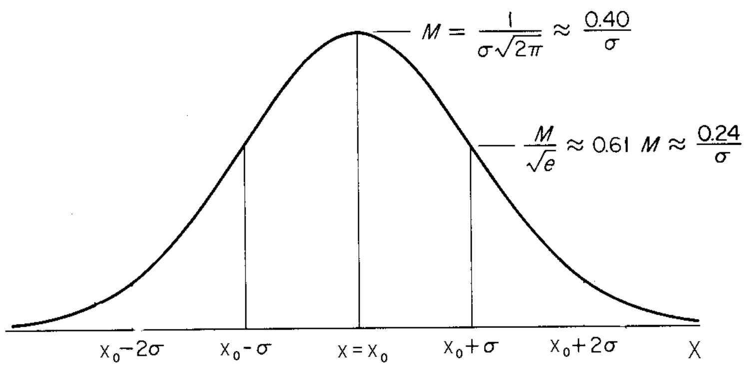
(7) $$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \quad \text{for} \quad -\infty < x < \infty$$

where *p(x)* denotes the probability density of the noise amplitude *x* to occur.

Since the amplitude certainly lies between $-\infty$ and $+\infty$, we have $$\int_{-\infty}^{\infty} p(x)dx = 1$$.

Therefore, the **maximum probability** of the Gaussian distribution function occurs at $\mathbf{x_0}$, and has the value

(8) $$p_{\max} = p(x_0) = \frac{1}{\sigma\sqrt{2\pi}}$$



From the diagram it can be seen that the **probability density** of the noise amplitude is **greatest about** $\mathbf{x = x_0}$ and symmetric about that point. The smaller σ, the greater the maximum density and the less the „spread“ of the density distribution.

The larger σ, the more the probability density is spreading out, with less concentration near x_0.

With **larger σ,** the values of the random noise amplitude become **„less predictable“** on any trial (see below for the meaning of σ).

The **first moment** of p(x) is defined as

(9) $$m_1 = \int_{-\infty}^{\infty} x\,p(x)dx$$

which can be shown to yield, after some calculation, $m_1 = x_o$.

I.e. the first moment is equal to the **average value, $\mathbf{x_o}$**

Similarly, the **second moment** of p(x)

(10) $$m_2 = \int_{-\infty}^{\infty} x^2\,p(x)\,dx$$

can be calculated to give

(10a) $$m_2 = x_0{}^2 + \sigma^2$$

The quantity $$\mu_2 = m_2 - x_0{}^2 = \sigma^2$$

is named the **variance** of the random signal, and **σ** is called **standard deviation**.

The **standard deviation** is equal to the **rms (root mean square)** value of the **noise voltage** (or current), whereas the **variance** represents the **mean square** value of the noise, corresponding to its **power**.

The Correlation Function

The key for the basic understanding of the **Lock-In "amplifier"** lies in the behavior of the so-called **Correlation Function**.

(4) $$R(\delta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) \cdot g(t+\delta)dt$$

Here we have a definite integral over the product of (generally) **any** two time dependent functions,
f and g, with a time parameter, δ, ranging from 0 ... ∞. R is a number indicating the correlation between f and g at time δ, being zero if f and g are completely "independent" from one another.

If you don't like waiting until infinity for the value of the integral, then you have to stop the integration before. R then also becomes a function of the upper integration limit, T :

(5) $$R(\delta,T) = \frac{1}{T} \int_0^T f(t) \cdot g(t+\delta)\,dt$$

In the case of **very weak signals from an experiment, g(*t*)**, with possibly very large noise superimposed, this relation allows for **looking for a correlation of g(*t*),** at a time δ, with a **"known" signal, f(*t*)**, conveniently called **"reference"** (see below).

If the Correlation Function, according to eq.(5), yields a value **unequal to zero**, then there must be a correlating, nonzero "signal", g(*t*), delivered by the experiment.

The experimental problem, in this view of affairs, is to **make the experiment respond** with the **frequency of the reference**. This can be achieved by some kind of **modulation** (or stimulation) of an interesting parameter of the experimental system with the reference frequency.

[<u>Note :</u> In terms of carrier frequency technology, the carrier frequency (= our "reference") is being amplitude modulated by the response signal from the experiment. The demodulation of the compound signal is done by "synchronous detection", i.e. by multiplicative mixing of the incoming signal with the original carrier frequency, exactly as we do it with our LockIn.]

The simplest case is to use a **harmonic function**, as the **reference**, $f(t) = a\sin(\omega_0 t)$ its frequency being ω_o .

If the "right" parameters of the experimental system are being modulated in this way, then the experiment will **respond with signals which also contain the frequency of the modulation**, i.e. simply

$g(t) = b\sin(\omega_0 t + \Delta)$ (+ „harmonics").

Here b is the amplitude of the signal you are looking for, **Δ is a phase shift** in the measurement signal with respect to the modulation function, caused by various „travelling times", or delays, within the experimental system.

This signal content can now be **detected** by applying the correlation function. Exactly this procedure is being employed in the Lock-In amplifier: In this way it is able to detect very **small**

periodic electric signals which may even be **completely buried in noise**.

[Note: We have taken here the two simplest harmonic functions just for the sake of simplicity. In principle, you may take any other periodic functions, like square wave, triangle, ... In the „technical realization of the Correlation Function“, the Lock-In amplifier, a **square wave** is often used as the "reference" function, f(*t*). The "signal" function, g(*t*), in many practical cases also may **deviate considerably** from the pure sine function taken above. These variations may, of course, change the value of the correlation function more or less significantly. We will further consider this problem in Part III of this series.]

The correlation function (called autocorrelation function due to the virtually equal basis functions below the integral) now takes the form

$$R(nT,\Delta) = \frac{ab}{nT}\int_0^{nT} \sin(\omega t)\cdot\sin(\omega t)dt \qquad (6)$$

Here, the upper integration limit is **nT**, where T is the **period of the frequency ωo**, and n is an integer. Since, as said above, no one likes waiting to infinity for an experimental value, we have to cut our measurement interval at **finite values of n**.

Eq.(6) , now, is the key to all the favorable properties of the Lock-In

as it will be deployed throughout this series.

In order to explore in detail the action of the correlation function with respect to its ability to **improve the Signal-to-Noise ratio** of an experimental signal, we will, however, in Part II, examine the behavior of eq.(6) with respect to its **frequency response**. This means that we will vary the frequencies of the **"signal" function**, g(ωt), with respect to the **"reference" function**, f(ω_0t). I.e., we will investigate the “outcome” of the (cross-)correlation function in the form

$$R(nT_0,\omega,\omega_0) = \frac{ab}{nT_0}\int_0^{nT_0} \sin(\omega_0 t)\cdot\sin(\omega t)dt \qquad (6a)$$

First however, you should have a look at the schematic of our "specimen" Lock-In.

Correlation Function and Fourier Transform

In Part Two, Exp. #1, you will investigate the **band pass** behavior of the Lock-In: you have to plot the output voltage from the Lock-In as a function of the frequency ω_1, or the frequency ratio ω_1/ω_0, respectively, where ω_0 is the (fixed) frequency of the „reference“, and ω_1 is the variable frequency of the „signal“.

The power, or voltage, as a **function of the frequency**, P(ω), or U(ω), is generally called a **spectrum**. In our correlation function we had, however, to deal with **time dependent** functions. From your math lessons you know how to *transform* **a time dependent function, f(t)** , into its **frequency domain „counterpart“, F(ω)** : This is done by means of the **Fourier Transform**.

(11)
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt$$

If we now simply take $f(t) = \sin(\omega t + \Delta)$ then we get

(12)
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt = \int_{-\infty}^{\infty} \sin(\omega t + \Delta)[\cos(\omega t) + i \sin(\omega t)] dt$$

the imaginary part being

(13)
$$\Im[F(\omega)] = \int_{-\infty}^{\infty} \sin(\omega t + \Delta) \cdot \sin(\omega t) dt$$

If we now *switch on* $f(t) = \sin(\omega t + \Delta)$ at t = 0, and *switch it off* at t = nT, then we get

(13a)
$$\Im[F(\omega)] = \int_{0}^{nT} \sin(\omega t + \Delta) \cdot \sin(\omega t) dt = R(nT, \Delta)$$

which corresponds, except for any amplitudes or „normalizing factors“, **exactly to our correlation function.**

This means that the **correlation function**, in our case with the **special functions** $f(t) = \sin(\omega t)$ **,** and $g(t) = \sin(\omega t + \Delta)$, and the **imaginary part of the Fourier transform** of , for t = 0 ... nT are **identical**.

So, what you will measure, in Part Two / Exp. #1, is in fact the (imaginary part of the) **Fourier spectrum of a switched sine wave** (with arbitrary amplitude and phase factors, respectively), depending on the **length of the switching interval, nT**.

Note that the **phase Δ** introduces a **phase shift** also in F(ω); for **Δ = 0**, it can be shown that the **real part of F(ω) vanishes**. In this case (Δ was set to zero by default in Part Two !), eq.(13a) represents the **complete Fourier transform** of the input signal at frequency ω.for t = 0 ... nT.

By varying the input frequency ω with respect to the reference frequency ω_0, or vice versa, with Δ = 0, and integrating from t = 0 ... nT_0, you will get $R(\omega,nT_0) = F(\omega,nT_0)$, the “spectral intensity” of the “switched” ω_0-signal at frequency ω.

If these data are plotted against ω, or the relative frequency $\omega/\omega_0 = f/f_0$ as in the V(f) plot in Exp. #1, then you will get the **spectral response of the LockIn filter**, with time constant nT_0. As you will see, the LockIn filter indeed shows the predicted **band filter characteristic** with **very small band widths depending on nT_0.**

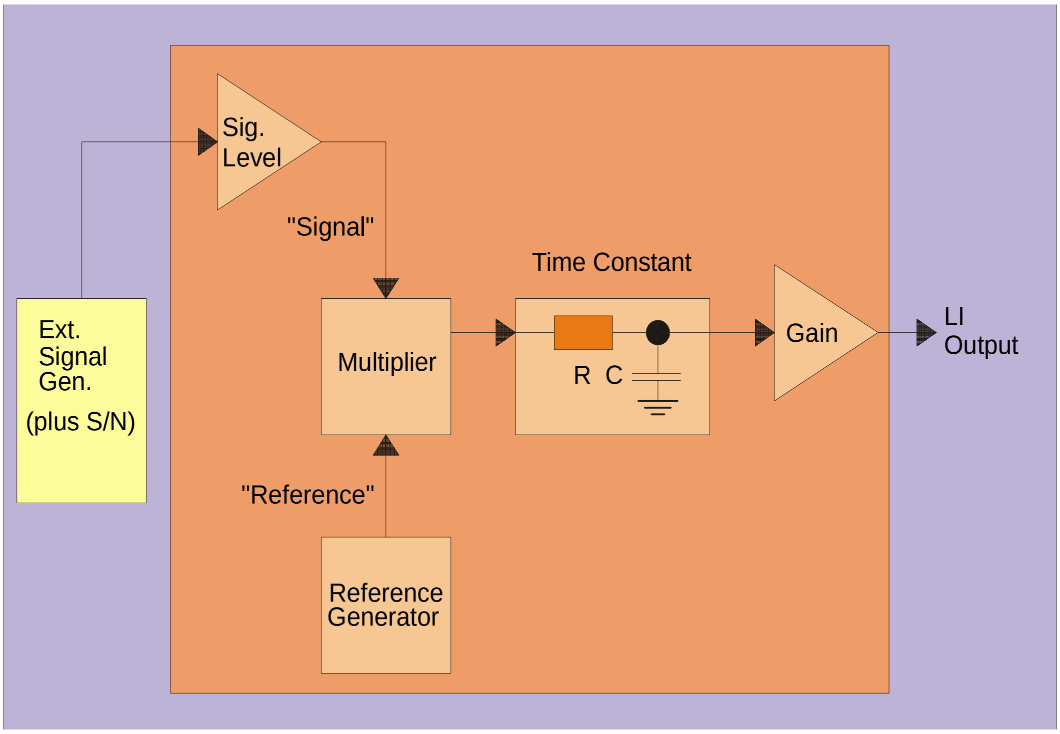
In order to **calculate the band width**, you have to **square the Fourier transform** data since the corresponding output power density is $S(\omega) = |F(\omega)|^2$ (W/Hz). **Integration** of the power density over the whole frequency range gives the **total power** transmitted. This may be thought of as an **ideal pand pass filter with bandwidth B** times the known power density over its "aperture".

In our case the **power density N_0** is **set to unity** by the given experimental conditions so that the total power transmitted is simply **N_0*B = B**. This bandwith is marked with **clear green** in the **power density spectra** you will generate in Exp. #1, V(f) plot. As one may "expect", the bandwidth of the LockIn filter is exactly the **half width**, i.e. the full width at half height, of the **first power density maximum.**

So, in spite of lack of a real band pass filter in the LockIn schematic, the instrument behaves in fact as a **band pass** filter. In this way, the mechanism of the LockIn's ability to improve significantly the Signal-to-Noise ratio of input signals may be calculated quantitatively.

For more details, see also Determination of the LockIn Bandwidth .

Schematic II



According to the requirements of eq.(6a), we need **two sine generators** with **variable frequencies** (or one with fixed frequency, and the other with variable frequency ...), called "Ext. Signal Gen.", with frequency ω, and "Reference Generator", with frequency $\omega_o = 1$ (Hz, kHz, MHz ...), respectively. In our virtual Lock-In below, we will vary the **frequency ratio**, ω/ω_0.

Furthermore, there must be a **Multiplier** with two inputs and an **Integrator**. These are the **two essential electronic components** in any Lock-In.

For the sake of simplicity, we have omitted here the phase shifter since it isn‘t needed in the present context, as you may see from the Correlation Function : If $\omega \neq \omega_o$, then the value of the Correlation Function is **independent** of **Δ**

In order to make the input signal **noisy** we have a noise generator (with **"white" noise** in this case) built into the signal source.

<u>Remember :</u>

$f(t) = a\sin(\omega_0 t)$ is the **reference** function, and

$g(t) = b\sin(\omega t + \Delta)$ is the **input voltage** from the External Signal Generator.

Note, that we now have **two different and independent signal sources**. The Lock-In input from the „experiment“ does not result from a „modulation“ with the reference. The main experimental task of this Part will be to take a **plot of the output voltage** of the Lock-In as a **function of the frequency ratio ω/ω_0** , or **f/fo**, resp.

An external "experiment" is shown, some parameter of which is being **modulated** by the reference signal, with frequency ω_o, as described in the previous section. Consequently, the output signal from the experiment also contains the stimulating frequency. This experiment also **adds noise** to the Lock-In input.

The schematic represents the **"final configuration"** of a **common Lock-In** amplifier, i.e. it contains all of the **essential components** of a real-world Lock-In instrument.

When you were to **build yourself** a LockIn amplifier with hardware components, the Multiplier / Integrator module could be made of a **Multiplier IC**, followed by an **integrator** circuit.

Technically, there exist **other construction principles for the PSD**, too, to name those where a **square wave** reference signal is used to **switch a transistor circuit** with the frequency of the input signal.

Both principles have their advantages and their drawbacks which shall not be discussed here (There will be, however, a **similar question to be answered**, anywhere during this course ...)

Noise Reduction by Narrow Band Filters : The Wiener-Khinchin Theorem

As will be shown in this Part Two, due to the special functions involved in the correlation function, the Lock-In behaves like a **band pass filter** with respect to its reference frequency, the **bandwidth** of the filter depending on the **integration time**. Intuitively this mechanism makes it clear how the Lock-In is capable to improve the Signal-to-Noise ratio significantly: The band pass filter „cuts" simply out a more or less broad part of the noise power spectrum in the vicinity of the input signal, hereby reducing the average relative noise power at the output, or the relative rms noise voltage, respectively. „Relative" means: with respect to the input conditions.

How can we get further information about the noise reduction numbers in our Lock-In ?
Especially:

How does the noise reduction depend on the integration time, or bandwidth, respectively ?

First, we have a random function of time, r(t), (the noise) added by the „experiment" to our „wanted signal", $g(t) = b\sin(\omega t + \Delta)$.

The correlation function now has the form

(14) $$R(nT,\Delta) = \frac{a}{nT}\int_0^{nT}\left[b\sin(\omega t + \Delta) + r(t)\right]\sin(\omega t)dt = R_0(nT,\Delta) + \frac{a}{nT}\int_0^{nT} r(t)\sin(\omega t)dt$$

The behavior of the first part, $R_0(nT,\Delta)$, is known from Parts One and Two. What can we do with the second part, now ?

Since we don't „know" r(t) analytically, we **can't evaluate the integral directly**. What we **do know**, however, about r(t) is its **rms value** (we can **measure** it with an **rms voltmeter**, or we can **analyze the noise spectrum**, as we did in **Part One**) and its spectrum (it is **white noise** with a **constant power spectral density**, independent of frequency, in principle ...)

The rms value of r(t) may be calculated by $\sqrt{\langle r^2(t)\rangle}$ where $\langle \ldots \rangle$ means the **time average**.

Instead of the **cross-correlation function** from Part Two, we now consider the **autocorrelation function** which is simply the correlation function of Part Two / eq.(1) with only **one function**, f(t), with the same argument :

(15) $$R(\delta) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\cdot f(t+\delta)dt$$

which can be interpreted as a **time average** : $\langle f(t)f(t+\delta)\rangle$.

The correlation function characterizes the **statistical relation** between the values of f(*t*) at time t and (t+δ), where δ can be both positive and negative as well as zero.

If f(t) and f(t+δ) are **statistically independent (real noise)**, then

(16) $$R(\delta) = \langle f(t)\cdot f(t+\delta)\rangle = \langle f(t)\rangle\langle f(t+\delta)\rangle = 0$$

For δ = 0, the autocorrelation function becomes

(17)
$$R(0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f^2(t)dt = \left\langle f^2(t) \right\rangle$$

which may be considered as the **mean power of the random process f(t)**, also known as its **variance** (see also in Part One : Gaussian or Normal Distribution). The mean amplitude of the **noise voltage** becomes $U_{rms} = \sqrt{\langle p \rangle R} = \sqrt{\left\langle f^2(t) \right\rangle R}$, where R is the resistance across which the „voltage drop" U_{rms} occurs.

The **power spectral density , S(ω)** (dimension: W/Hz), of an **aperiodic funtion f(t)** is defined by

(18)
$$S(\omega) = \lim_{T\to\infty}\left\langle \frac{|F(\omega)|^2}{2\pi T} \right\rangle$$

where F(ω) is the **Fourier transform** of f(t).

Combining eqs.(4) and (5) yields the **average power** of the process expressed by f(t):

(19)
$$R(0) = \left\langle f^2(t) \right\rangle = \frac{1}{2\pi}\int_0^\infty S(\omega)d\omega = \left\langle p \right\rangle$$

This is a special form of the famous **Wiener-Khinchin Theorem** which permits the determination of the power spectral density from a given correlation function, and conversely.

Generally, the **Wiener-Khinchin Theorem** states that the **autocorrelation function R(δ)** and the **spectral power density S(ω)** are the **Fourier transform**, or the **inverse Fourier transform**, respectively, of **each other** :

(20a)
$$R(\delta) = \frac{1}{2\pi}\int_0^\infty S(\omega)e^{i\omega\delta}d\omega$$

and

(20b)
$$S(\omega) = \int_0^\infty R(\delta)e^{-i\omega\delta}d\delta$$

(The proof of this theorem is not difficult to perform but a bit lengthy, so that it will be omitted here.)

We will use this theorem to **calculate the noise reduction by the Lock-In** a little bit later.

Band Limited White Noise

White noise is defined as a **random process** with **constant spectral power density**, S(ω) = N_o [W/Hz], independent of frequency. This is, of course, a purely **mathematical fiction**, since

(21) $$\langle p\rangle = R(0) = \frac{1}{2\pi}\int_0^\infty S(\omega)d\omega = \frac{1}{2\pi}\int_0^\infty N_0\, d\omega \quad \Rightarrow \quad \infty$$

which makes **no sense, physically**.

It is, however, useful to introduce the concept of **„band limited white noise"** whose spectral density is given by

$$S(\omega) = N_o \qquad \text{for} \qquad \omega_1 < \omega < \omega_2$$

and zero outside of this range.

In this case

(22) $$\langle p\rangle = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} N_0\, d\omega = N_0 B$$

remains certainly finite; $B = (\omega_2 - \omega_1)/2\pi$ is the **bandwidth** of the band limited noisy process.

The Transfer Function

When a „signal" is **applied to the input of a linear** network, the **same signal** may appear **different at the output**. If we apply, for example, a simple harmonic function $f(t) = V_o \cos(\omega t)$ at the input, the response at the output will generally be of the form

$$g(t) = V_0 A(\omega)\cos\left[\omega t + \Delta(\omega)\right]$$

or

$$g(t) = H(\omega)\cdot f(t) \quad ,$$

with

$$H(\omega) = A(\omega)\,\exp\left[i\Delta(\omega)\right]$$

being the **Transfer Function** of the network.

It can be shown that the **output spectral density $S_o(\omega)$** is related to the **input spectral density $S_i(\omega)$** by

(23) $$S_0(\omega) = \left|H(\omega)\right|^2 S_i(\omega)\,.$$

Bandfilter

We have now all the tools together to **calculate the noise reduction by our Lock-In „filter"**. For this purpose we consider the application of white noise to a band filter of bandwidth $\Delta\omega \ll \omega_o$ (ω_o is the center frequency). For the sake of simplicity, the filter characteristic is assumed to be of the **ideal rectangular shape**.

In this case, the transfer function

$$H(\omega) = K = \text{const} \quad \text{for} \quad \omega_0 - \frac{\Delta\omega}{2} < \omega < \omega_0 + \frac{\Delta\omega}{2}$$

and

$$\mathrm{H}(\omega) = 0 \quad \text{elsewhere.}$$

The output spectral density is now related to the input density by eq.(10) :

$$S_0(\omega) = |H(\omega)|^2 \, S_i(\omega) = N_0 K^2 \quad \text{for} \quad \omega_0 - \frac{\Delta\omega}{2} < \omega < \omega_0 + \frac{\Delta\omega}{2}$$

and zero elsewhere.

By using the **Wiener-Khinchin Theorem**, the autocorrelation function becomes now

(24) $$R(\delta) = \frac{1}{2\pi}\int_0^\infty S_0(\omega) e^{-i\omega\delta} d\omega = \frac{N_0 K^2}{2\pi} \int_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} \exp(-i\omega\delta)\, d\omega$$

$$= N_0 K^2 \frac{\Delta\omega}{2\pi} \frac{\sin(\Delta\frac{\omega}{2}\delta)}{\Delta\frac{\omega}{2}\delta} \cdot \cos(\omega_0\delta)$$

which reduces to

(24a) $$R(0) = N_0 K^2 B = \langle p \rangle$$

for $\delta = 0$, which is basically the same relation as eq.(22).

Eq.(24) again contains the familiar **sin(*x*) / *x*** relation, which is **maximum for δ = 0**.

In the above calculations, the **transfer function** was assumed to be a **constant, K**. In the Lock-In we may have, however, a **different „transfer profile“** which should be taken into account in a more exact calculation. If you will do so, using again the Wiener-Khinchin Theorem, you will notice that the **essential results** of our above calculation remain **unaltered** :

The **auto-correlation** function, and hence the **transmitted average noise power**, is directly **proportional to the bandwidth** BW = $\Delta\omega/2\pi$. Hence the **rms noise voltage** at the output is **proportional to the square root of the band width**.

This is the **principal content of the Wiener Theorem**.

For our special LockIn case, we come to an **equal result** using the **cross-correlation** function,, and its meaning as the **Fourier transform** of a switched, i.e. non-continuous, sine wave (cf. Correlation Function and Fourier Transform and Determination of the LockIn Bandwidth).

This is the **answer to our question at the beginning of this chapter**. You should **check your experimental data** whether or not they fulfill these theoretical considerations.

Resonance Absorption

In the previous sections of this course you - hopefully ... - learned a lot about the basic function principles of the Lock-In Amplifier. Now, in Part Three, we will see how the device works within a typical **experimental setup.**

In our Advanced Physics Lab at the University of Konstanz we use the Lock-In in four experiments:

(A) the "classical" (cw) Electron Spin Resonance experiment,
(B) detection of Surface Plasmons in an evaporated metal film,
(C) in a "phonon assisted" tunnelling experiment, and
(D) in our "Noise Analysis" experiment where the Lock-In is demonstrated "in praxi".

(We could have added here the Raman effect, but here we actually use photon counting instead of Lock-In detection.)

In the first two experiments, the physical systems respond to an **external stimulation** by an **electromagnetic radiation field** with a **resonant absorption of radiation power**. Without going too much into detail, let us only state that a **surface plasmon** is generated for a **definite angle of incidence** onto the glas prism carrying a gold film on its base. Under these conditions, the reflected light beam is more or less **attenuated**. In the case of the **ESR**, the absorption process occurs **at a certain magnetic field**, the frequency of the electromagnetic radiation being kept constant (given, e.g., by the dimensions of the resonant cavity).

In both cases, the desired response of the system occurs only under **special conditions** :

These conditions are experimentally searched for, in general, by **slowly sweeping** one of the variable input parameters of the experiment, e.g. the **angle of incidence** in the case of the **plasmon resonance**, or the **magnetic field** in the case of the *ESR*, respectively. When the correct conditions are encountered, the response of the physical system can be detected e.g. as **loss of energy** of electromagnetic radiation applied to the system which can be detected by rectifying the output radiation. This loss can be **very small**, resulting in a very small variation of the rectified dc output. Very small variations of relatively large dc signals, however, can be **extremely difficult to detect** due to possibly **strong noise** components and **inevitable drifts** within the system.

So, with our knowledge of the Lock-In we can **improve our situation** considerably : We simply superimpose a small **periodic variation** onto our slowly variable ("dc") "search parameter", i.e. we **modulate** this input parameter to the system.

As a consequence, the output signal from the physical system will show the periodicity of the modulating signal (plus **„harmonics“** of it, as you will see in Part Three/2), and can now further be processed with the known advantages of the Lock-In, namely the significant **improvement of the Signal-to-Noise ratio** of ac signals as shown in the previous Parts of this series.

The **actual experimental situation** is shown on the „user interface“ of the applet.

Again, you have **two screens** representing now a **"System" window** and an **output window**, respectively.

On the system screen, labelled **Experiment**, the **response of the system** under investigation is depicted in the form of a **"System Function"**.

In the present version of the experiment, you have the **choice** between a **Lorentzian resonant**

absorption curve, $1/[(x - x_o)^2 - fwhm^2/4]$, where fwhm = full width at half maximum of the resonance curve is a convenient measure of the **"quality" of the resonance process** under investigation (remember the quality factor of a band pass filter in Part Two) , and a **series of straight lines**, interconnected by **adjustable "kinks"** (or "bends", if you prefer ...).

In order to generate the above mentioned **surface plasmon**, we have to **vary the angle of incidence** onto the glas prism which carries the gold film on its "back" side. In our real lab experiment this is done by a **mirror** mounted on a small **rotation table** which reflects the beam of a laser diode onto the prism. A **mechanical vibration** is added to the **continous rotation** movement by fixing the mirror at the end of a ***piezoelectric bimorph*** which is driven **sinusoidally** by the built-in **reference generator** of the Lock-In Amplifier.

When we approach the **resonance condition** by the **continuous variation** of the angle of incidence, we **"scan"** across the (Lorentzian) resonance curve by the **mechanical vibration** of the mirror.

This situation is shown, as an example, on the **Experiment** Screen: The **light input** to the plasmon generating system **-** the gold film **-** is visualized by a **red vibrating "beam"** at the bottom of the screen. A **sinusoidal angular vibration** is superimposed as the beam **scans continuously** across the resonance curve. The actual **output** of the system,the reflected light intensity as detected by a photodiode, is shown in yellow at the right side of the input screen.

As you will easily notice, this output is **everything but sinusoidal**, in general, due to the **nonlinearity** of the scanned **system response curve**.

This **yellow** output signal is now **fed into the input** of a Lock-In and processed by the PSD, exactly in the same way as the input signal in the Lock-In Schematic I in Part One of this series.

The **output from the Lock-In** is shown on the **righthand "Lock-In Output"** screen.

In the **tunnelling experiment** mentioned above we don't have a resonant absorption curve, but, instead, a series of **"straight lines"**, with **different slopes**, embedded in the **I(V) curve of a silicon backward diode**. The most interesting features with this situation are best investigated in Part III/2 of this series with the "2f-mode" of the Lock In, but, for **comparison with the resonance line**, you should have a look at the output in this case, too.

The Lock-In settings:

The following parameters are invariable and pre-set :

the phase $\Delta = 0$, i.e. maximum output from the PSD,
Input Signal Level = 1V,
Time Constant = 1T.

The **adjustable parameters** are as follows:

The vibration, or **modulation amplitude** (the **red** sine curve at the bottom of the Input Screen), adjustable from 0.1 to 4*fwhm (see above),
The Lock-In **output gain**, from 0.5 to 30.
You can add "static" **noise** to the system's response function.
You may change the **system function** between **"Lorentzian"** and **"Straight Lines"**.

In the **Straight Lines** system, you can **adjust the situation at will by "dragging"** the **markers at the kinks**, with the **left mouse key pressed**.

For our actual **Lorentzian** (in green on the left hand screen) the **fwhm is set to 20 pixels**. The

peak-to-peak amplitude of your **modulation signal** in the applet can be **adjusted in units of fwhm**.

The actions of the buttons on the applet's front are, hopefully, self explaining.
Try them out !

<u>**Note:**</u> **At first glance**, the **output** from the Lock-In looks like the **derivative** of the original function. But, in general, this is **not** the case. You can test it by **integrating the output curve** and visual **comparison with the original function**, the Lorentzian. As you will see, the **shape of the output plot** depends very much on the **amplitude** of the modulation. This behavior can also be investigated very well by using the **Straight Lines** as the system function.

Exploring the "System Function"

In a typical experiment where lock-ins are employed for **measuring tiny signals**, some parameter of the system under investigation is slowly „scanned" across a certain range of that parameter in which there occurs any „anomaly", leading to a **characteristic output signal** of the lock-in. Typical examples are **resonance processes** or **discontinuities** due to some physical process, respectively, in an elsewhere smooth characteristic, simply called „sytem function" in the following. For the first kind of applications, you will most probably employ the „normal" working mode of the instrument, while the latter is best investigated using the „2f" mode.

Some effort has to be made to **reconstruct** the original system function from the lock-in output signal, however. So, how does the system function transform into the output signal ?

As was said above, some parameter, p, is scanned slowly, p(t), across the system function f(p,...); at the same time, the parameter is **modulated periodically**, with frequency ω, and modulation amplitude a. The modulation signal, a cos(ωt), is conveniently taken from the **reference output** of the lock-in.

So we have

$$p(t,\omega) = p(t) + a\cos(\omega t) \tag{25}$$

The **response of the system** at any time, g(t), may be described approximately by Taylor´s expansion of f(p,...) around the actual value of the „steady" part, p(t), of eq. 1 :

$$
\begin{aligned}
g(t) &= f\left[p(t),...\right] + \left.\frac{\partial f}{\partial p}\right|_{p(t)} a\cos(\omega t) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial p^2}\right|_{p(t)} \left[a\cos(\omega t)\right]^2 + \frac{1}{3!}\left.\frac{\partial^3 f}{\partial p^3}\right|_{p(t)} \left[a\cos(\omega t)\right]^3 + ... \\
&= f\left[p(t),...\right] \\
&\quad + \left[\left.\frac{\partial f}{\partial p}\right|_{p(t)} \cdot a + \frac{3}{4\cdot 3!}\left.\frac{\partial^3 f}{\partial p^3}\right|_{p(t)} \cdot a^3 + ...\right]\cos(\omega t) \\
&\quad + \left[\frac{1}{2}\left.\frac{\partial^2 f}{\partial p^2}\right|_{p(t)} \cdot a^2 + \frac{4}{2\cdot 4!}\left.\frac{\partial^4 f}{\partial p^4}\right|_{p(t)} \cdot a^4 + ...\right]\cos(2\omega t) \quad + \quad ... \\
&= f\left[p(t),...\right] + A_1\cos(\omega t) + A_2\cos(2\omega t) + ...
\end{aligned} \tag{26}
$$

Since lock-ins detect **ac signals** only, the dc part, f(p(t),...), will not contribute to the output signal, but the „harmonic" (with ω, 2ω, ...) parts will do so.

From eq.(26) you may also clearly see that the output of the lock-in in its „normal“ mode - with ω as the reference frequency - as well as in the „2f“ mode strongly depends on the **amplitude of the modulation** signal.

For **very small amplitudes**, the output will be proportional to the **first derivative** of the system function in the **normal** operation mode, and proportional to the **second derivative** in the **2f mode**. With larger modulation amplitudes, more and more **nonlinear** terms add to the „pure“ derivatives, deforming hereby the curve form of the output. This effect can be investigated very impressively in the applets.

To restore the original system function, you have to **integrate** the output curves once or twice, respectively.

A few remarks on the "System Function"

I. In the chapters called "Applications in Science" we introduced the - somewhat unusual but more intuitive - term *system function* to describe the interesting property of the system under investigation. In fact, the *system* function is a **response** function : a response to an externally applied **perturbation** of any kind. This, in general complex, response function is generally called the "**susceptibility**" of the system.

In the case of nuclear magnetic resonance, e.g., the interaction energy of the nuclear magnetic moment, $\boldsymbol{\mu} = \gamma \mathbf{J}$ ($\mathbf{J} = \hbar \mathbf{I}$ being the angular momentum of the nucleus, **I** the angular momentum operator), with an externally applied magnetic field **H** is given by the eigenvalues of the simple Hamiltonian

(1) $$\mathcal{H} = -\mu \cdot H,$$

namely

(2) $$\mathbf{E_m = -\gamma\hbar H_o m} \qquad m = I, I-1, \quad , -I.$$

These energy levels are equally spaced. For a transition to occur between adjacent levels, you have to apply an energy of

(3) $$\mathbf{\Delta E = |E_m - E_{m-1}| = \hbar\omega = \hbar\gamma H_o}, \text{ at}$$

(4) $$\mathbf{\omega = \omega_o = \gamma H_o}$$

So, in order a resonance condition to be met, you may leave the excitation frequency, ω_o, fixed, and vary the magnetic field H, or alternatively you may sweep ω, and leave H_o fixed. At resonance, rf energy will be absorbed by the nuclear spin system, a condition which you can detect, e.g., by monitoring the rf amplitude across the exciting coil in the case of resonance.

Since these effects are generally very small, and accompanied, or even hidden, by relatively high noise levels, you may favorably employ lock-in detection techniques in this case.

In order to do so, you have basically two experimental choices :

1. Leave H_o fixed and sweep ω across the resonance. The periodic modulation of a system parameter required by the lockin can be achieved by periodic *low frequency* (*not amplitude* !) *modulation* of ω.

This procedure results in a rather complicated rf excitation spectrum and is used preferably where sweeping the main field H_o is inconvenient, or impossible, e.g. with superconduchting magnets.

Much more common, however, is the second choice :

2. Leave $\omega = \omega_o$ fixed, and sweep H, at the same time periodically modulating it. This can be done very easily by superposing H with a weak low frequency alternating magnetic field produced by, e.g., a pair of modulation coils, or a solenoid.

While sweeping H across the "resonance" under these conditions, an absorption signal at $H=H_o$ in the form of the above mentioned *system function* may be recorded.which is nothing else than - as will be shown below - the *imaginary part of the rf susceptibility of the nuclear spin system* under investigation.

II.

According to what was said above, any resonance is characterized by a *complex susceptibility* expressing the *linear relationship* between a **system parameter response** and a **-** generally time variable **- driving force**, which may be "out of phase", resulting in the "complex" response. In the case of magnetic resonance : the magnetization of the spin system and the applied rf magnetic field,

$$(5) \qquad \Rightarrow \qquad \mathbf{M(t) = \chi\, H(t) = (\chi' - i\chi'')H(t)}$$

It is not the intention of this introductory text to present a thorough derivation of the spin susceptibility relations here, the interested reader may refer to, e.g., Charles P. Slichter : Principles of Magnetic Resonance, Harper and Row, N.Y..

A short guideline may be presented as follows : In the case of magnetic resonance, a precession of the nuclear magnetic moments around a "rotating" coordinate frame is generally considered, the total magnetic field the spin experiences being

$$(6) \qquad \mathbf{H(t) = i\, H_1 \cos \omega_z t + j\, H_1 \sin \omega_z t + k\, H_o}$$

with Ho being the static (= "swept") magnetic field, H_1 the amplitude of the rf magnetic field, and ω_z the component of ω along the z axis.

The interaction Hamiltonian for a spin ensemble, in principle eq. (1), with this magnetic field, inserted into the Schrödinger equation of the spin system gives the famous **Bloch equations**

$$(7) \qquad \begin{aligned} \frac{dM_x}{dt} &= \gamma(\boldsymbol{M}\times\boldsymbol{H})_x - \frac{M_x}{T_2} \\ \frac{dM_y}{dt} &= \gamma(\boldsymbol{M}\times\boldsymbol{H})_y - \frac{M_y}{T_2} \end{aligned}$$

where $\boldsymbol{M}_x$ and $\boldsymbol{M}_y$ denote the magnetization of the spin ensemble in the rotating coordinate system due to the interaction of the spins with the rf magnetic field. T_2 is a "transversal relaxation" time for establishing some "mean value" in M_x and M_y of the spin system.

The solution of eqs.(7) with low H_1 shows that the magnetization is a complex constant in the *rotating reference frame* :

(8)

$$M_x \propto \frac{(\omega_o - \omega)T_2}{1+(\omega-\omega_o)^2 T_2^2}\cdot H_1$$

$$M_y \propto \frac{1}{1+(\omega-\omega_o)^2 T_2^2}\cdot H_1$$

For the x component of M (M_X) in the *laboratory frame* - in which any measurements take place - we get

(9) $$M_X = M_x \cos\omega t + M_y \sin\omega t$$

or, with $H_X(t) = H_{Xo}$ cos ωt and eq. ( ) :

(10) $$\boldsymbol{M_X(t) = [\,\chi' \cos\omega t + \chi'' \sin\omega t\,]\, H_{Xo}}$$

defining χ' and χ''as the magnetizations in perpendicular directions in the rotating frame :

$$\chi' \propto \frac{(\omega_o - \omega)T_2}{1+(\omega-\omega_o)^2 T_2^2}$$

and (11)

(**"Lorentzian line"**) $$\chi'' \propto \frac{1}{1+(\omega-\omega_o)^2 T_2^2}$$

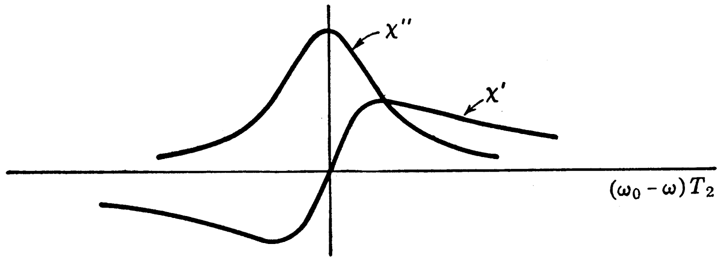


FIG. 2.11. χ' and χ'' from the Bloch equations plotted versus $x \equiv (\omega_0 - \omega)T_2$.

(Plot taken from C.P.Slichter, loc.cit.)

It can be shown, now, that the average rf power, <P>, absorbed by the spin ensemble, with volume V, amounts to

$$<P> = \frac{1}{2} \omega H_{Xo}^2 \chi'' V \tag{12}$$

This expression verifies what was said above :

The “system function” in the case of magnetic resonance is proportional to the *imaginary* (= *absorptive*) part χ'' of the complex magnetic rf susceptibility of the spin system under investigation.

Exactly this situation is simulated in Part III/1 of the Lock-In Project.

The real part of the susceptibility χ' is called the *dispersive part* (“*dispersion*”) of the complex rf susceptibility.

The „2f“ Mode

Before proceeding, you should be familiar with Part III/1.

As we have seen there, the **output from the experiment** (the input signal into the Lock-In) results from a **"mirroring"** of the sinusoidal **modulation** of the experimental "scan" parameter at the **"system function"** of the experiment. In the case of a nonlinear system function, as is the Lorentz function, or the "straight lines" system at the kinks, this process leads to a **deformation** of the output signal with respect to the original sine; in other words, there will be a **characteristic amount of admixture** of **harmonics** to the fundamental modulating sine, the amplitudes of the harmonics clearly depending on the **nonlinearity** - or **curvature** - of the system function.

These **harmonics may be detected by phase sensitive detection**, just as it was the case with the fundamental frequency. In this case, of course, you have to apply a **harmonic** frequency to the **reference input** of the PSD. For the detection of the first harmonic content, you feed the double frequency, 2f, into the reference input of the PSD, for the second harmonic 3f, and so on.

The effect of **harmonics generation** can best be seen in the previous applet (Part III/1) at the **minimum point** of the model system function, the Lorentzian absorption curve, and at the **kinks** of the Straight Lines system. Here, as one can see from a **Fourier analysis** of the yellow experimental output signal on the left screen, there is no longer any contribution from the fundamental, but, instead, a **very strong signal with the double frequency**. This effect leads to a **zero output** signal - the "zero crossing" - in the **"1f"** Lock-In output curve.

If you apply, however, **2f** to the reference input of the PSD, you will get a **maximum** Lock-In signal output at this special point.

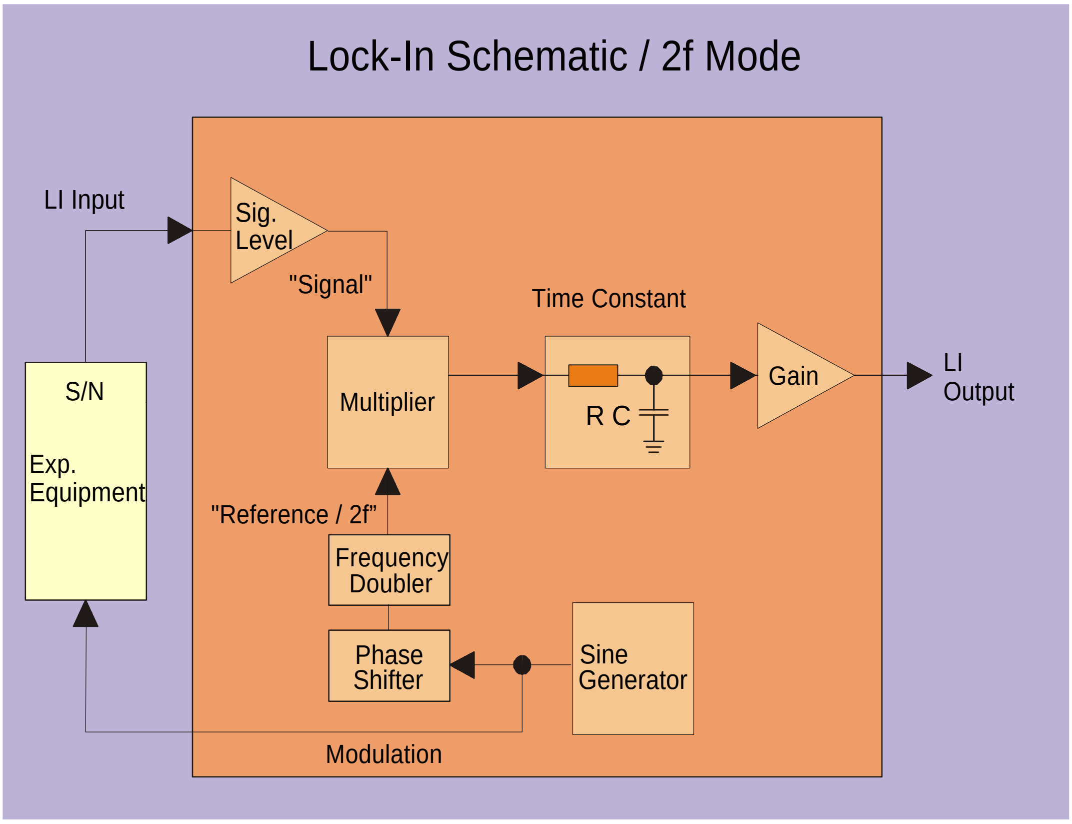
As one may argue from this experiment - and can show exactly by analytical methods - the **2f output** corresponds to the **second derivative (curvature) of the system function**, under the ***same conditions*** that hold for the experimental parameters to make the fundamental Lock-In output to be proportional to the **first derivative** (i.e. for very small modulation amplitudes).

This feature is of special interest in the case of system functions which exhibit **kinks**, or points where the slope of the function changes. By differentiation you will get a "step" in the first derivative and a **spike** in the second one (as you probably already know from question #5 in the previous Part ...).

In our Physics Lab, we encounter this case in the **"Phonon Assisted Tunneling"** experiment, where electrons in Silicon, at a temperature of 4.2 K, are accellerated by an applied voltage and gain an appropriate momentum to "tunnel" through the energy gap between the valence and conduction band in Si. At specific energies, eU, there opens a new tunneling, or conductivity, channel which **steepens the slope** of the I(U) function of the backward diode. These situation can favorably be investigated with the **2f mode** of a Lock-In amplifier, resulting in "peaks" where the changes in conductivity occur.

This situation is nicely simulated with the „Straight Lines“ system function in this Part.

As usual, you should first have a look at the 2f mode schematic.



Here, the **2f reference** is generated **between the phase shifter and the PSD input** by **doubling** the output from the Sine Generator. At the output of the frequency doubler, the 2f reference is by default set to be "in phase" with the 1f signal input and to have the same amplitude ($1V_{eff}$, to name it).

The **experiment** is **modulated** - as before - with frequency **1f !**

Note that the Phase Shifter acts on the 1f sine from the Sine generator. This means that a phase shift of , e.g., π in 1f corresponds to a shift of 2π in the frequency doubled signal. For maximum LI output in the present 2f case, the phase on the Lock-In panel could have to be re-adjusted.
(The **correct phase setting**, of course, **depends on the place where the frequency doubler is located: before or behind the phase shifter**; in our case, it's behind).

This situation is **not very easy to understand**, so you may think a little bit about it ...

The Lock-In Analyzer

As we have seen in Part Two, the output of the Lock-In amplifier depends on the cosine of the pase difference between the "signal" and the "reference" input into the Phase Sensitive Detector (assuming the two signals to have the same frequency). So, in order to get the maximum output amplitude, you have to adjust the setting of the phase shifter very carefully to $\Delta = 0$. But, given a very small, noisy input signal, how do you get this special setting ?

Well, of course, you can **try** (and **hope** ....).

More promising, however, is the following **procedure** :

Choose an arbitrary phase setting and watch the output signal. Then vary the phase until the output becomes zero. Now you have a phase difference of 90°or 270° between the input and the reference signals. The rest is easy: Make a phase change of ± 90°, and you have got it ....

This was in fact the proper, old fashioned, method to find out the optimal phase setting .... until the invention of the Lock-In Analyzer. This instrument **doesn't need any phase setting from the user's side**.

How does the Analyzer work ?

In Part One, we used the auto-correlation function

(27) $$R_1(\Delta) = \frac{ab}{nT}\int_0^{nT} \sin(\omega t)\cdot\sin(\omega t + \Delta)\, dt$$

with the result

(28) $$R_1(\Delta_1) = U_1 = \frac{ab}{2}\cdot\cos\Delta_1$$

If we now **shift** the phase angle Δ_1 in eq.(28) by an amount of $\pm\pi/2$, then we can use this **new phase setting** $\Delta_2 = \Delta_1 \pm \pi/2$ equivalently in our Correlation Function

(29) $$R_2(\Delta_2) = \frac{ab}{nT}\int_0^{nT} \sin(\omega t)\cdot\sin(\omega t + \Delta_2) dt = \frac{ab}{nT}\int_0^{nT} \sin(\omega t)\cdot\sin(\omega t + \Delta_1 \pm \frac{\pi}{2}) dt$$

with the result

(30) $$R_2(\Delta_1) = U_2 = \pm\frac{ab}{2}\cdot\sin\Delta_1$$

Now, we have **two output voltages**, U_1 and U_2, depending on Δ_1 and Δ_2, respectively, which we now may **add as polar coordinates** to get the **output voltage**

(31) $$U_{out} = \sqrt{U_1^2 + U_2^2} = \sqrt{\left(\frac{ab}{2}\right)^2 \left[\cos^2\Delta + \sin^2\Delta\right]} = \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = a_{rms} \cdot b_{rms}$$

Similarly, we now can calculate the **phase difference** between the input and the reference signals:

(32a) $$\tan\Delta = \frac{\sin\Delta}{\cos\Delta} = \frac{U_2}{U_1}$$

or

(32b) $$\Delta = \tan^{-1}\left(\frac{U_2}{U_1}\right)$$

If you adjust the reference voltage, $a_{rms,}$ to unity, then the **output reading U_{out}** will be b_{rms}, that is: the **rms signal** you are looking for, without any phase adjustments.

In this way, the Lock-In Analyzer gives you **two** output voltage values :

one for the **rms value of the input signal**, and the other one for the **phase difference** between signal and reference.

"In reality", the necessary mathematical operations in the above calculations are performed by specialized integrated analog circuits,the so-called "Multifunction Converters" (eg. Burr-Brown 4302).

University of Konstanz
Department of Physics

Advanced Physics Online Lab

The Lock-In Amplifier

In scientific as well as in many engineering measurements, frequently very small electrical signals occur which may even be totally „buried“ in surrounding noise, becoming “invisible” this way. In order to make such signals „visible“ again, measuring devices have to be used which do not only not contribute to additional noise themselves but – favorably – do even reduce already present noise, or, in technical terms : **improve the Signal-to-Noise (S/N) ratio**. Essentially three kinds of devices are being deployed for this purpose :

1. the **Lock-In Amplifier**,
2. the **Boxcar-Integrator**, and
3. the Time Averaging Computer.

Out of these three, the boxcar integrator and the time averaging computer form *time average values* of individual subsequent measurements, the boxcar in an analog way, the averaging computer digitally. In this way statistical signal fluctuations (“noise”) are *„averaged out*”. In contrast to this, the lockin acts as a **band pass filter** with **extremely narrow** bandwidth. The accompanying broad band noise spectrum is recected this way : The Signal-to-Noise ratio increases … significantly, up to 60 dB (!)

Please note that the lockin does *not* have a „common“ frequency dependent *band pass filter* built in. As you can see from the correspondent schematics (see Schematic 1, e.g.), it only contains a simple *low pass filter* the cut-off frequency of which is *independent* (!) from the center frequency of the band pass filter, i.e. the measuring frequency used in the experiment. You will explore the „myth“ of this function in detail in Part Two of this Project.

An other frequently deployed feature of the lockin is the capability of *phase sensitive detection* of (small) ac voltages. Both, the improvement of the S/N ratio as well as the phase sensitive detection feature, make the lockin one of the most interesting and most widely used measuring devices in science and technology.

In our Advanced Lab the “hardware” lockin is used with five „real“ experiments, one of them - the Phonon Spectroscopy - also being accessible over the internet.

For the first time, this complete “virtual experiment” is not being executed with a real measuring device in the lab but – for our local students also as a part of our Advanced Lab - *via the internet* with a simulation of the device. It should be noted, however, that the data worked with in the experiment are statistical data. For these it is unimportant, in principle, whether they are generated “naturally” – by a noisy device, e.g. - or „artificially“, by a satisfactory algorithm. They may be treated simply as *statistical experimental data*, with *corresponding errors* which have to be *considered seriously* in the work ! So this virtual experiment may act as a *scientific “specimen experiment”* in which all skills of the researchers are asked for a proper elaboration of the work.

All you need to perform the virtual experiment is a relatively fast computer, 1 GHz + are sufficient, an internet access, a 56k modem will do the work, and a Java-2 enabled browser with the *actual Java PlugIn* (1.4.x) from Sun installed. Since during the simulation extensive numerical calculations will be done, the faster the processor, however, the better.

Appropriate PCs are at the students' disposition in the rooms of the lab at the university as well as at many PC pools of the university.

Before performing the experiment, please become familiar with the following :

Mechanisms of noise generation : white noise, colored noise, 1/f noise, telgraph noise;
Gaussian distribution, Wiener-Khinchin theorem (one of the basic laws of communication theory : the experiment as communication process between researcher and "nature");
Basics of the mathematical description of statistical processes;
Correlation functions : auto-correlation, cross-correlation;
„Measure ratios" in electrical measures : decibel (dB), Signal-to-Noise ratio, noise factor with amplifiers;
Structure and function of high/low/band pass filters, filter charcteristics;
Charakteristic units of noise and ac voltages/currents : amplitude, rms value

References :

Noise in general :

(1) A. Ambrozy: Electronic Noise, McGraw-Hill, 1982

(2) H. Beneking: Praxis des elektronischen Rauschens, BI-Hochschulskripten 734 (German)

(3) M.J.Buckingham: Noise in Electronic Devices and Systems, Ellis Horwood, 1983

The actual topic :

(4) L.A.Wainstein and V.D.Zubakov: Extraction of Signals from Noise, Prentice Hall, 1962

(5) T.H.Wilmshurst: Signal Recovery from Noise in Electronic Instrumentation, Adam Hilger, 1990

(6) M. Stachel Advanced Lab Virtual Lock-In Project, http://www.lockin.de

Experiments and exercises:

You will find the experiment on the internet at

http://www.lockin.de/

If your institution takes part in the *World Wide Educational LockIn Project (WELIP)* supported by the Advanced Online Lab, please type your corresponding personal identification, issued to you by your institution in the **Stud ID** area, otherwise simply leave this field free. In the **Institution/University** field please indicate the *"center subdomain"* of your institution's URL, e.g. www.*uni-konstanz*.de.

The experiment consists of four "Parts" each one dealing with a special feature of the LockIn :

Part One **Noise Reduction and Phase Sensitive Detection**
Here the basic application functions of the lockin will be investigated.

Part Two **The Correlation Function**
Here you will explore the background of the noise reducing action of the lockin, and how the phase sensitivity comes about.

Part Three **Application of the lockin in physics**
You will detect very small noisy signals from a resonant absorption process, Examples : Magnetic spin resonance, plasmon resonance, etc...
Furthermore the often used „2f" mode of the lockin will be examined. If you like, you may check your knowledge and skills *in praxi* with one of our **real time online experiments**.

Part Four **The LockIn Analyzer**
The final step – until now - in lockin evolution.

The tasks proposed here are essentially a one-to-one translation of the corresponding German "working sheet" for our local physics students.

Part One

(I.1) Adjust the input signal level to S = 1 V_{rms} .
Reduce the S/N ratio to -20 dB: Your input signal – as visualized on the left screen on the lockin working panel – is completely hidden in noise.
Also is the output signal : only noise ...
Now improve the S/N ratio by means of the lockin. For that, rise the time constant, TC, and determine the corresponding noise amplitude, using the TC values generated *individually for you*, in the TC choice field.

Note : For this purpose you may use a subroutine invoked by clicking the **NAD** button on the lockin working panel („Noise Amplitude Distribution"). Here a preset number of output samples from the Integrator Output stage will be taken and depicted as a histogram over the noise amplitude.
If you do so for an infinitely long time, you will get an exact Gaussian distribution of the noise amplitudes. Since we may not want to wait such long we have limited the number of samples to max. 10 000 which, however, you should make use of whenever possible. Of interest for you is the mean amplitude as a function of TC, i.e. the *standard deviation* of the corresponding Gausian distribution.

With „**Fit Function**" you may invoke a Gaussian distribution function which you may „adjust", as a "guide to the eye", to your data. From this "optical fit" you may get the proper standard deviation, or "noise voltage", for a certain TC (see „Gaussian or Normal Distribution"). Remember : Your signal level S has been adjusted (hopefully …) to "1 V".

On an suitable drawing grid depict the *improvement* of the S/N ratio as a *function of TC*.
For this, calculate the ratio **S/N(nT) / S/N(T).**

Note : Also for this purpose we support you with an appropriate drawing tool :
Please click the **S/N(TC)** button …

Since S/N(1) is your basis for all subsequent calculations, please take at least *5 NAD diagrams at TC = 1T* , and calculate the average value $V_{sig} / \langle V_{noise} \rangle|_{TC=1T}$.

Exercises : (a) How large is the improvement of S/N(T), in dB, as compared to S/N(0) ?

(b) Determine (graphically) the *functional dependence S/N(nT) / S/N(T).*

(Note : Within the „Function Fit" diagram you may create a straight line through the origin (1,1), the slope of which may be adjusted by pressing the right mouse button and "dragging" it closely along the right hand plot border.)

© What S/N value do you expect for TC = 10^5T with respect to your *original* S/N ratio of *- 20 dB* ?
(Note : Don't forget to take into consideration the improvement of your original S/N by the *initial integration S/N(T)*)

(d) What TC will yield an S/N improvement of 60 dB ?

(I.2) Verify eq.(1) in the introductory text. For this, please have a look at the deduction of the original equations in Part Two : The Correlation Function (eq. 4 … 6).

At what phase values will there be maximum output voltage levels ?
At what phase values will the output voltage become zero ?

(I.3) Why does one need an adjustment range of 360° for the phase ?

(I.4) Does the phase setting have an influence on the output noise voltage, for fixed TC settings ? Check your answer experimentally and explain your result.

(I.5) Commercially available lockin amplifiers generally make use of a *square wave reference* instead of a sine. What difference does this make with respect to the output signal, or, asked inversely : what is the advantage of a sine reference ?

(I.6) Can you deploy the lockin also for working with *time dependent* input signals (apart from the modulation and noise ...) ? Under what conditions with respect to the changes ?

Part Two

(II.1) Measure the output voltage of the lockin as a function of the frequency ratio f/fo (f = input frequency, fo = reference frequency) for different integration times, TC. Plot your data, for each TC, in an appropriate drawing grid (see Note below).

For this task you best use the "Noise OFF" setting from the S/N choice field.
Input signal level again 1V.

Note : Here you need a special - „octave“, not decade - log drawing grid for a “visual linearization“ of your data. You may invoke such a drawing area by clicking the **V(f)** button. Draw your data with a left mouse click in the grid. You may delete wrong data with a right mouse click (please aim exactly ...). Clicking the „Reset“ button will delete the complete plot, so : please attention.

Printing of the plot with a screendump routine, or by Alt/Print + MS Paint (for Windows ...)

(II.2) Your V(f) diagrams show the typical structure of *band pass* filters, the *bandwidth depending on TC.*
For the evaluation of your data, a click on “Fit Function” will invoke again an appropriate graphical tool with a couple of parameters to be adjusted visually.

Exercise : Determine the band widths (full width at -3 dB (=0.707) from maximum) of your filter characteristics as a function of TC.

The commonly used power transmission functions P(f) – **Spectral Power Density** – of the filters are obtained by squaring the voltage characteristic V(f) [$P=V^2/R$], in this case : band width = full width at half maximum (fwhm).

(II.3) Draw your band with ratios BW(TC)/BW(1) over TC.

For this you can make use of the drawing aid „BW(nT)/BW(1)“ (**BW-Plot** button).

Exercises : (a) What *functional dependence* do you get ? („visual fit“ as in I.1, S/N-Plot)
(b) What - absolute – band widths will you get for 10^4T and 10^5T ?

Compare your S/N(TC) dependence from Part I with your band width data :
(c) What correlation exists between the S/N ratio and the band width, i.e. S/N(BW) ?
(d) Do your results „confirm“ the Wiener-Khinchin-Theorem ?

(II.4) Please choose from the Time Constant menu “none”, and have a look at the product of two harmonic functions, i.e. eq. (6) , without integration.

Exercises : (a) What is the frequency of the output signal for fo=1 ?
(b) What happens when you change the frequency ?
(c) Please try to explain the „undershots“ (negative output voltage values) in your V(f) characteristics.

Note : Please have a look at the *Fourier transform* of the respective products.

(II.5) According to eq.(2 and 3) the output voltage from the lockin has the value of a definite integral, i.e. it is a *dc voltage*.
However, at the output of the multiplier stage you see an *ac voltage*. What is the reason of the subsequent “rectification” of the voltage ?

(II.6) According to your previous investigation the lockin in fact behaves as a narrow band pass filter the band width depending on TC.

Is there any connection between the pass frequency and TC ?

(II.7) In Part One you worked with a kind of „white“ noise. In general, however, rather a so-called „1/f“ noise will be encountered, a type of noise with strong low frequency components.

Does, to your present knowledge, the lockin also reduce these low frequency components from the remaining output noise ?

How would you explain the fact that low frequency noise components appear still much stronger in the output signal than higher frequency components ?

Hint : Compare the power density spectra of white noise vs. 1/f noise.

(II.8) In the introductory text we boldly pretend that the phase parameter Δ in eq.(3) is unimportant for our experiment. Why is this valid only for $f \neq fo$, resp. $\omega \neq \omega o$?

Part Three/1

(III.1) „At first sight" the output characteristic of the lockin looks like the first derivative of the "system function". In general, however, this is *not* the case as you may convince yourself by varying the (red) "Modulation Amplitude".

For this purpose, please take several output plots for varying modulation amplitudes, and determine the respective fwhm values (see above).

Under what circumstances does the lockin output represent in fact the first derivative of the system function ?

(III.2) Create a noisy system function by adding noise to it.
With rising modulation amplitude, the S/N ratio on the output screen increases, whereas the line shape will be more and more distorted.

Find an „optimum" modulation amplitude where there is :
A suffucientliy large output signal, a large S/N ratio, and an adequate reproduction of the line shape of the system function. (This will be one of the standard problems when working with the lockin).

(III.3) Consider a real experimental situation : You get a very small noisy output signal from your experiment ...

Questions : (a) What initial modulation amplitude would you use to „find" the signal ?
(b) This initial amplitude is not very well suited in general for the subsequent measurements, why ?
© What are the criteria for a maximum modulation amplitude, with regard to "output signal quality" ?

(III.4) As you know from Part Two of this Project, the output signal from the LI becomes zero with vanishing input signal, and/or at a phase difference of $\pm 90°$ compared with the reference wave. As you may see easily from the yellow part of the drawing on the left hand system screen, none of these conditions is, however, fulfilled when the (blue, right hand) output signal passes through zero : The input signal is larger than zero, and the preset phase difference is equal to 0°.

Question : Why will the output signal be zero in this case ?

(III.5) The yellow curve on the left hand screen represents something like a sine wave, mirrored at the system function. For this reason it contains many *non-linearities* of the system function, in short : *harmonic waves* ("harmonics"):

Question : What happens when the phase sensitive detector will be fed with a *harmonic*, instead with the *fundamental frequency* of the modulation ? What is the lockin sensitive for, in this case ?

(III.6) What is the difference in shape of the output signal, if any, when using a square wave as the reference instead of a sine wave ? Make use of the square wave reference tool to investigate this problem.

(III.7) Choose the „Straight Lines“ as the system function. You may change the shape of the input function by “dragging” around the “kinks” with the left mouse button.
What is the shape of the output signal at the straight lines, what at the kinks/bends ?
On what “kink propert”y does the output signal level depend at the kinks ?
What line shape do you expect after a differentiation of your original lockin output ?

Part Three/2

(III.7) Why does the maximum output signal of the lockin at a phase difference of 45° instead at 0°, as in the „1f“ case ?

Hint : Compare the yellow output from the experiment at the minimum of the resonance with the Fourier series of f(x)=|sin(x)| ...

(III.8) Interchange the places of the phase shifter and the frequency doubler in the “2f” schematic : At what phase difference would you expect a maximum lockin output signal in this case?

(III.9) Under what circumstances will a „true“ 2nd derivative of the system function appear at the lockin output ? Would you really like to work this way ?

(III.10) In the appendix you find three original screenshots of our Phonon Spectroscopy online real time experiment :
fig. 1 shows the I(U) characteristic of a backward tunnel diode at room temperature,
fig. 2 the same but for the actual experiment situation, i.e. at 4.2 K (liquid Helium).

Simulate a system function with the „**Straight Lines**“ tool of the applet corresponding to the *characteristic features of fig.2* (in comparison to fig.1) [the cornerpoints of the straight lines may be adjusted properly by dragging them as described above ...].
Draw the corresponding lockin output voltage run.

Compare your resulting lockin output with the original 2f output plot, fig. 3.
Please discuss the common features and differences of both plots (“theory” and “reality“ …).

Try to reproduce your original system function by double integration of the lockin output (variation of the intergration constant by “IC1” slider).

Invert your initial phase setting (e.g. - 45° instead of + 45°). Why can’t you reconstruct the left hand original shape any more ?

(III.11) What params are responsible for the height of the peaks in the lockin output for the straight lines input configuration ?

Part Four

(IV.1) What is the reason of the instability of the phase reading at ± 90° ?

(IV.2) Are there phase readings available for the whole phase regime of 0 ... 360° ? Reasons ?

(IV.3) What output voltage shapes would you expect if you scan the system functions of Part Three by means of a lockin analyzer ?

(IV.4) For what kind of measurements would you prefer the lockin analyzer, for what the "common" lockin ?

Appendix

Fig. 1

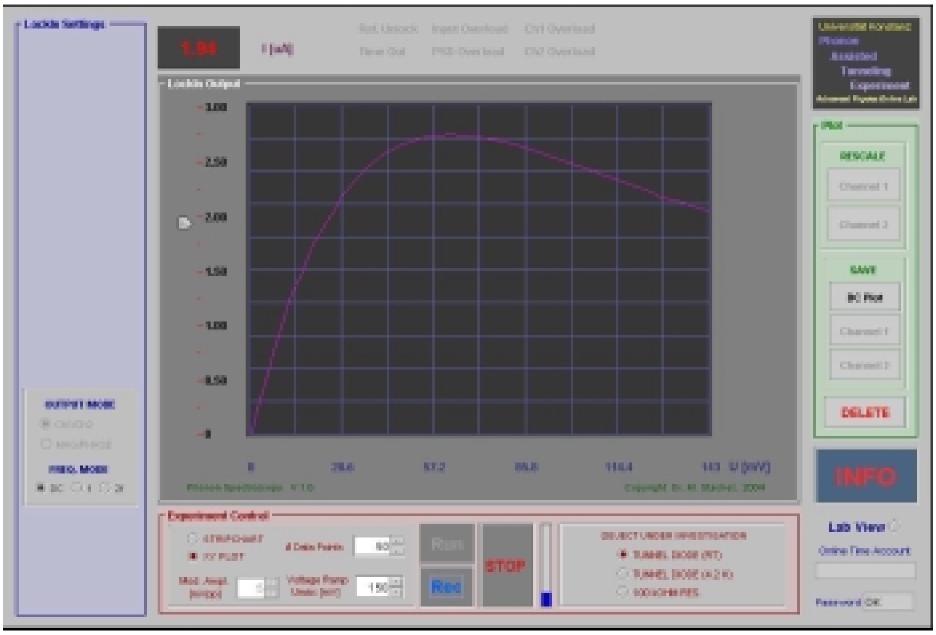


Fig. 2

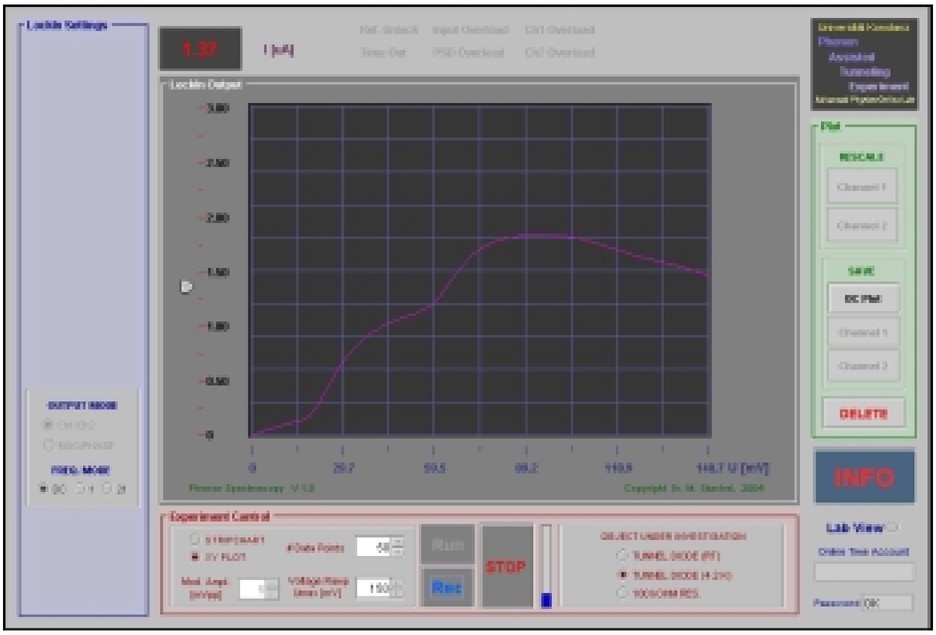


Fig. 3