Frustrating geometry: Non Euclidean plates

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Outline:

• Growth and geometric frustration.
• Incompatible 3D hyperelasticity.
• Non Euclidean plates: Examples.
• Reduced 2D elastic theory of non Euclidean plates.
• Non Euclidean plates: Analysis.
• An application: Almost minimal surfaces.
Outline:

- **Growth and geometric frustration.**
- Incompatible 3D hyperelasticity.
- Non Euclidean plates: Examples.
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- An application: Almost minimal surfaces.
Plant growth mechanics
• Residual stress due to local growth.
• Biological response to mechanical stress.
• Manipulating growth regulation causes morphological changes.

Growth

Arabidopsis mutant  Peach leaf curl  CINCINNATA mutant,
[Nath et al 2003]
Time scale separation

\[ \tau_{\text{Elastic}} << \tau_{\text{Shaping}} \]

Plant growth (Arabidopsis):

- Cell division: hours
- 2 % area growth by expansion: hours
- Acoustic time: 1 µs
- Cantilever mode typical time: 100 µs
Spontaneous growth and geometric frustration

Why local growth is likely to result in residual stress?

Given 8 points 28 rods of \textit{arbitrary} lengths, you will (probably) not be able to connect every pair once.

“think in rod length rather than points”
Spontaneous growth and geometric frustration

Why local growth is likely to result in residual stress?

Example: Isotropic non-homogeneous growth. Every point initially at rest expands isotropically by a factor $\lambda(\vec{r})$. 
Spontaneous growth and geometric frustration

Why local growth is likely to result in residual stress?

Example: Isotropic non-homogeneous growth. Every point initially at rest expands isotropically by a factor $\lambda(\vec{r})$.

The only expansion factor which does not result in residual stress:

$$ \lambda(\vec{r}) = \frac{C}{|\vec{r} - \vec{r}_0|^2} $$
Soft active deformations

Residual stress must be addressed when describing motion by auto-deformation.

Making use of unused “residual work”.
Signature of geometric frustration

**Tempered glass fragments cannot be “re-joined”**
When tempered glass shatters, each of its fragments deforms to relax internal stresses. The relaxed fragments do not fit one another.
Signature of geometric frustration

The helicoidal form of a Bauhinia pod is residually stressed. Slicing along its length generate helices of a higher pitch.
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Hyper-elasticity

[Truesdell 1952]

A measure of local deformation: Strain

+ Local elastic energy density as a function of the deformation

= Hyperelastic description

\[ E = \iiint W(\varepsilon) dV \]

Stress is obtained as a Frechet derivative of the elastic energy density.

\[ S = \frac{\partial W}{\partial \varepsilon} \]
A measure of deformation: The Strain

The displacement field \( u = r - x \)

\[
\varepsilon = \frac{1}{2} \left( \nabla u + (\nabla u)^T + (\nabla u)^T \nabla u \right)
\]

Cauchy St-Venant strain
A measure of deformation: The Strain

The displacement field $u = r - x$

Rely on the existence of a stress free configuration
Differential geometry I : Three dimensions

- The metric tensor: \( g_{ij} = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j} \)

- Length element: \( ds^2 = g_{ij} dx^i dx^j \)

A 3 by 3 matrix is the metric of a body in Euclidean space.

All components of the Riemann curvature tensor vanish.

Such metrics are called:

- Euclidean
- Compatible
- Embeddable

If two embeddable metrics are identical, the bodies they describe differ by a rigid motion.
The energy stored within a deformed elastic body is a volume integral of an elastic energy density which depends only on the local metric tensor and on tensors that characterize the body, but are independent of the configuration.

3D “incompatible” hyperelasticity

3D metric determine a configuration uniquely.

Our adaptation of Truesdell’s hyperelasticity principle is formulated in terms of metric.
3D “incompatible” hyperelasticity

- The energy density depends on the configuration through the metric.

- For every $x$ there exists a unique SPD tensor $\bar{g}(x)$ for which the energy density vanish, i.e.

- $\bar{g}(x)$ is called the reference metric.

- The Cauchy-St. Venant strain

\[ \epsilon = \frac{1}{2}(g - \bar{g}) \]

current configuration metric  \hspace{2cm} reference metric

\[ W(\epsilon, x) = 0 \iff \epsilon = 0 \]
When a stress free configuration exists, we may set $\bar{g} = I$.

In such coordinates the Cauchy-St. Venant strain adopts the familiar form

$$\varepsilon = \frac{1}{2} (g - I) = \frac{1}{2} \left( (\nabla r)^T \nabla r - I \right) = \frac{1}{2} \left( \nabla u + (\nabla u)^T + (\nabla u)^T \nabla u \right)$$
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\[
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\]

For such a case the elastic energy density of an isotropic material is of the form

\[
W = A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + O(\varepsilon^3)
\]

\[
A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)
\]

\text{Lame coefficients}
3D “incompatible” hyperelasticity

When a stress free configuration exists, we may set $\bar{g} = I$.

In such coordinates the Cauchy-St. Venant strain adopts the familiar form

$$\varepsilon = \frac{1}{2} \left( g - I \right) = \frac{1}{2} \left( (\nabla r)^T \nabla r - I \right) = \frac{1}{2} \left( \nabla u + (\nabla u)^T + (\nabla u)^T \nabla u \right)$$

For such a case the elastic energy density of an isotropic material is of the form

$$W = A'_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + O(\varepsilon^3)$$

$$A'_{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)$$

A stress free configuration may not exist.
Locally (only at a point) we may set new coordinates in which \( \bar{g}' = I \)

\[
W = \frac{1}{2} A^{ijkl} \varepsilon'_{ij} \varepsilon'_{kl}
\]

\[
\varepsilon'_{ij} = \frac{1}{2} \left( g'_{ij} - \delta_{ij} \right)
\]

\[
A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)
\]
3D “incompatible” hyperelasticity

Locally (only at a point) we may set new coordinates in which \( \bar{g}' = I \)

\[
W = \frac{1}{2} A^{ijkl} \varepsilon'_{ij} \varepsilon'_{kl} \\
\varepsilon'_{ij} = \frac{1}{2} (g'_{ij} - \delta_{ij}) \\
A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})
\]

Tensor transformation rules
(Covariance becomes valuable)

\[
\Lambda^k_i = \frac{\partial x'^k}{\partial x^i} \\
(\Lambda^{-1})^k_i = \frac{\partial x^k}{\partial x'^i}
\]
3D “incompatible” hyperelasticity

Locally (only at a point) we may set new coordinates in which $\bar{g}' = I$

$$W = \frac{1}{2} A^{ijkl} \varepsilon'_{ij} \varepsilon'_{kl}$$

$$\varepsilon'_{ij} = \frac{1}{2} (g'_{ij} - \delta_{ij})$$

$$A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})$$

Tensor transformation rules

(Covariance becomes valuable)

$$\Lambda^k_i = \frac{\partial x^k}{\partial x^i}$$

$$\left(\Lambda^{-1}\right)^k_i = \frac{\partial x^k}{\partial x'^i}$$

$$W = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2} (g_{ij} - \bar{g}_{ij})$$

$$A^{ijkl} = \lambda \bar{g}^{ij} \bar{g}^{kl} + \mu (\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk})$$
**3D “incompatible” hyperelasticity**

Locally (only at a point) we may set new coordinates in which $\bar{g}' = I$

$$W = \frac{1}{2} A^{ijkl} \varepsilon'_{ij} \varepsilon'_{kl}$$

$$\varepsilon'_{ij} = \frac{1}{2} \left( g'_{ij} - \delta_{ij} \right)$$

$$A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)$$

Tensor transformation rules

(Covariance becomes valuable)

$$\Lambda^k_i = \frac{\partial x^k}{\partial x^i}$$

$$\left( \Lambda^{-1} \right)_i^k = \frac{\partial x^r}{\partial x'^i}$$

$$\bar{g}_{ij} = \Lambda^k_i \Lambda^l_j \delta_{kl}$$
3D “incompatible” hyperelasticity

Recapitulation

\[ E = \iiint W \sqrt{g} \, dx^1 \, dx^2 \, dx^3 \]

\[ \varepsilon_{ij} = \frac{1}{2} \left( g_{ij} - \overline{g}_{ij} \right) \quad \text{ } \quad W = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \varepsilon_i^i \varepsilon_k^k + \mu \varepsilon_i^i \varepsilon_k^k \]

Strain is measured with respect to a reference metric.

[Effati Sharon & Kupferman, JMPS 2009]
3D “incompatible” hyperelasticity

Recapitulation

\[ E = \iiint W \sqrt{|g|} dx^1 dx^2 dx^3 \]

\[ \varepsilon_{ij} = \frac{1}{2} \left( g_{ij} - \bar{g}_{ij} \right) \quad \text{and} \quad W = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \varepsilon_i^i \varepsilon_k^k + \mu \varepsilon_i^i \varepsilon_k^k \]

Strain is measured with respect to a reference metric.

[\text{Efrati Sharon & Kupferman, JMPS 2009}]

The reference metric need not be immersible.
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Environmentally responsive gels.
• N-Isopropylacrylamide undergoes a large volume reduction when heated above 33 °C.
• The volume reduction ratio depends strongly on monomer concentration.
• When injected cold with a catalyst, the gel polymerizes in a few minutes, “freezing” the monomer concentration gradient.

[Klein Efrati & Sharon. Science 2007]
Experimental realization of Non Euclidean plates

[Klein Efrati & Sharon. Science 2007]
Experimental realization of Non Euclidean plates

- Does not shrink
- Shrinks considerably

\[ \text{Shrinkage homogeneous across the thickness} \]

Large thickness: Flat and strained.
Small thickness: Buckled.
Experimental realization of Non Euclidean plates

- Does not shrink
- Shrinks considerably

Shrinkage homogeneous across the thickness

Large thickness: Flat and strained.
Small thickness: Buckled.
Experimental realization of Non Euclidean plates

Hyperbolic metric (saddle like everywhere)
Experimental realization of Non Euclidean plates

The only shaping mechanism is the Prescription of a Non-Euclidean Metrics
Shaping by metric prescription

Leaf shaping

Plastic deformation

Crocheting & knitting

Uncontrolled tearing

Controlled tearing

Crocheted segment of the hyperbolic plane [Taimina 1997].
Computer simulations

Defects in flexible membranes with crystalline order
[Seung & Nelson 1988]

Recent implementation to macroalgae blades
[Koehl et al 2008]

Three layers of identical array of springs of spatially varying rest length.
[Marder et-al 2004]
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Differential geometry II : Surfaces

The first quadratic form (metric) \[ a_{\alpha\beta} = \partial_\alpha r \cdot \partial_\beta r \]

The second quadratic form (curvatures) \[ b_{\alpha\beta} = \partial_\alpha r \cdot \hat{N} \]

The mean curvature

\[ H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{1}{2} \text{Tr}(b_\alpha^\beta) \]

The Gaussian curvature

\[ K = \kappa_1 \kappa_2 = \det(b_\alpha^\beta) \]

\[ K = F(a_{\alpha\beta}, \partial_\gamma a_{\alpha\beta}, \partial_\gamma \partial_\delta a_{\alpha\beta}) \]

Gauss’ Theorema Egregium
Differential geometry II : Surfaces

The first quadratic form (metric) \( a_{\alpha\beta} = \partial_{\alpha} \mathbf{r} \cdot \partial_{\beta} \mathbf{r} \)

The second quadratic form (curvatures) \( b_{\alpha\beta} = \partial_{\alpha\beta} \mathbf{r} \cdot \hat{N} \)

The mean curvature
\[
H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{1}{2} Tr(b_\beta^\alpha)
\]
The Gaussian curvature
\[
K = \kappa_1 \kappa_2 = \det(b_\alpha^\beta)
\]

\[
K = F(a_{\alpha\beta}, \partial_\gamma a_{\alpha\beta}, \partial_\gamma \partial_\delta a_{\alpha\beta})
\]
Gauss’ Theorema Egregium

\[
0 = G_{1,2}(a_{\alpha\beta}, \partial_\gamma a_{\alpha\beta}, b_{\alpha\beta}, \partial_\gamma b_{\alpha\beta})
\]
Peterson Mainardi Codazzi eq.
The first quadratic form (metric) \( a_{\alpha\beta} = \partial_{\alpha} r \cdot \partial_{\beta} r \)

The second quadratic form (curvatures) \( b_{\alpha\beta} = \partial_{\alpha\beta} r \cdot \hat{N} \)

Given two quadratic forms which satisfy all three GPMC equations, they define a surface uniquely.

\[
K = F(a_{\alpha\beta}, \partial_{\gamma} a_{\alpha\beta}, \partial_{\gamma} \partial_{\delta} a_{\alpha\beta})
\]

Gauss’ Theorema Egregium

\[
0 = G_{1,2}(a_{\alpha\beta}, \partial_{\gamma} a_{\alpha\beta}, b_{\alpha\beta}, \partial_{\gamma} b_{\alpha\beta})
\]

Peterson Mainardi Codazzi eq.
A toy model
Trapezoidal inclusion

Large thickness: Flat configuration. The trapezoid is under compression.

Small thickness: Buckled configuration. The trapezoid is bent to accommodate the excess length.

Similar lengths

Too long
A toy model

No stress free configuration

- “Rest lengths” are continuous
- The body possess no internal structure across the thin dimension
Neither plates nor shells

A plate may be considered as a stack of identical surfaces

A shell may be considered as a continuous collection of non-identical surfaces

Similarly to plates: no structural variation across the thin dimension

Unlike plates: 2D metric of mid-surface non-Euclidean
Neither plates nor shells

A plate may be considered as a stack of identical surfaces

A shell may be considered as a continuous collection of non-identical surfaces

Existing plate/shell theories do no apply

Similarly to plates: no structural variation across the thin dimension

Unlike plates: 2D metric of mid-surface non-Euclidean
Neither plates nor shells

\[-\frac{t}{2} \leq x^3 \leq \frac{t}{2}\]
denotes the coordinate along the thin dimension

Non-Euclidean plate metric

\[
\bar{g} = \begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} & 0 \\
\bar{a}_{21} & \bar{a}_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \partial_3 \bar{a}_{\alpha\beta} = 0.
\]

\[
K(\bar{a}_{2D}) \neq 0 \iff R_{ijkl}(\bar{g}_{3D}) \neq 0
\]
Reduced 2D elastic energy

Goal: Describe the elastic behavior, using only mid-surface properties; First and second fundamental forms.

\[ E = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left( \frac{Y}{2(1+\nu)} \int \int \left[ \varepsilon_i^k \varepsilon^i_k + \frac{\nu}{1-2\nu} \varepsilon^i_k \varepsilon^k_i \right] \sqrt{g} \left| d\mathbf{x}^1 d\mathbf{x}^2 \right| \right) d\mathbf{x}^3 \]

• Integrate out the thin dimension.

• Assumption: Faces of sheet are free of forces.
Reduced 2D elastic energy

Goal: Describe the elastic behavior, using only mid-surface properties; First and second fundamental forms.

\[
E = t \int \int \frac{1}{8} A^{\alpha\beta\gamma\delta} (a_{\alpha\beta} - \overline{a}_{\alpha\beta})(a_{\gamma\delta} - \overline{a}_{\gamma\delta})\sqrt{\|a\|} \, dx^1 \, dx^2 + \\
t^3 \int \int \frac{1}{24} A^{\alpha\beta\gamma\delta} b_{\alpha\beta} b_{\gamma\delta} \sqrt{\|a\|} \, dx^1 \, dx^2 + h.o.t
\]

\[
A^{\alpha\beta\gamma\delta} = \frac{Y}{1+\nu} \left( \overline{a}^{\alpha\gamma} \overline{a}^{\beta\delta} + \frac{\nu}{1-\nu} \overline{a}^{\alpha\beta} \overline{a}^{\gamma\delta} \right)
\]

\[
a_{\alpha\beta} = \partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r}
\]

The first quadratic form (metric) of the mid-surface

\[
b_{\alpha\beta} = \partial_\alpha \mathbf{r} \cdot \hat{\mathbf{N}}
\]

The second quadratic form (curvatures) of the mid-surface
Reduced 2D elastic energy

\[ E = t^2 \hat{e}_S + t^3 \hat{e}_B \]

\[ e_S = \frac{1}{8} \int \int A^{\alpha \beta \gamma \delta} (a_{\alpha \beta} - \overline{a}_{\alpha \beta})(a_{\gamma \delta} - \overline{a}_{\gamma \delta}) \sqrt{\overline{a}} \, dx^1 \, dx^2 \]

Stretching content measures deviations from the given 2D metric. Vanishes only if the midsurface is an isometric embedding of \( \overline{a} \).

\[ e_B = \frac{1}{24} \int \int A^{\alpha \beta \gamma \delta} b_{\alpha \beta} b_{\gamma \delta} \sqrt{\overline{a}} \, dx^1 \, dx^2 \]

Bending content measures the magnitude of curvatures. Vanishes only for flat configuration.

Generalization of F.V.K and Koiter theories for large displacements and arbitrary intrinsic geometry.
Reduced 2D elastic energy

\[ E = t e_S + t^3 e_B \]

\[ e_S = \frac{1}{8} \int \int A^{\alpha\beta\gamma\delta} (a_{\alpha\beta} - \bar{a}_{\alpha\beta})(a_{\gamma\delta} - \bar{a}_{\gamma\delta}) \sqrt{|a|} \, dx^1 \, dx^2 \]

\[ e_B = \frac{1}{24} \int \int A^{\alpha\beta\gamma\delta} b_{\alpha\beta} b_{\gamma\delta} \sqrt{|a|} \, dx^1 \, dx^2 \]

Large thickness:
- Bending term dominates
- Flat solutions (unbuckled)

Small thickness:
- Stretching term dominates
- 2D isometric solution

Note: \( a_{\alpha\beta} \) and \( b_{\alpha\beta} \) are not independent variables.

[Éfrati Shron & Kupferman JMPS 2009]
Reduced 2D elastic energy

Current literature: Elastic substrate, gravitational force, capillary forces, tension, confinement or some other mechanism necessitates buckling. Competition between the two energy terms yields the buckling length scales.
**Reduced 2D elastic energy**

Current literature: Elastic substrate, gravitational force, capillary forces, tension, confinement or some other mechanism necessitates buckling. Competition between the two energy terms yields the buckling length scales.

No external force; Confinement is replaced by an embedding problem.
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• Growth and geometric frustration.
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• Non Euclidean plates: Examples.
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• Non Euclidean plates: Analysis.
  - Buckling
  - The vanishing thickness limit
  - Equipartition
• An application: Almost minimal surfaces.
Hemispherical plate

We numerically minimize the elastic energy \( E = te_S + t^3 e_B \)

\[
\bar{a} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(x^1) \end{pmatrix}
\]

We use a reference metric of a spherical cap

\( 0.1 \leq x^1 \leq 1.1 \) The solution is assumed to be axially symmetric
Hemispherical plate

\[ E = t e_s + t^3 e_B \]
Hemispherical plate

\[ E = t e_s + t^3 e_B \]
Hemispherical plate

$E = t e_s + t^3 e_B$

Almost isometric embedding
Hemispherical plate

\[ E = t e_s + t^3 e_B \]

Almost isometric embedding

Slightly buckled

Out of plane displacement vs. Thickness

Thickness

Almost isometric embedding

Slightly buckled
Hemispherical plate

Analytic results:

• The unbuckled state must contain both compression and tension.
• If $K(\bar{\alpha}) \neq 0$, there exists a finite buckling thickness.
• Classification in terms of the plane stress in the unbuckled configuration to super-critical and sub-critical bifurcations.
Hemispherical plate

We have proved that in the \( t \to 0 \) limit \( a_{\alpha\beta} = \overline{a}_{\alpha\beta} \)
i.e. an isometric embedding of mid-surface.
Hemispherical plate

We have proved that In the $t \to 0$ limit $a_{\alpha\beta} = \bar{a}_{\alpha\beta}$
i.e. an isometric embedding of mid-surface.

The isometry is (usually) not unique.
The limit configuration will minimize the bending content amongst all isometries.

Recent Gamma limit proof for non-Euclidean plates directly from 3D
[Lewicka & Pakzad 2009].
Hemispherical plate

No mutual zero to the two terms; No equipartition.

\[ E = t e_s + t^3 e_B \]

\[ e_s \xrightarrow{t \to 0} 0 \]

\[ e_B \xrightarrow{t \to 0} e_B^0 \]
Hemispherical plate

\[ E = t e_s + t^3 e_B \]

\[ e_s \xrightarrow{t \to 0} 0 \]

\[ e_B \xrightarrow{t \to 0} e_B^0 \]

The variation of the bending energy about its limit

\[ t^3 (e_B^0 - e_B) \propto t^{7/2} \]
The power $7/2$ comes from a boundary layer whose thickness scales as

$$l_{B,L} \propto \sqrt{\frac{t}{k_\parallel}}$$

The variation of the bending energy about its limit

$$t^3(e_B^0 - e_B) \propto t^{7/2}$$

[Efrati et-al PRE 2009]
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Solving the vanishing thickness limit

Provided there exists an isometric embedding with bounded energy content, the vanishing limit configuration minimizes the Willmore energy among all isometric embeddings.

Gaussian curvature \( K = k_1 k_2 \)  
Mean curvature \( H = \frac{1}{2} (k_1 + k_2) \)

\[
\frac{1}{4} \int \int \left( k_1^2 + k_2^2 + 2 \nu k_1 k_2 \right) dA = \int \int \left( H^2 - \frac{1-\nu}{2} K \right) dA
\]
Solving the vanishing thickness limit

Provided there exists an isometric embedding with bounded energy content, the vanishing limit configuration minimizes the Willmore energy among all isometric embeddings.

\[ \int \int H^2 \, dA \]

[Willmore 1986]

Constrained minimization problem: The GMPC equations.

Gaussian curvature \( K = k_1 k_2 \) \hspace{1cm} \text{Mean curvature } \( H = \frac{1}{2} (k_1 + k_2) \)
Differential geometry III: Isometric embedding of surfaces in 3D

The Gauss Peterson Mainardi Codazzi compatibility conditions may be considered as evolution equations for the curvature form, $b_{\alpha\beta}$.

- For $0 < K$ elliptic set of P.D.Es.

- For $K < 0$ hyperbolic set of P.D.Es.

\[ K = \frac{1}{9} cm^{-2} \]

\[ t = 0.075 cm \quad t = 0.06 cm \quad t = 0.025 cm \quad t = 0.019 cm \]
Almost minimal surfaces

Minimal surfaces are surfaces of vanishing mean curvature \( H = 0 \)

- Constitute trivial minima of the Willmore energy. \( \iint H^2 \, dA \)
- Not every metric can be embedded as a minimal surface.
Almost minimal surfaces

Minimal surfaces are surfaces of vanishing mean curvature $H = 0$

- Constitute trivial minima of the Willmore energy. $\iint H^2 \, dA$
- Not every metric can be embedded as a minimal surface.

The mean curvature of any hyperbolic metric can vanish along a curve
Almost minimal surfaces

We considering a thin narrow hyperbolic strip \( t \ll w \ll L \)

\[
E_W = \int_{-\frac{w}{2}}^{\frac{w}{2}} \int H^2 \, dA
\]
Almost minimal surfaces

We considering a thin narrow hyperbolic strip \( t \ll w \ll L \)

\[
E_W = \int \left( e_0 + e_1 w + e_2 w^2 + e_3 w^3 + \ldots \right) dx
\]
Almost minimal surfaces

We are considering a thin narrow hyperbolic strip \( t << w << L \)

\[ E_w = O(w^5) \]

- On the mid-line of the strip the mean curvature vanishes.
- The resulting ribbons resemble minimal surfaces.
- Both chiralities possess the same energy, thus are equally probable.
Almost minimal surfaces

Inspired by the result that every (narrow enough) hyperbolic metric can be embedded as an almost minimal surface, we found new family of pseudospherical embeddings (surfaces of constant Gaussian curvature: $K=-1$).
Almost minimal surfaces

Inspired by the result that every (narrow enough) hyperbolic metric can be embedded as an almost minimal surface, we found new family of pseudospherical embeddings (surfaces of constant Gaussian curvature: $K=-1$).

The new family of pseudospherical embeddings also admits embeddings of very large mean curvature.
Almost minimal surfaces
Prescribing a hyperbolic geometry on a strip using responsive gels

Do not Shrink

Shrinks
Almost minimal surfaces
Prescribing a hyperbolic geometry on a strip using responsive gels
Almost minimal surfaces

- The mean curvature quadratic in the width coordinate.
Almost minimal surfaces

- The mean curvature quadratic in the width coordinate.

- Results become irrelevant for wide strips.
Conclusion

• Residual stress inevitable in local shaping mechanisms.

• “Incompatible” elasticity is the theoretical tool to deal with residual stress.

• Non-Euclidean plates: - Derivation from 3D elasticity.
  - Buckling, Boundary layer.
  - The isometric embedding problem.

• Application: Almost minimal surfaces.
Conclusion

- Residual stress inevitable in local shaping mechanisms.
- “Incompatible” elasticity is the theoretical tool to deal with residual stress.
- Non-Euclidean plates: - Derivation from 3D elasticity.
  - Buckling, Boundary layer.
  - The isometric embedding problem.
- Application: Almost minimal surfaces.
- Natural extension: Non-Euclidean shells.
  Elastic theory of non-Euclidean plates and shells [submitted to Nonlinearity 2009]
- Geometric applications: Folding and controllability of non Euclidean plates.
The End