Efimov Physics Beyond Three Particles

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on
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Illustration: Harald Ritsch/IQOQI

Universality

- Consider particles interacting through 2-body potential with range R.
- Classically, the particles 'feel' each other only within the potential range.
- But, in the case of resonant interaction, the wave function has much larger extent.
- At low energies, the 2-body physics is govern by the scattering length, a.

$$\lim_{k\to 0} k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}r_0 k^2$$

- When $|a| \gg R$ the potential details has no influence: *Universality*.

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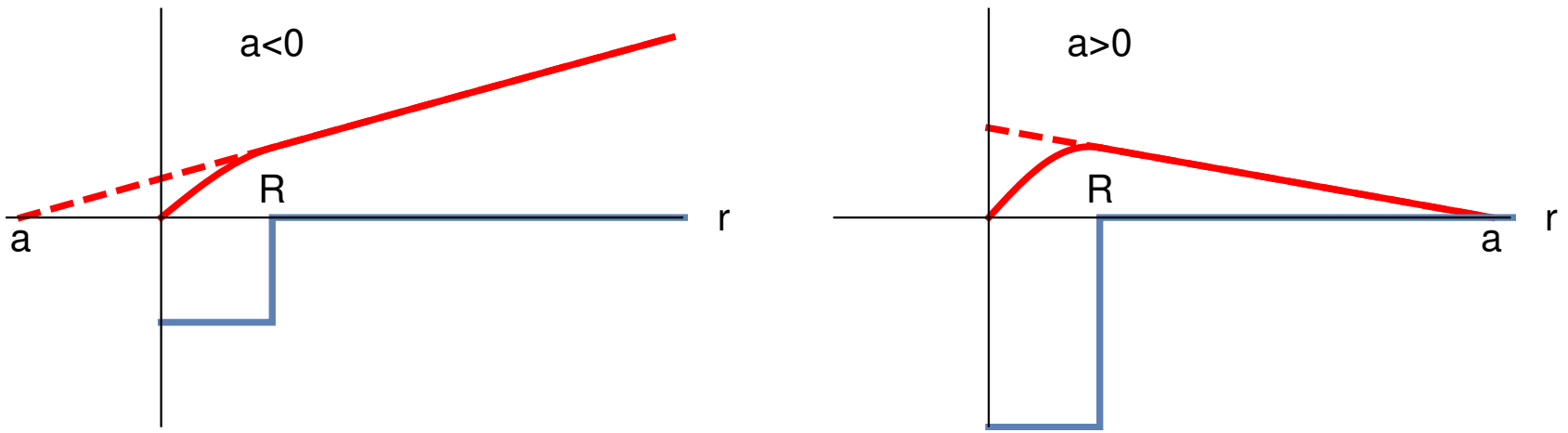

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Universality

- Naturally, $a \approx r_0 \approx R$.
  Universal systems are fine-tuned to get $a \gg r_0, R$.
- Corrections to universal theory are of order of r_0/a and R/a.
- For $a > 0$, we have universal dimer with energy $E = -\hbar^2/ma^2$.
- Nucleus: $a_s \approx -23.4$ fm, $a_t \approx 5.42$ fm, $R = \hbar/m_\pi c \approx 1.4$ fm.
  Deuteron binding energy, 2.22 MeV, is close to $\hbar/ma_t^2 \approx 1.4$ MeV.
  cf. Ronen Weiss, Gautam Rupak, Sebastian König, ... talks
- Ultracold atoms near a Feshbach resonance,

$$a(B) = a_{bg}\left(1 + \frac{\Delta}{B - B_0}\right)$$

- ^{4}He atoms: $a \approx 95$ Å$\gg r_{vdW} \approx 5.4$ Å.
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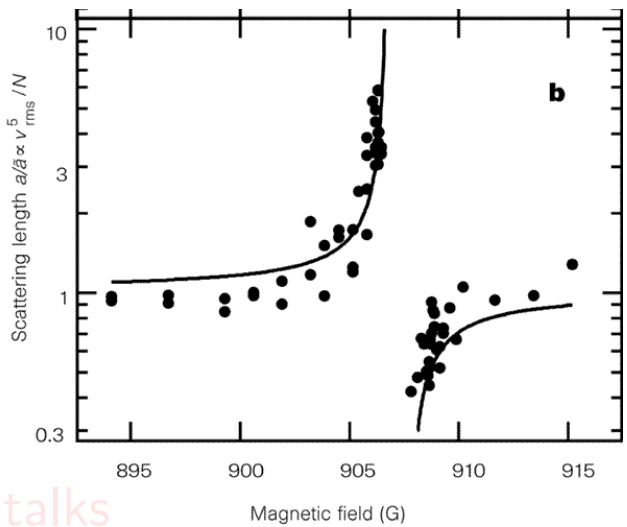

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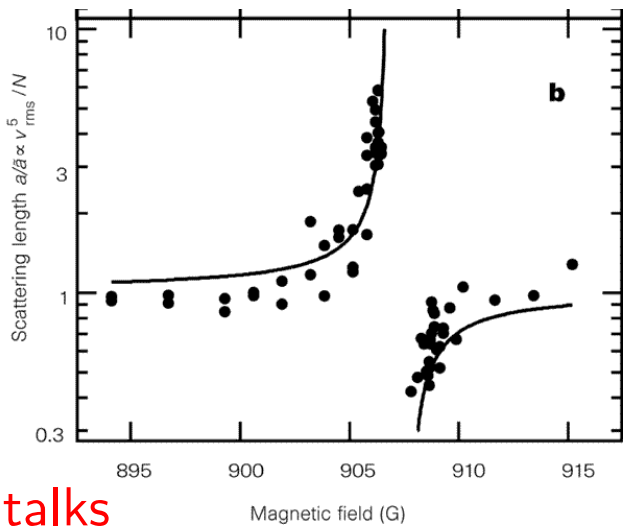

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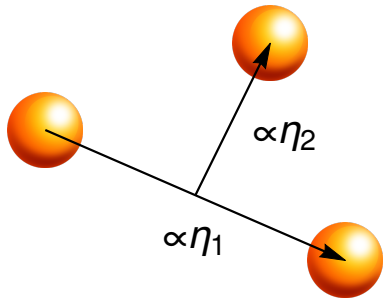
Efimov Physics for 3 Identical Bosons I

- In the zero-range approximation, one can replace the interaction by the Bethe-Peierls boundary condition,

$$\frac{\partial \log(r_{ij}\psi)}{\partial r_{ij}} \xrightarrow[r_{ij}\to 0]{} -\frac{1}{a}$$

to be applied to the Faddeev component ψ.



- In the *unitary limit*, $a \longrightarrow \infty$, the hyper-radial $\rho^2 = \eta_1^2 + \eta_2^2$ equation decouples,

$$\left(-\frac{d^2}{d\rho^2} + \frac{{s_0}^2 - 1/4}{\rho^2}\right) R(\rho) = E R(\rho)$$

zero-range 2-body interaction $\Longrightarrow$ long-range 3-body interaction!

- For 3 identical bosons, $s_0 \approx 1.00624i$: fall of a particle to the center in ρ^{-2} potential $\Longrightarrow$ scale invariance $\Longrightarrow$ Thomas collapse

Thomas, Phys. Rev. 47, 903 (1935)

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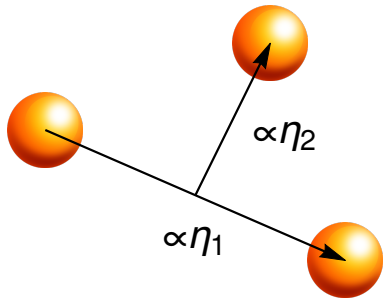

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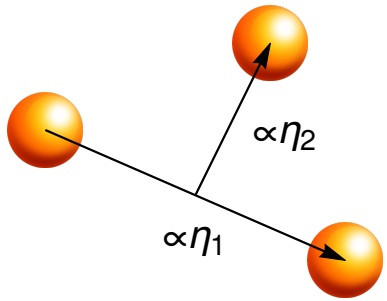

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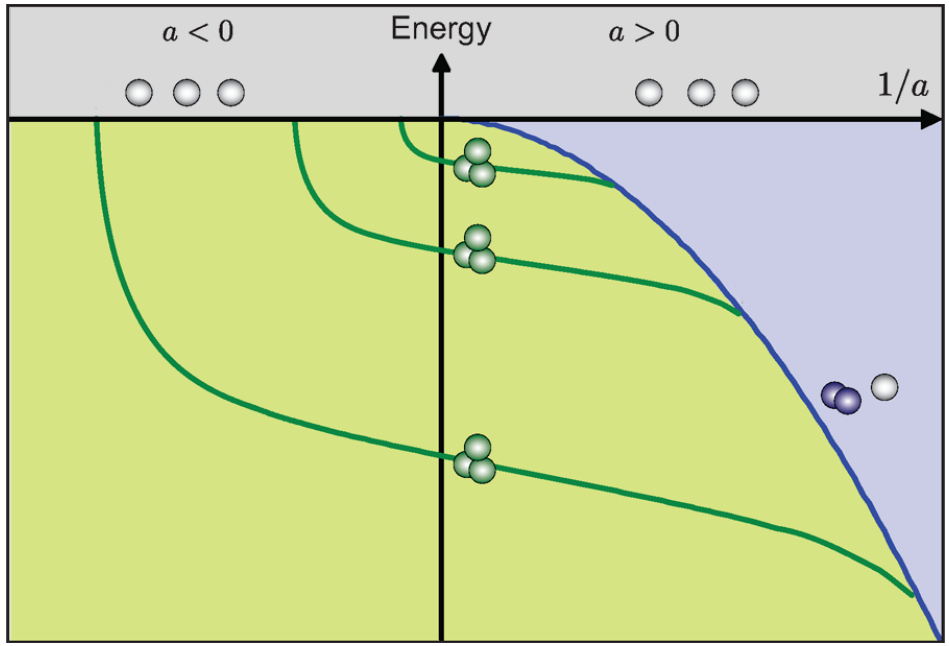
Efimov Physics for 3 Identical Bosons II

- Assume 3-body potential barrier at finite ρ_0
- infinite number of bound states $E_n = E_0 e^{-2\pi n/|s_0|}$ with $e^{2\pi/|s_0|} \approx 515$
- discrete scale invariance: $\kappa_n = \kappa_0 e^{-\pi n/|s_0|}$; $e^{\pi/|s_0|} \approx 22.7$
- Borromean binding

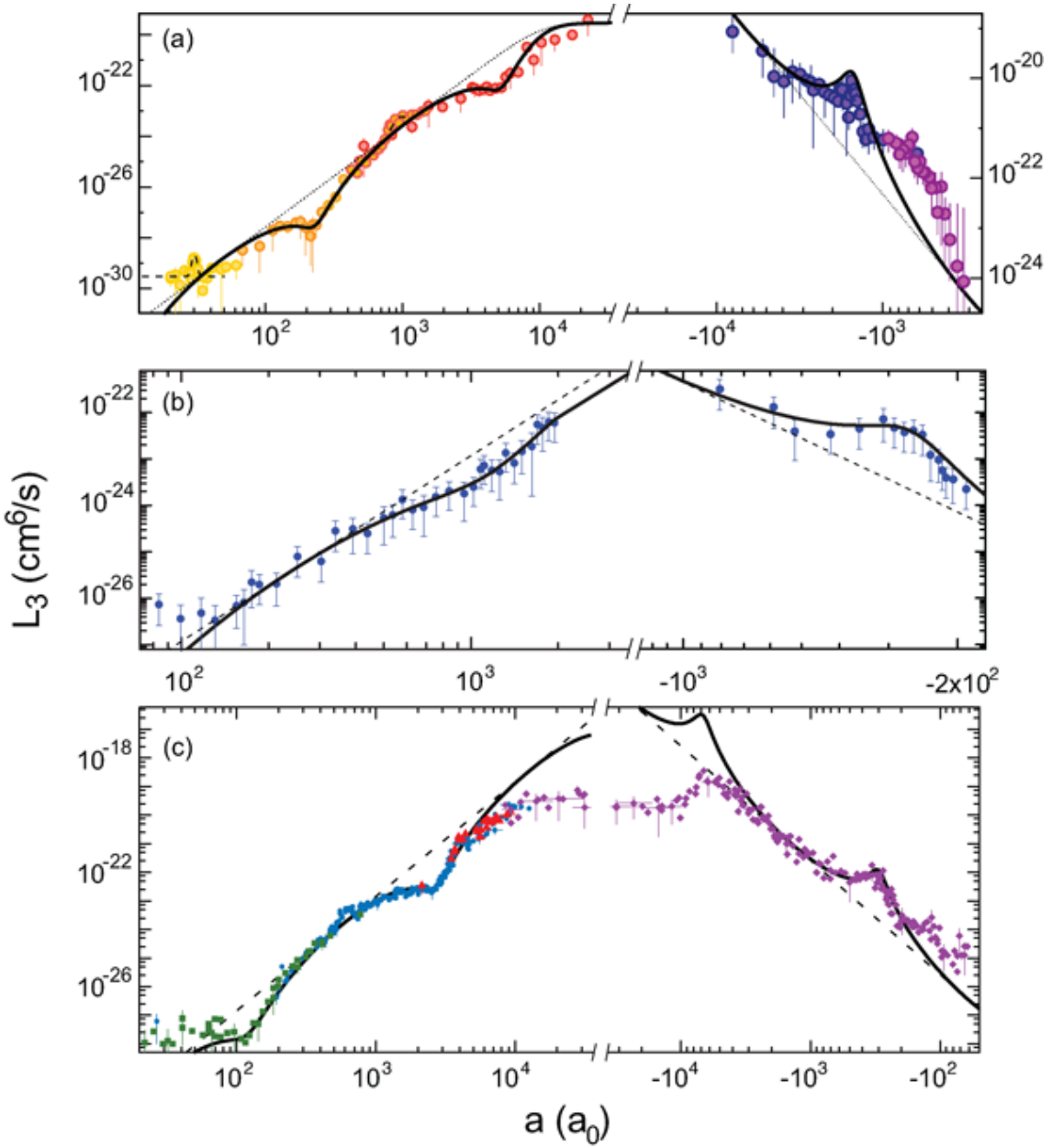


Ferlaino and Grimm, Physics **3**, 9 (2010)

Efimov, Phys. Lett. B **33**, 563 (1970)

Review: Naidon and Endo (2017)

Efimov Physics in Ultracold Atoms



Ferlaino and Grimm, Physics **3**, 9 (2010)

- ^{39}K
  Zaccanti *et al.*,
  Nature Phys. **5**, 586 (2009).
- ^{7}Li
  Gross *et al.*,
  Phys. Rev. Lett. **103**, 163202 (2009).
- ^{7}Li
  Pollack *et al.*,
  Science **326**, 1683 (2009)

cf. Bing Zhu talk

Efimov Physics Beyond 3 Particles?

- There is no Efimov effect for four or more particles.

  Amado & Greenwood, Phys. Rev. D **7**, 2517 (1973)

- Two tetramers are tied to each Efimov trimer

  Hammer & Platter, Eur. Phys. J. A **32**, 113 (2007)

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Physics **2**, 26 (2009)

Ferlaino et al., PRL **102**, 140401 (2009)

ultracold Cs gas

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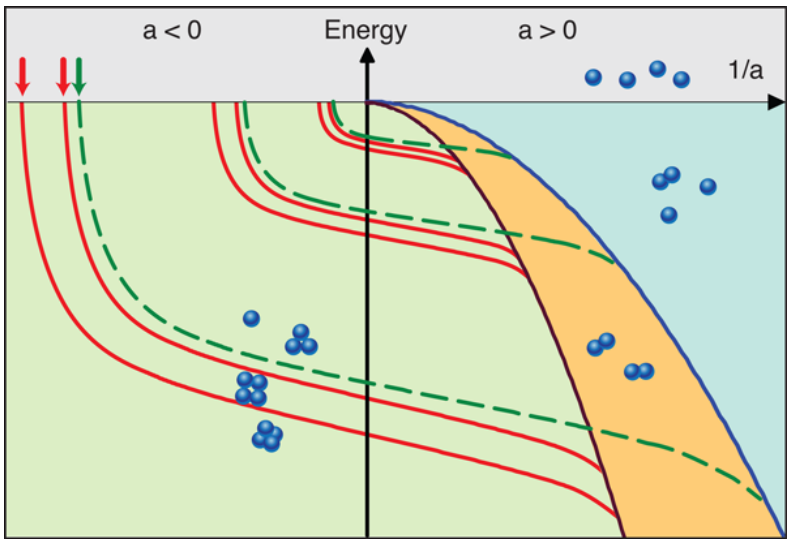

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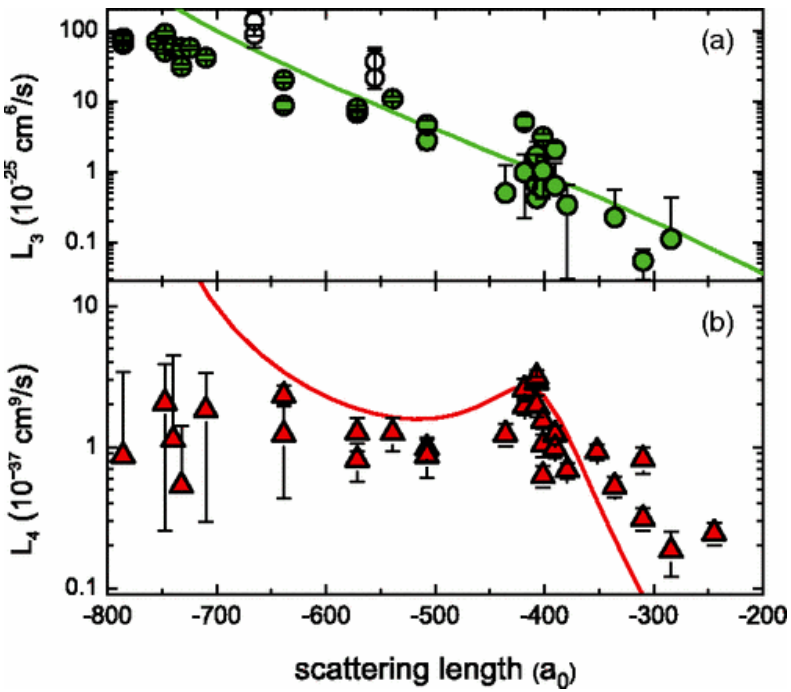

Ferlaino et al., PRL **102**, 140401 (2009)

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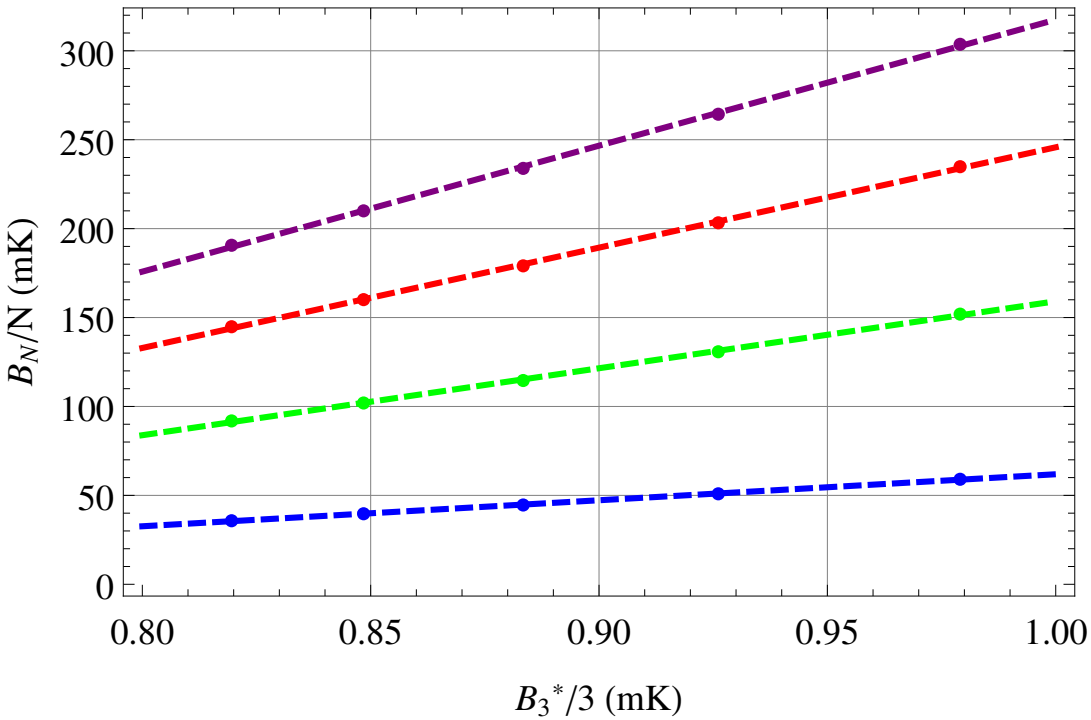
Clusters of He atoms in short-range EFT

- Tjon line: correlation between triton and alpha binding energies.
  Tjon, Phys. Lett. B **56**, 217 (1975)
- Therefore, there is no need for four-body parameter at leading order.
  Platter, Hammer, Meissner, Phys. Lett. B **607**, 254 (2005)
- Same is true for 5- and 6-body clusters, also attached to an Efinov trimer.

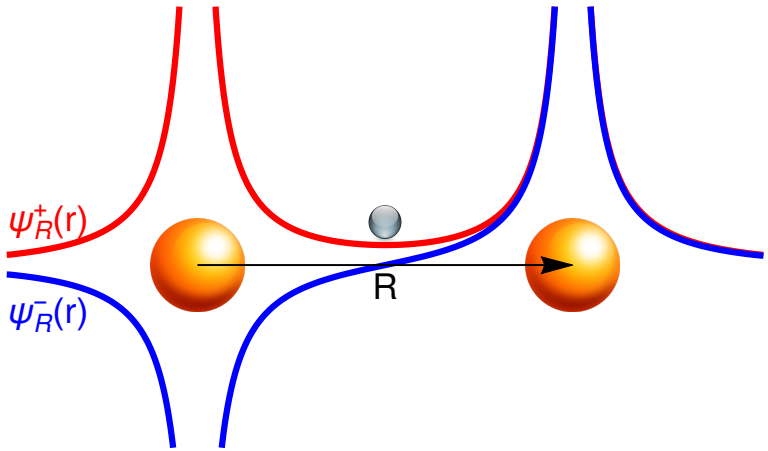


BB, Eliyahu and van Kolck, PRA **94**, 052502 (2016)

cf. Alejandro Kievsky talk

2+1 Case: Born-Oppenheimer Approximation

Consider two heavy atoms interacting with a light atom.



$$\psi_{\boldsymbol{R}}^{\pm}(\boldsymbol{r}) \propto \frac{e^{-\kappa(R)|\boldsymbol{r}-\boldsymbol{R}/2|}}{|\boldsymbol{r}-\boldsymbol{R}/2|} \pm \frac{e^{-\kappa(R)|\boldsymbol{r}+\boldsymbol{R}/2|}}{|\boldsymbol{r}+\boldsymbol{R}/2|}$$

$$\kappa_{\pm}(R) \mp \frac{e^{-\kappa_{\pm}(R)R}}{R} = 1/a$$

The light atom induces effective potential between the heavy atoms:

$$\epsilon_+(R) = -\frac{\kappa_+^2(R)}{2m} \approx \begin{cases} -\frac{0.16}{mR^2} & R/a \ll 1 \\ -\frac{1}{2m}\left(\frac{1}{a^2} + \frac{\exp(-R/a)}{aR}\right) & R/a \gg 1 \end{cases}$$

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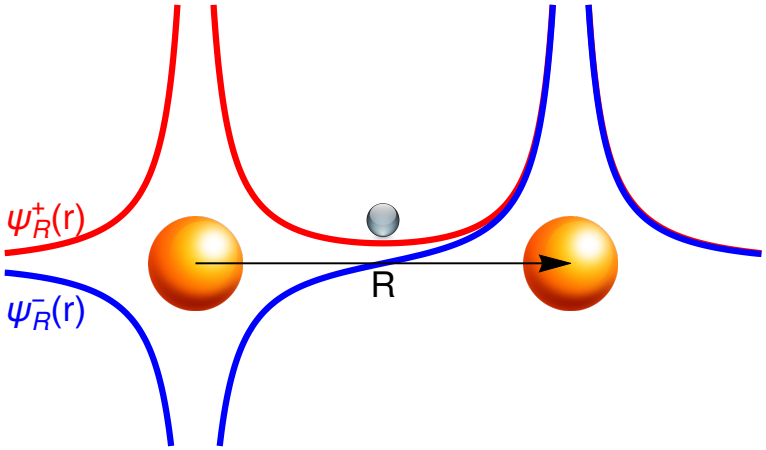

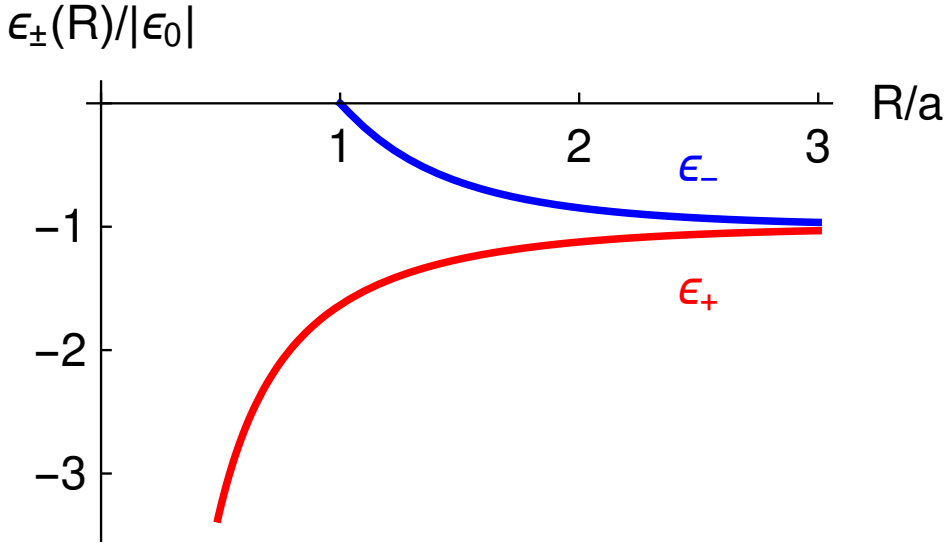

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$$s^2 = l(l+1) - 0.16M/m + 1/4 \qquad u(r) \propto R^{1/2 \pm s}$$

- For $s^2 < 0$, $u(R) \propto \sqrt{R}\sin[|s|\log(R/R_0)]$, where R_0 is the three-body parameter.
- $l = 0$ for bosons giving purely attractive $-1/mR^2$ effective potential.
  cf. Marcelo Yamashita, Lucas Platter talks
- $l = 1$ for fermions adding centrifugal barrier, $l(l+1)/MR^2$.

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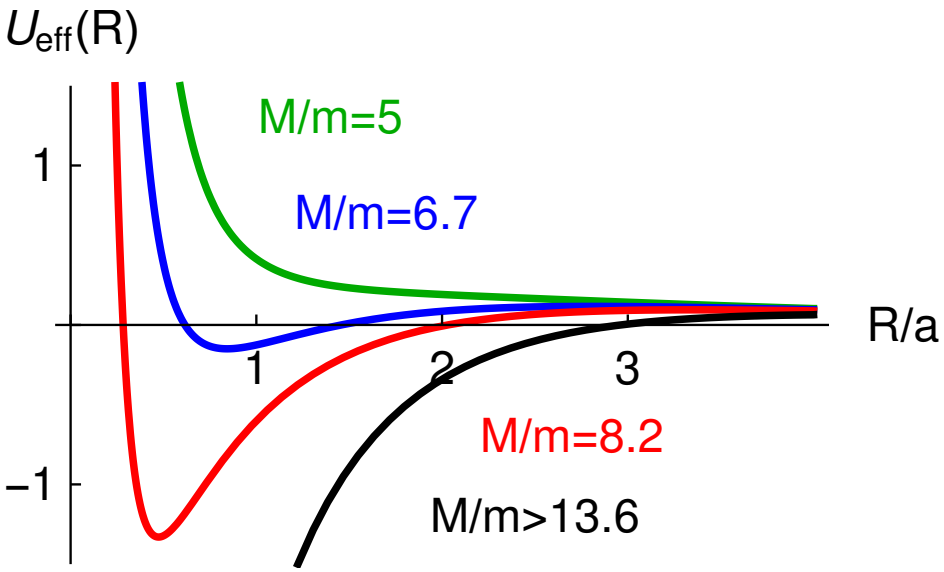


^{40}K-^{6}Li p-wave resonance: Jag *et al.*'14

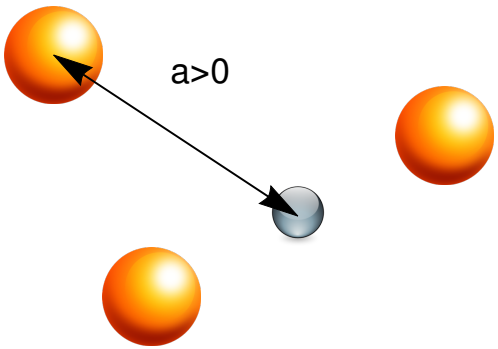
Universal trimer:
Kartavtsev & Malykh '07
cf. Anastasiya Malykh talk

Efimovian trimers: Efimov '73

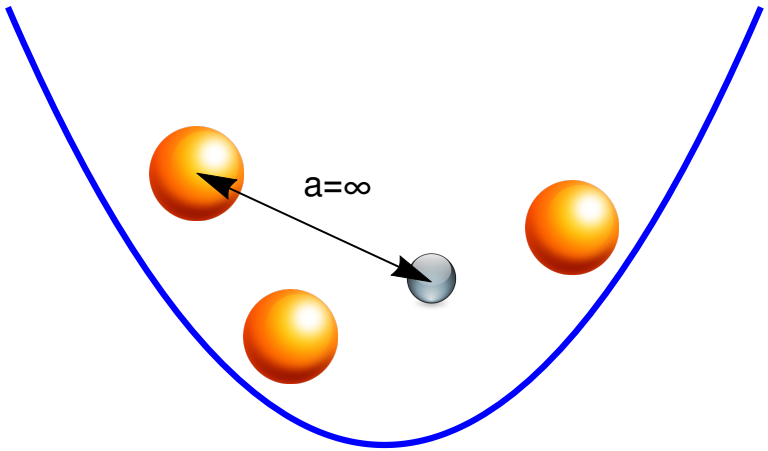
$$E_n = E_0 e^{-2\pi n/|s_0|}$$

Free-Space and Trapped (N+1) System

Free space



Harmonic trap



$$\psi(R) \propto R^{-3N/2+1+s}$$

$$E = \hbar\omega(s + 2n + 1)$$

$s^2 > 0$ universal non-Efimovian state

$s^2 < 0$ tower of Efimov states

Both can be used to calculate the scale factor s!

Tan, arXiv:cond-mat/0412764 (2004); Werner and Castin, Phys. Rev. A **74**, 053604 (2006).

The Fermionic (N+1) Problem

How many heavy fermions can be bound by a single light atom?

system	L^π	free space binding	trap crossing	Efimov
2+1	1^-	8.173	8.619	13.607
		KM '07	Petrov '03	Efimov '73
3+1	1^+	≈ 9.5	?	13.384
		Blume '12		CMP '10
4+1	?	?	?	?
5+1	?	?	?	?

Efimov, Nucl. Phys. A **210**, 157 (1973).
Petrov, Phys. Rev. A **67**, 010703(R) (2003).
Kartavtsev and Malykh, J. Phys. B: At. Mol. Opt. Phys. **40**, 1429 (2007).
Castin, Mora, and Pricoupenko, Phys. Rev. Lett. **105**, 223201 (2010).
Blume, Phys. Rev. Lett. **109**, 230404 (2012).

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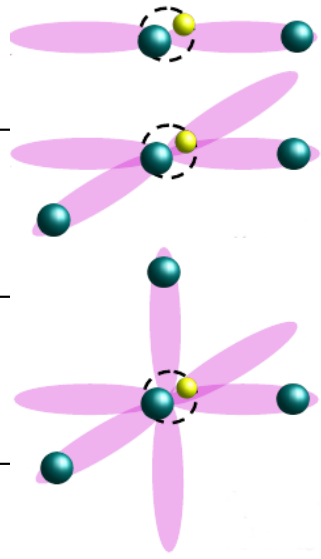
				CMP '10
4+1	0^-	9.672	9.41	13.279
5+1	0^-	?	?	?



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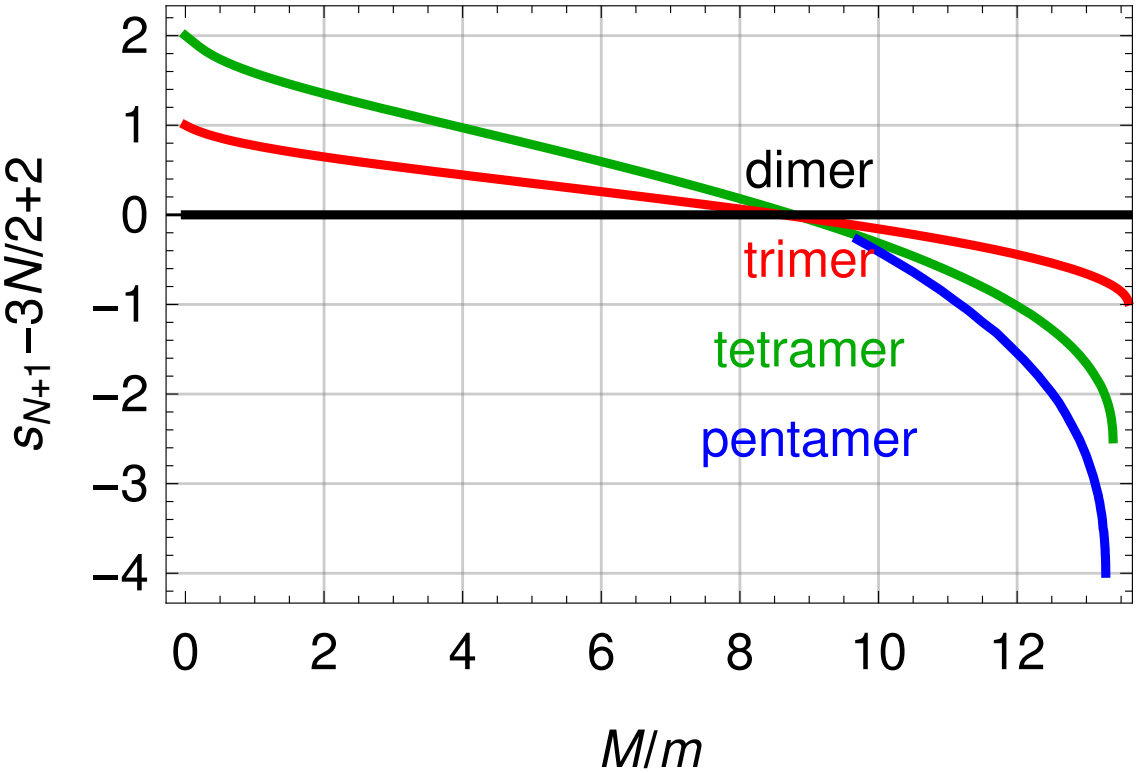
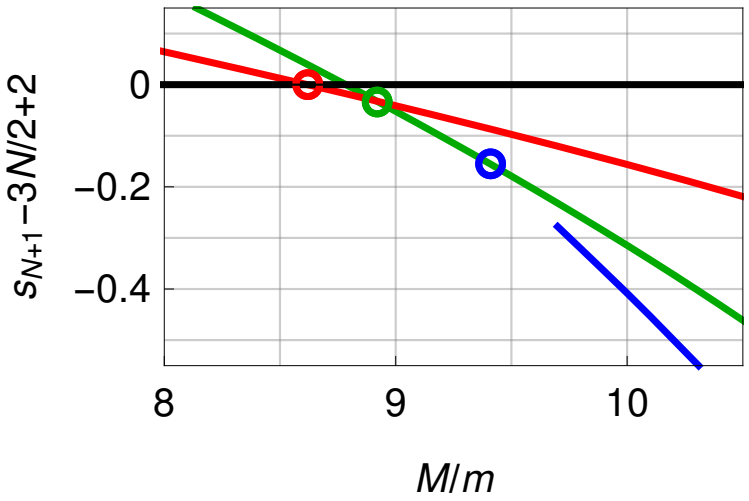
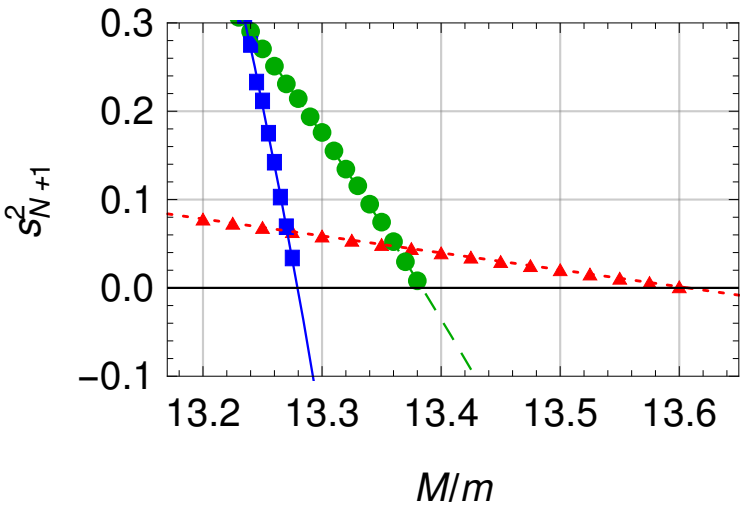
BB and D. S. Petrov, Phys. Rev. Lett. **118**, 083002 (2017).

Results: the Scale Factor

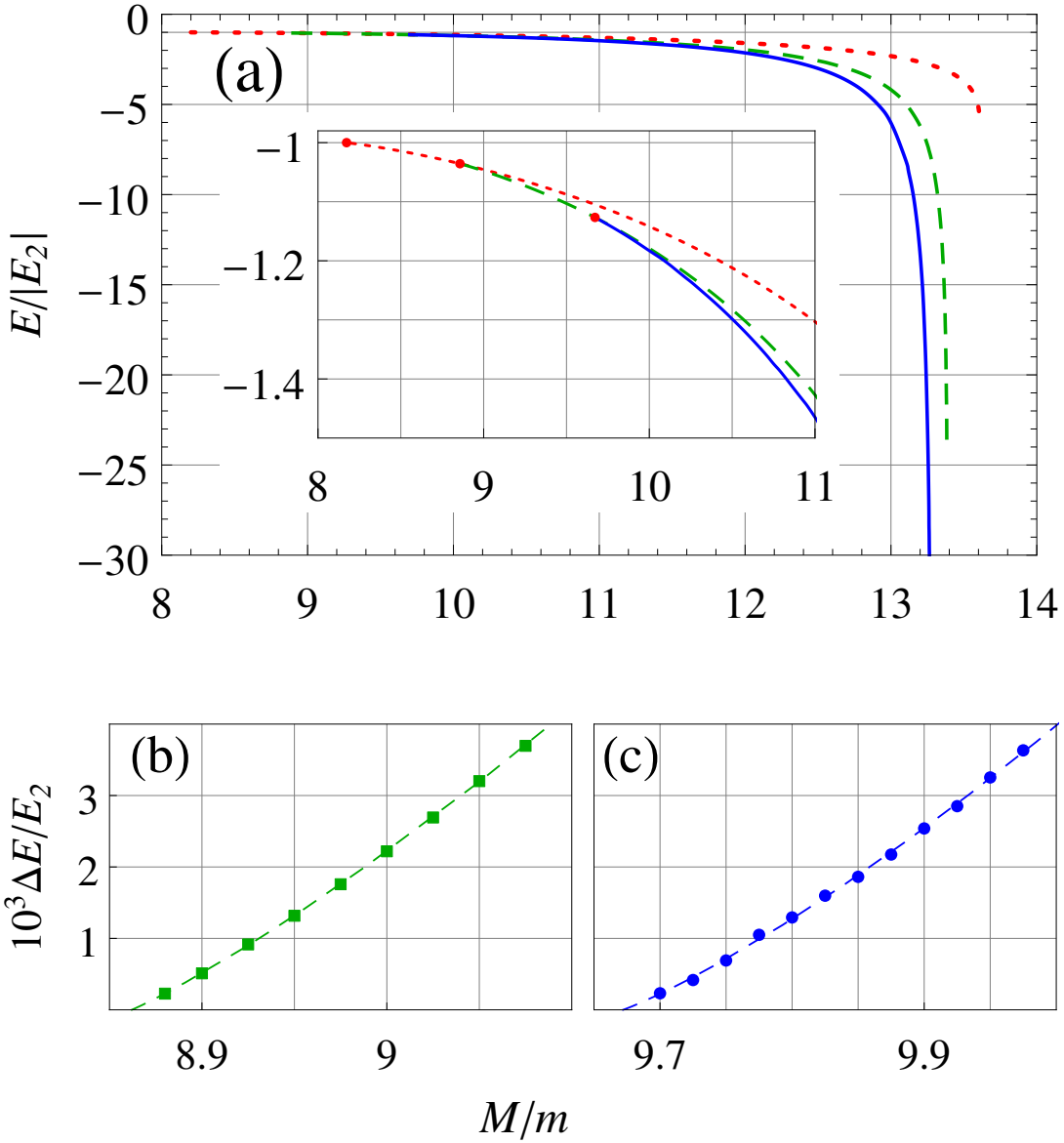






A pure (4+1) Efimov state exists for M/m>13.279!

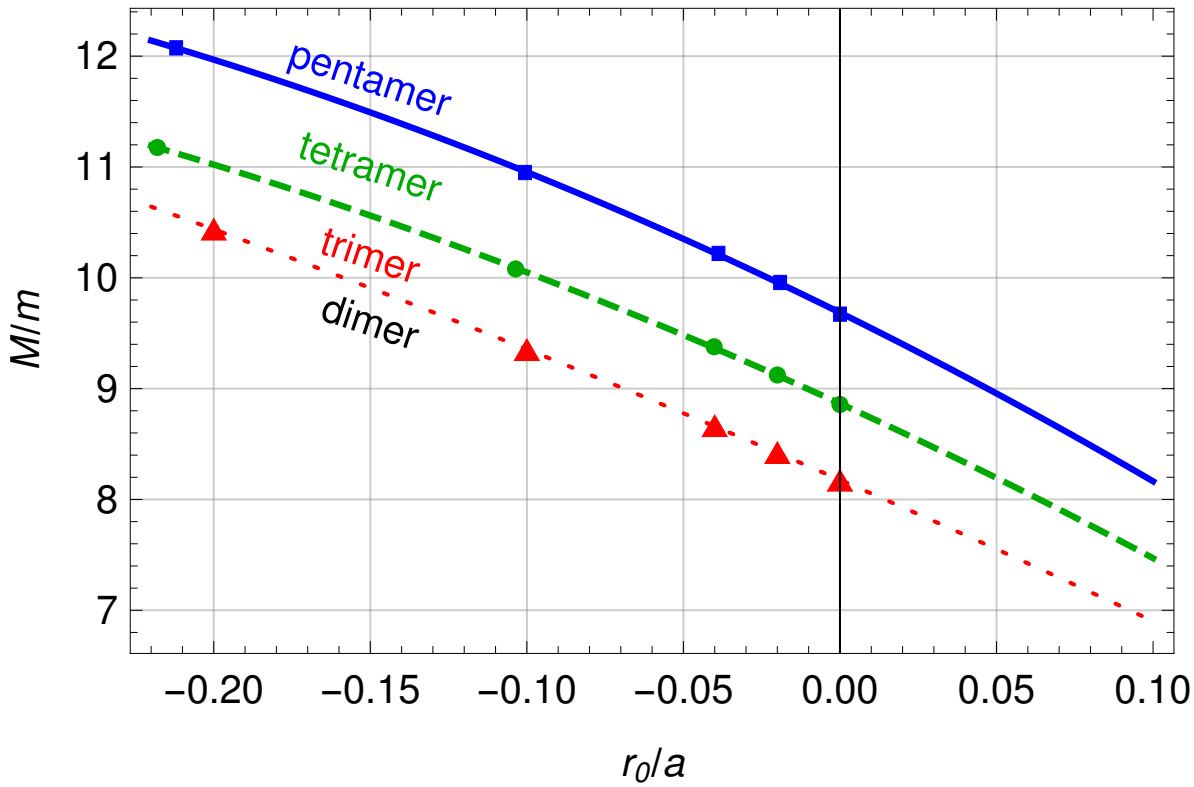
Results: the Energies



A universal (4+1) non-Efimovian state exists for M/m>9.672!

Effective-Range Effects

$$-\frac{1}{a} \longrightarrow -\frac{1}{a} + \frac{r_0}{2}k^2$$



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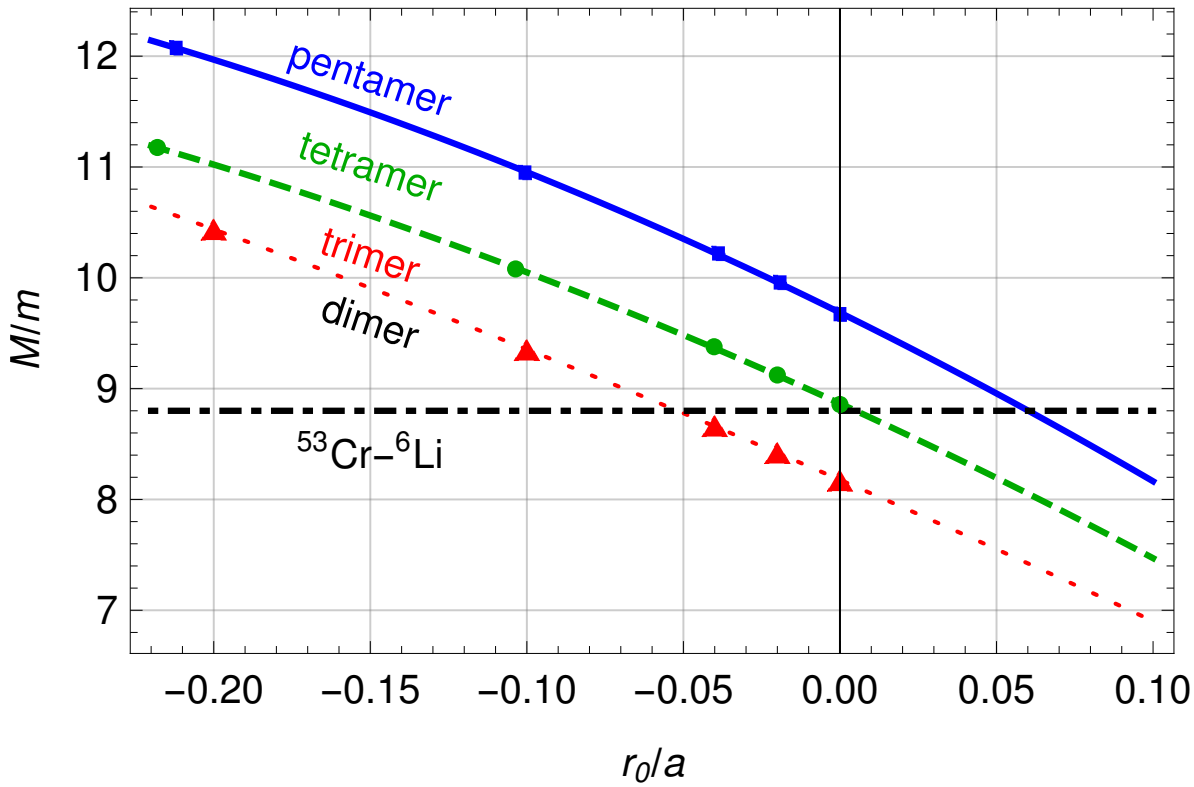


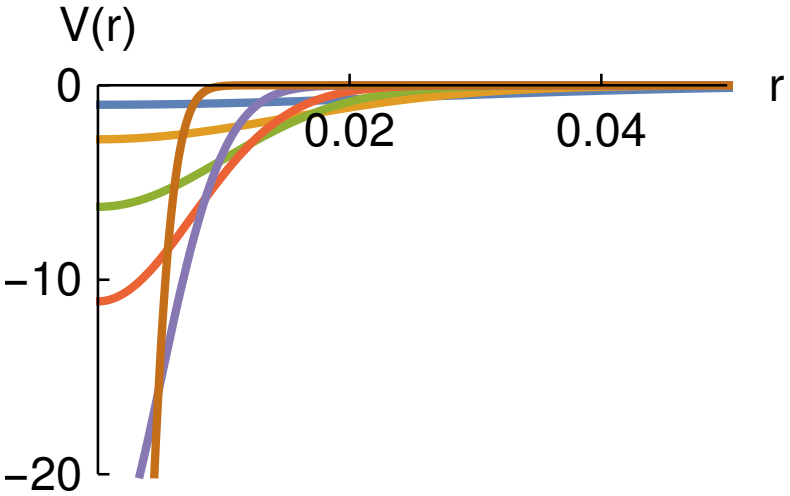
Does the 5+1 Efimov state exist?

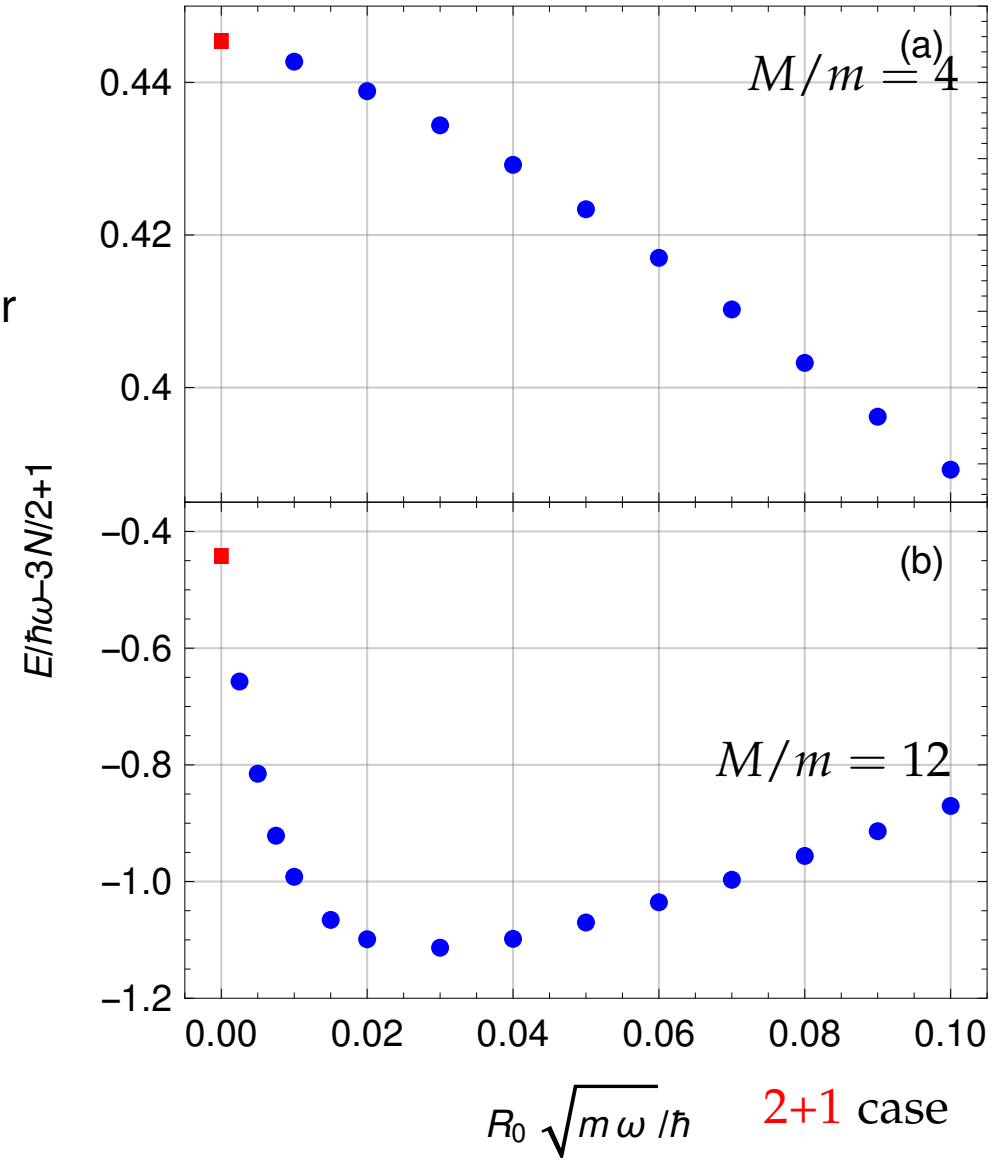
- The STM equation, works in the zero-range limit, is too complicated to be solved in deterministic method.
- Stochastic method suffers from sign problem due to radial node.
- Let's try to use finite-range potential and extrapolate to the zero-range limit...

Finite-Range Potentials

Extrapolating the zero-range limit with finite-range potentials

$$V(r) = -V_0\, e^{-\frac{r^2}{2R_0^2}}$$





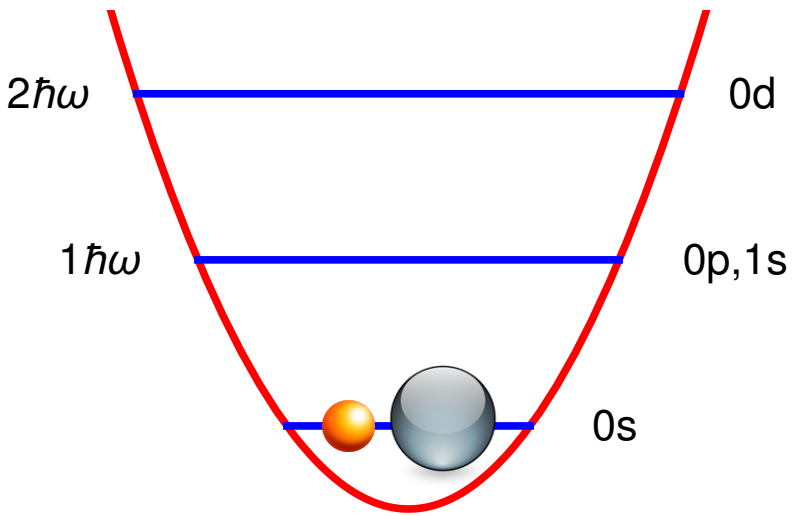
2+1 case

BB, Phys. Rev. A **96**, 022708 (2017)

The (N+1) System in a Trap: M/m=0 Limit

- The infinite-heavy impurity problem reduces to N trapped fermions scattering on a zero-range potential at the trap center.
- At unitarity, $E_{nl} = (2n + l - \delta_{l,0} + 3/2)\hbar\omega$

  Busch *et al.*, Found. Phys. **28**, 549 (1998)
- The zero-point energy $\frac{3}{2}N\hbar\omega$ is subtracted and energy is measured with respect to the dimer energy, $\epsilon = E/\hbar\omega - 3N/2 + 1$



system	ϵ	π	L	configuration
1+1	0	+	0	0s
2+1	1	+	0	0s 1s
		−	1	0s 0p
3+1	2	+	1	$0s\,0p^2$
		−	1	0s 1s 0p
4+1	3	+	1	$0s\,1s\,0p^2$
		−	0	$0s\,0p^3$
5+1	4	−	0	$0s\,1s\,0p^3$

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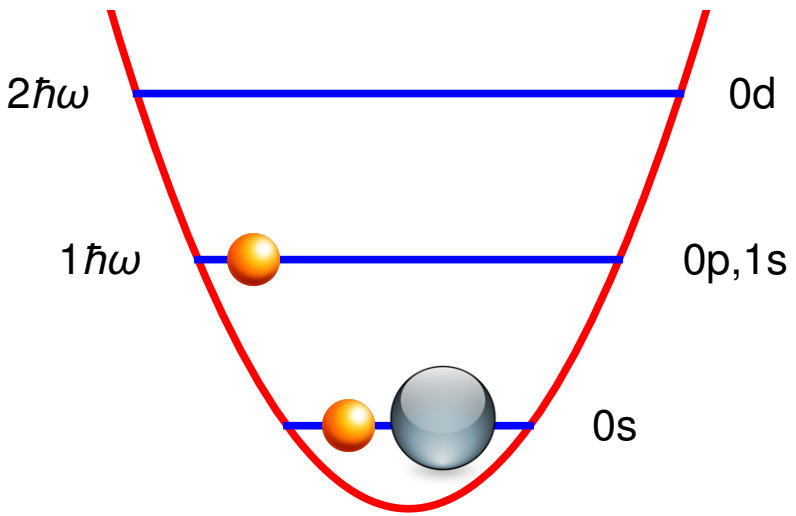

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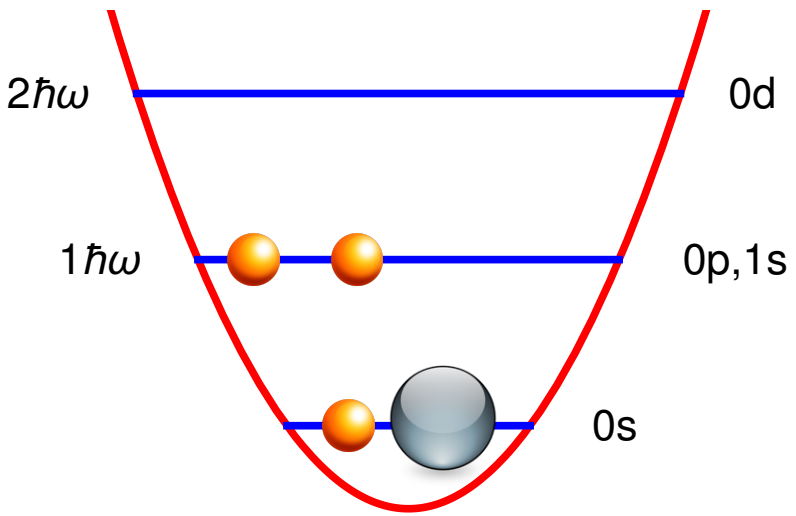

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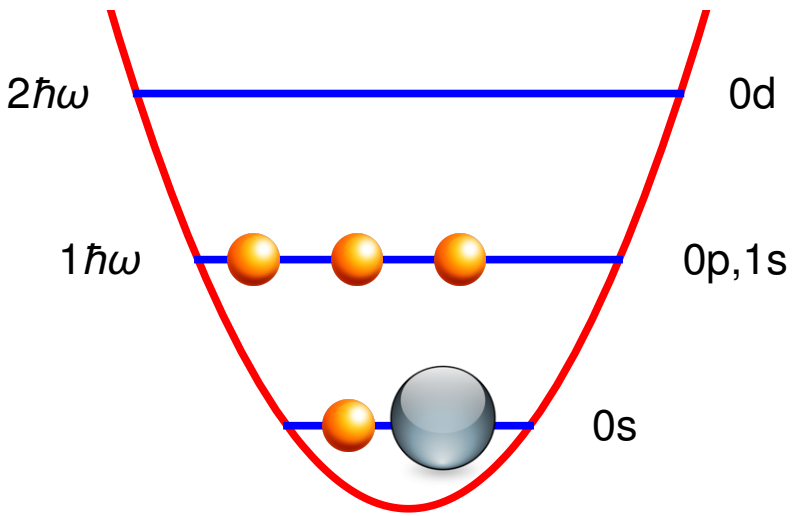

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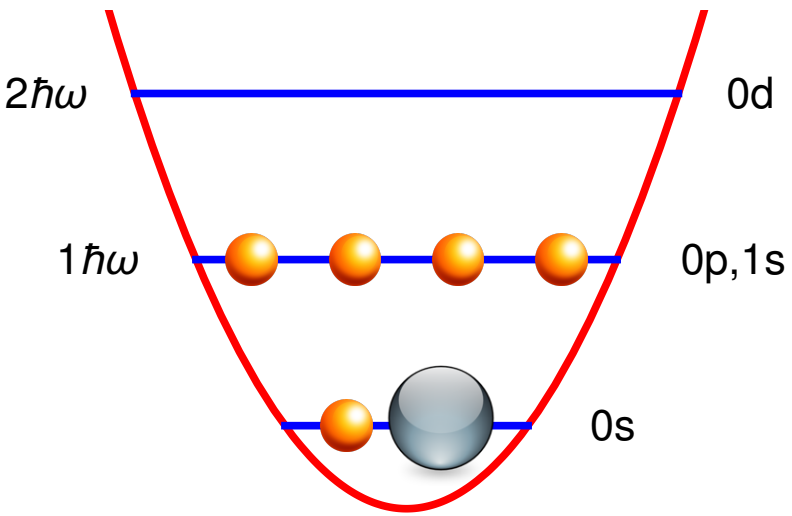

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The (2+1) System in a Trap

- Natural parity states:

$$\frac{2}{\Gamma(a-1/2)\Gamma(b-1/2)} + \frac{(-\gamma)^L}{\sqrt{\pi}\Gamma(c)}\,{}_2F_1\left(a,b;c;\gamma^2\right) = 0$$

$$a = 1+(L-s)/2 \quad b = 1+(L+s)/2 \quad c = L+3/2 \quad \gamma = M/(M+m)$$

Rittenhouse, Mehta, and Greene, Phys. Rev. A **82**, 022706 (2010).

$0s\,2s \quad 0^+$

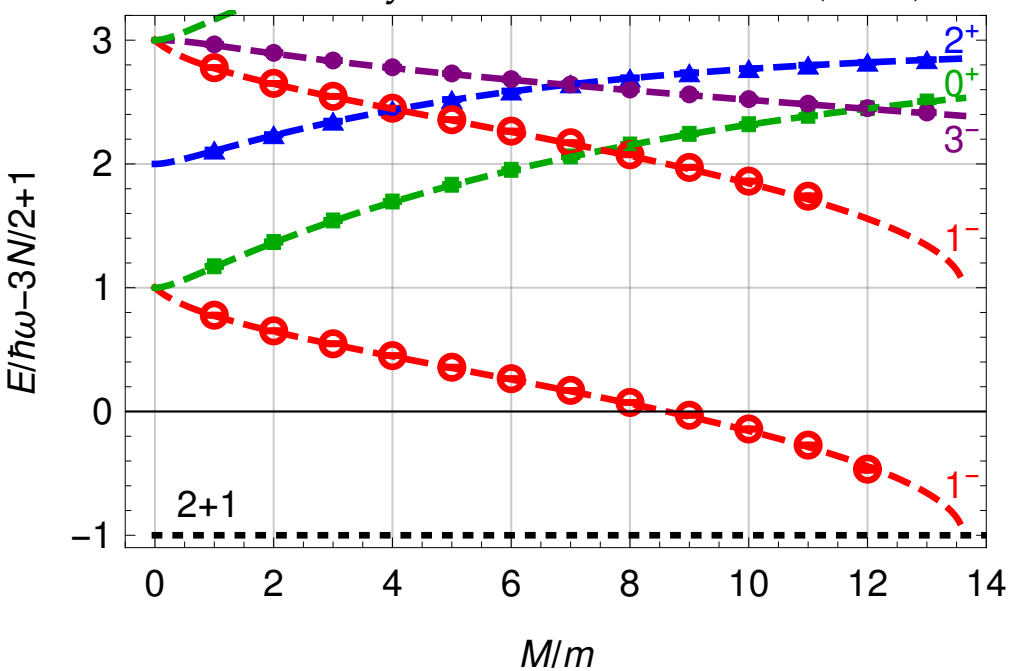
$0s\,0f \quad 3^-$

$0s\,1p \quad 1^-$

$0s\,0d \quad 2^+$

$0s\,1s \quad 0^+$

$0s\,0p \quad 1^-$



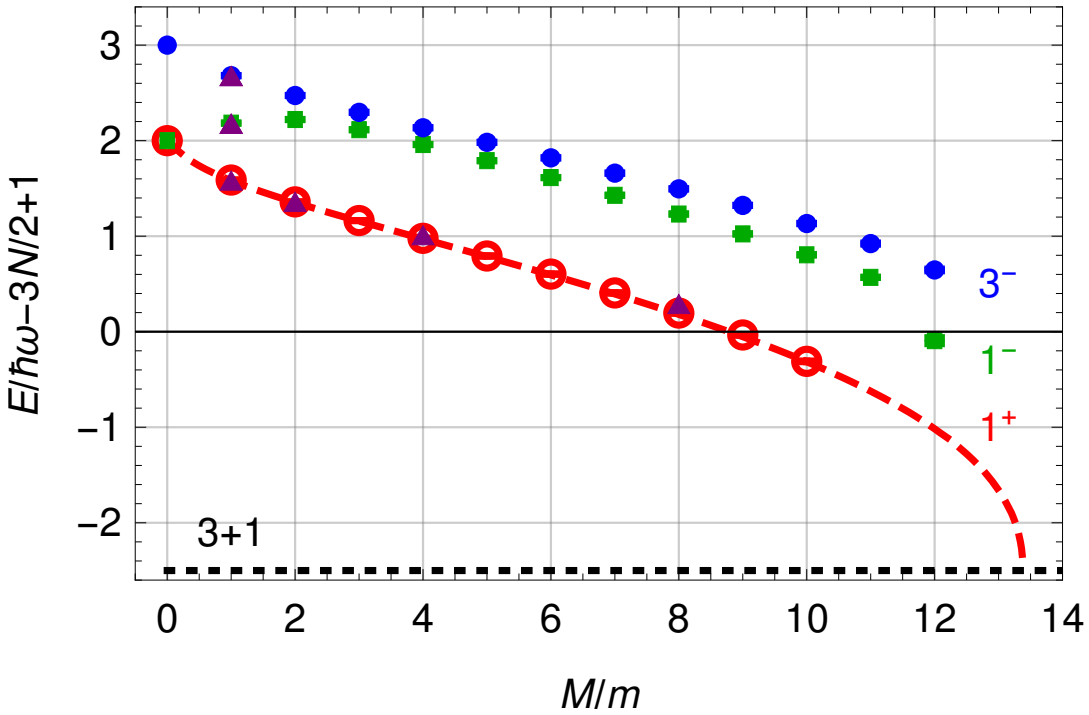
- BO lesson: light particle induces attraction for odd L and repulsion for even L.
- Unnatural parity states are non-interacting!

The (3+1) System in a Trap

$0s\,0p\,0d \quad 1^-, 2^-, 3^-$

$0s\,0p\,1s \quad 1^-$

$0s\,0p^2 \quad 1^+$



Triangles: Blume and Daily, Phys. Rev. A **82**, 063612 (2010); Rakshit, Daily, and Blume, Phys. Rev. A **85**, 033634 (2012).

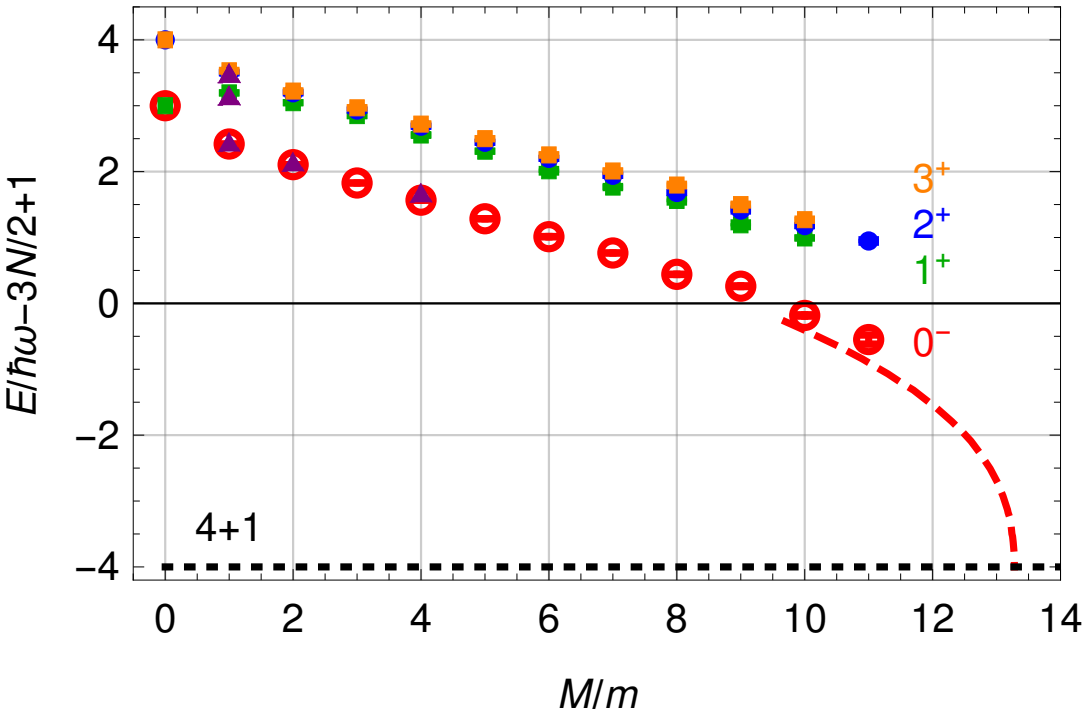
Dashed line: **BB** and Petrov, Phys. Rev. Lett. **118**, 083002 (2017).

The (4+1) System in a Trap

$0s\,0p^2\,0d \quad 1^+, 2^+, 3^+$

$0s\,0p^2\,1s \quad 1^+$

$0s\,0p^3 \quad 0^-$



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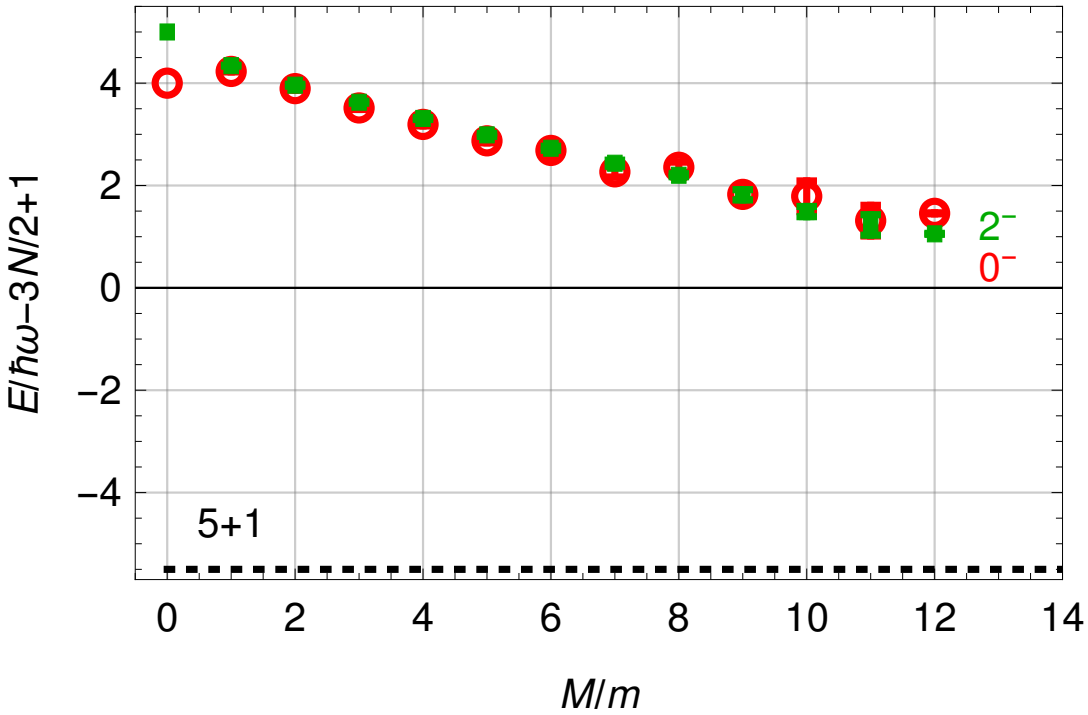
The (5+1) System in a Trap

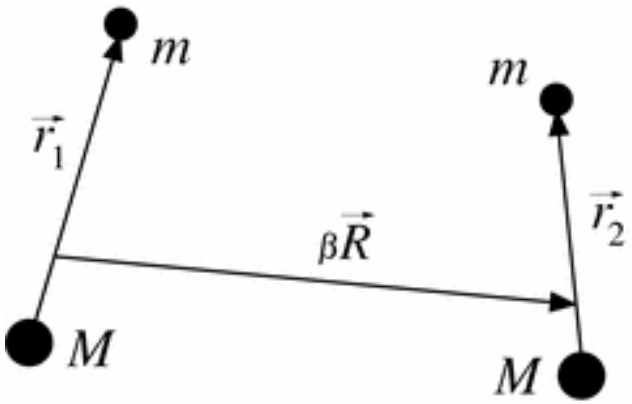
$0s\,0p^3\,0d \quad 2^-$

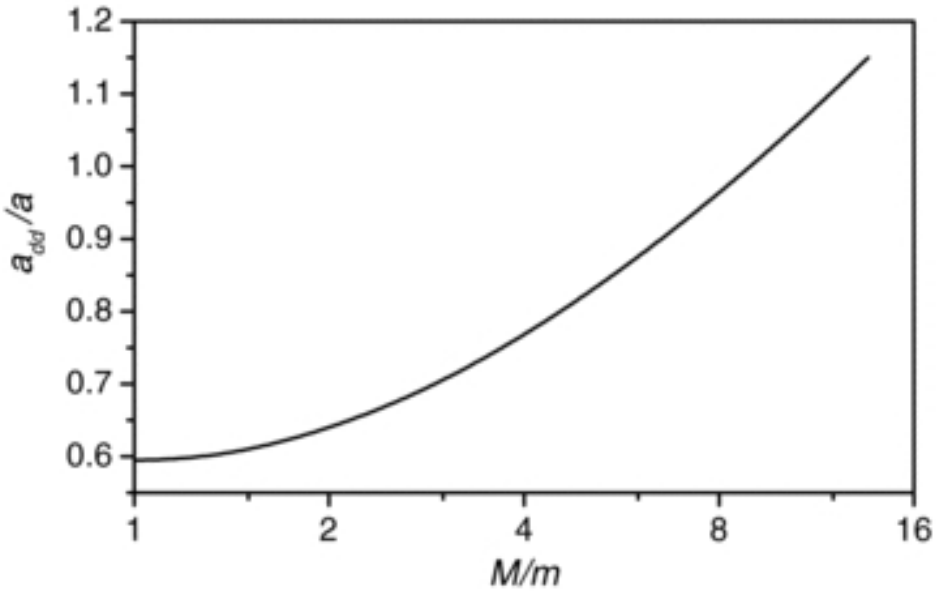
$0s\,0p^3\,1s \quad 0^-$



- The physics of a dynamic light-impurity is very different from that of the static heavy-impurity case; transition becomes sharper for larger N.
- The existence of $(5+1)$ Efimov state is unlikely.

The (2+2) case





- For the fermionic case,
  - No 4-body weakly bound states; $a_{dd} = 0.6a$
    Petrov, Salomon, and Shlyapnikov, PRL **93**, 090404 (2004);
    J. Phys. B: At. Mol. Opt. Phys. **38**, S645 (2005)
  - No four-body Efimov effect for the 2+2 fermions
    Endo and Castin, Phys. Rev. A **92**, 053624 (2015)
- For the bosonic case, cf. Pascal Naidon talk.

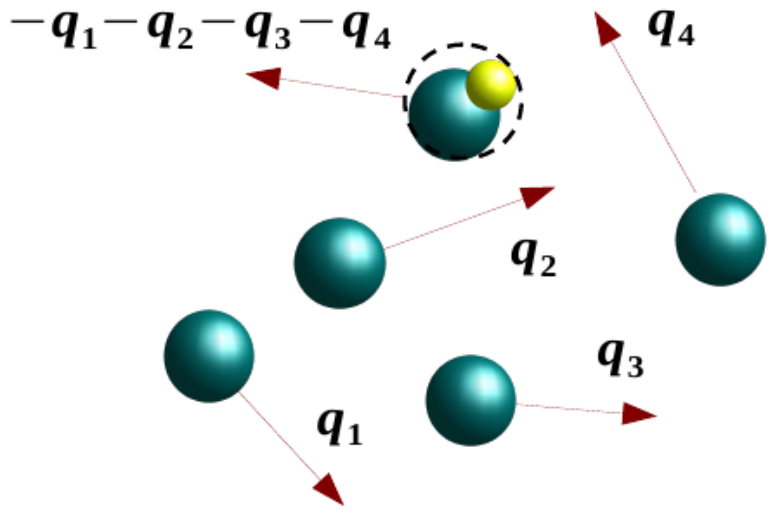
Methods

- Zero-range: The STM Equation
  - Deterministic Methods
  - Stochastic Methods: DMC-STM
- Finite-range: The Stochastic Variational Method

The STM Equation

Skorniakov and Ter-Martirosian, '56

$$\frac{1}{4\pi}\left(\frac{1}{a}-\kappa\right)F(\mathbf{q}_1,\ldots,\mathbf{q}_{N-1})=\int\frac{d^3q_N}{(2\pi)^3}\frac{\sum_{i=1}^{N-1}F(\mathbf{q}_1,\ldots,\mathbf{q}_{i-1},\mathbf{q}_N,\mathbf{q}_{i+1},\ldots,\mathbf{q}_{N-1})}{-\frac{2\mu E}{\hbar^2}+\frac{\mu}{M}\sum_{i=1}^{N}q_i^2+\frac{\mu}{m}\left(\sum_{i=1}^{N}\mathbf{q}_i\right)^2}$$



$$\kappa=\sqrt{-\frac{2\mu E}{\hbar^2}+\frac{\mu}{M}\sum_{i=1}^{N-1}q_i^2+\frac{\mu}{M+m}\left(\sum_{i=1}^{N-1}\mathbf{q}_i\right)^2}$$

- (2+1) case, 1^+:
  $F(\mathbf{q}_1)=f(q_1)\mathbf{q}_1\cdot\hat{z}$
- (3+1) case, 1^+:
  $F(\mathbf{q}_1,\mathbf{q}_2)=f(q_1,q_2,u_{12})\mathbf{q}_1\times\mathbf{q}_2\cdot\hat{z}$

Skorniakov and Ter-Martirosian, Zh. Eksp. Teor. Fiz. **31**, 775 (1956);
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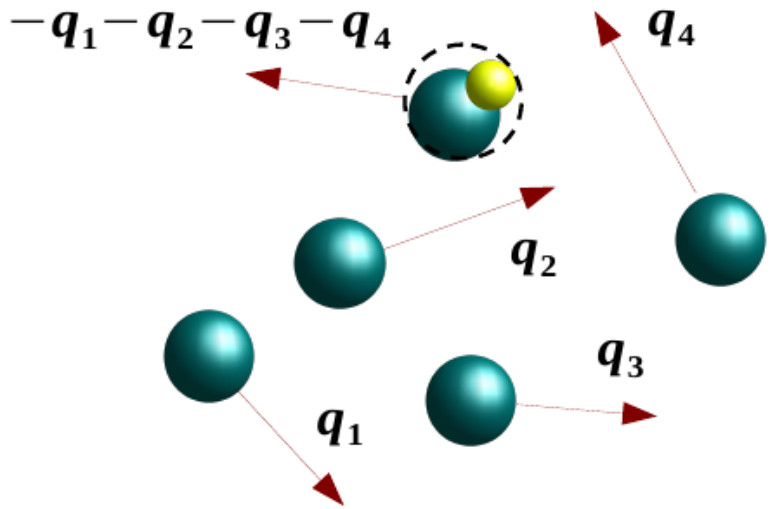

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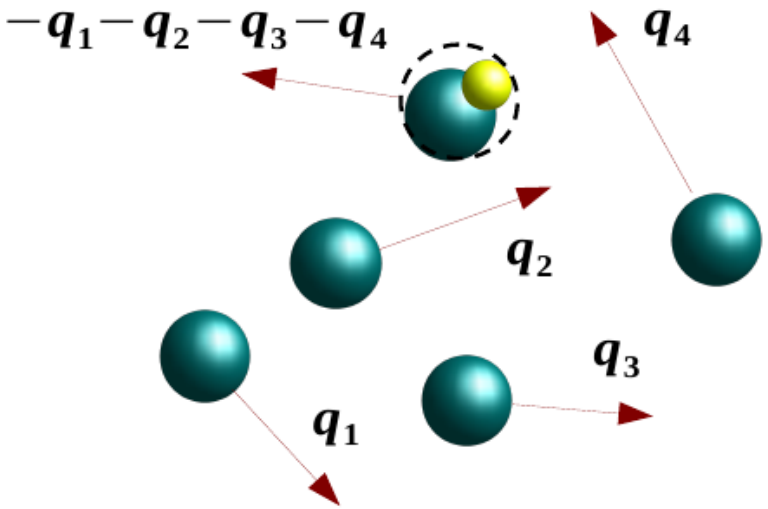

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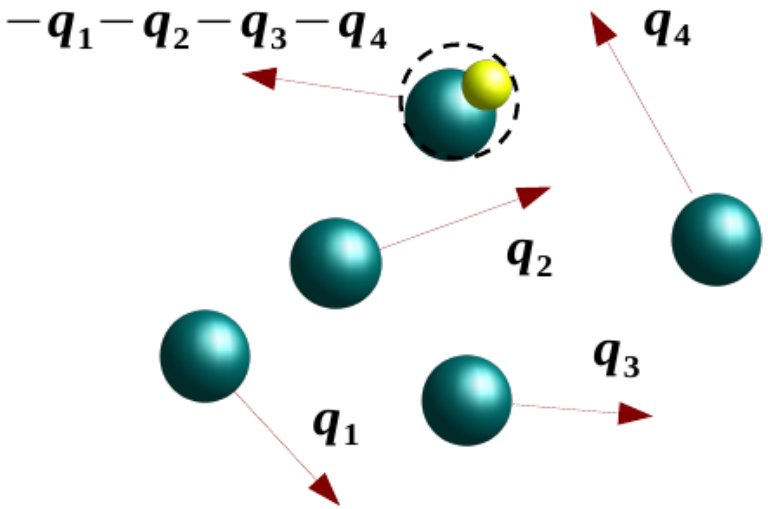

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Deterministic Methods

- To calculate the scale factor s, put $a^{-1} = 0 = E$ and use $F(\lambda q_i) = \lambda^{1-3N/2-s} F(q_i)$.
- For (2+1) case, one gets

$$\frac{2}{\Gamma(a-1/2)\Gamma(b-1/2)} + \frac{(-\gamma)^L}{\sqrt{\pi}\Gamma(c)} {}_2F_1\left(a, b; c; \gamma^2\right) = 0$$

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- For (3+1) case, one gets 2-dimensional equation, which can be solved on a grid.

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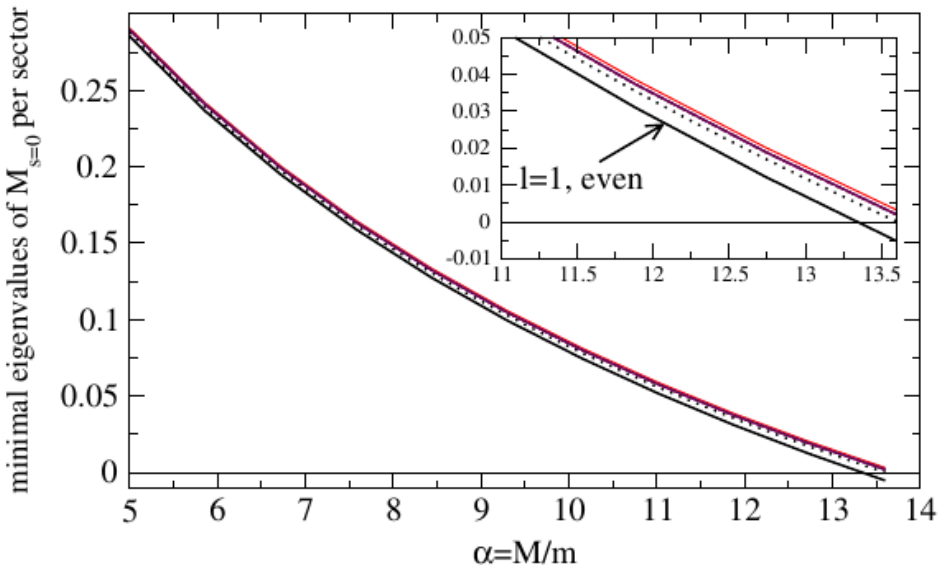

Castin, Mora, and Pricoupenko, Phys. Rev. Lett. **105**, 223201 (2010)

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- For (4+1) case, dimension is too large for grid-based method.
- Positive f can be interpreted as density distribution function.
- Organize a diffusion process for which the STM equation,

$$f(\vec{Q}) = \int K(\vec{Q}, \vec{Q}') f(\vec{Q}') d^{3(N-1)} Q'$$

  is the detailed-balance equation
- Diffusion of N_w walkers,

$$\{\vec{Q}_1^j, \vec{Q}_2^j, ... \vec{Q}_{N_w}^j\} \longrightarrow \{\vec{Q}_1^{j+1}, \vec{Q}_2^{j+1}, ...\}$$

  Each walker is branched according to $W(\vec{Q}_i) = \int K(\vec{Q}, \vec{Q}_i) f(\vec{Q}') d^{3(N-1)} Q$
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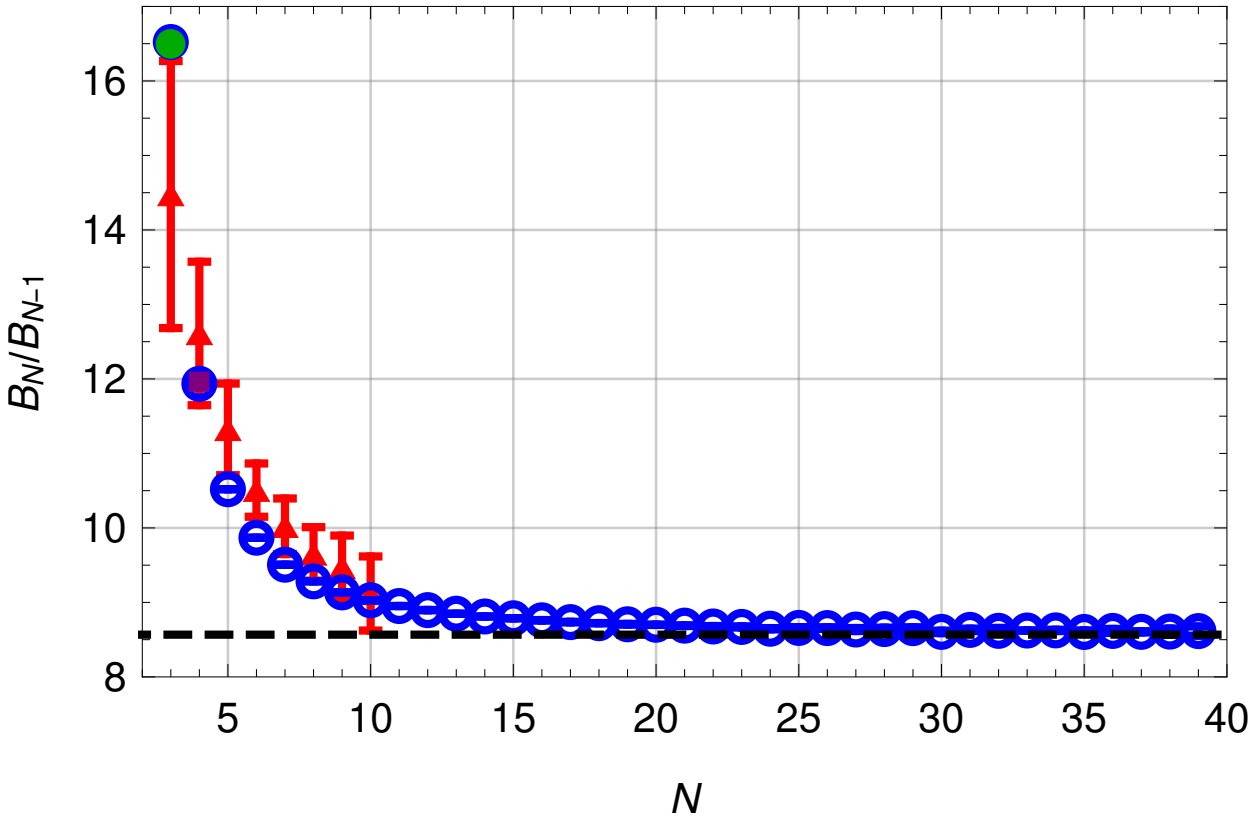
$2D$ Bosons Droplets



$B_3/B_2 = 16.522688(1)$ Hammer and Son, Phys. Rev. Lett. **93**, 250408 (2004).

$B_4/B_2 = 197.3(1)$ Platter, Hammer, and Meissner, Few-Body Syst. **35**, 169 (2004).

Lattice EFT calculations Lee, Phys. Rev. A **73**, 063204 (2006).

$B_N/B_{N-1} \xrightarrow{N\to\infty} 8.567$ Hammer and Son, ibid.

DMC-STM **BB** and Petrov, New J. Phys. **20**, 023045 (2018)

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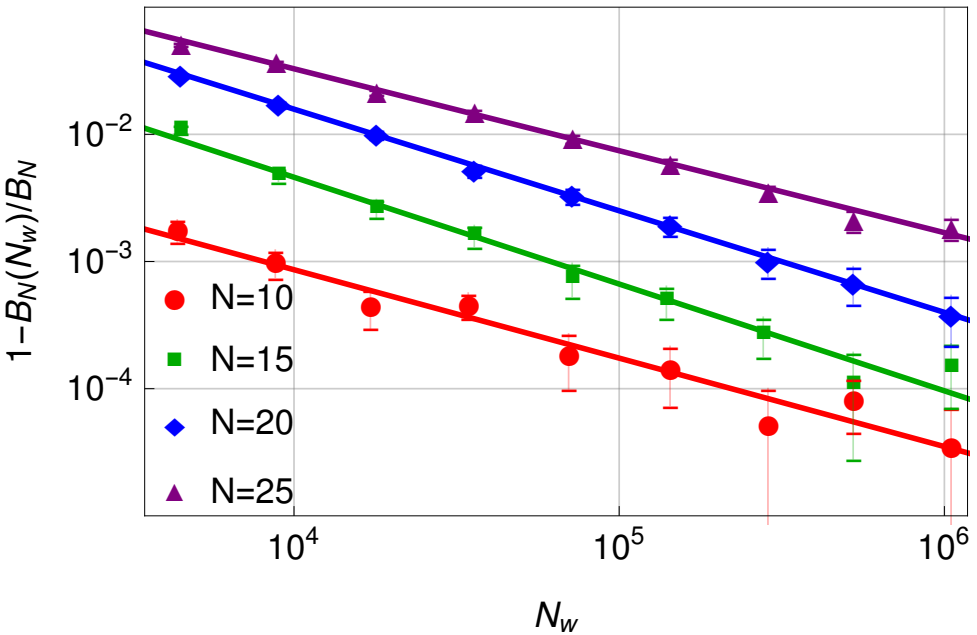

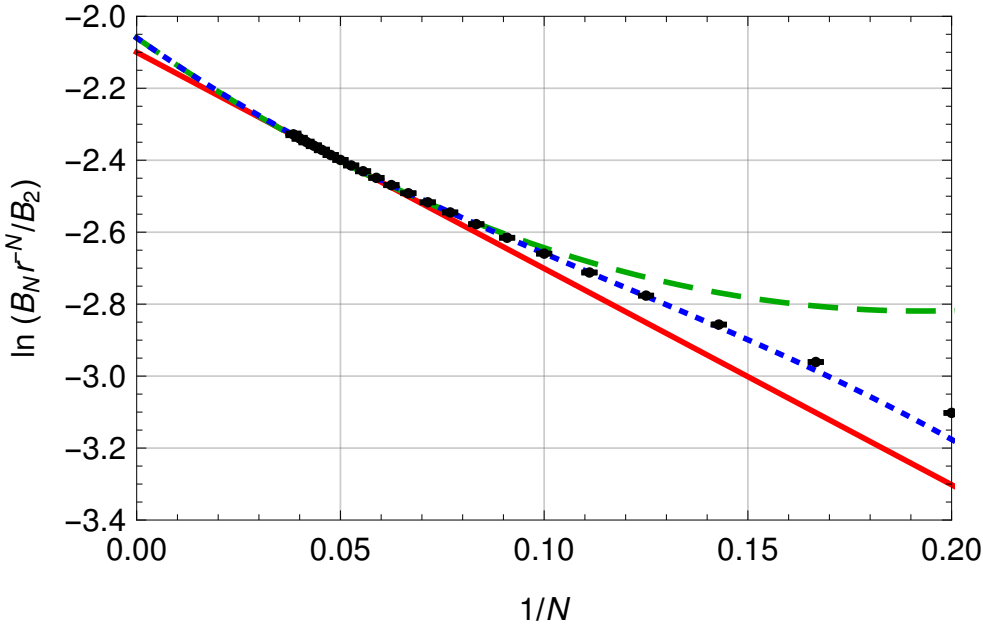

$$B_N/B_2 \approx 8.567^N \exp(c_1 + c_2/N),$$

$$c_1 = -2.06(4)\ c_2 = -8(2)$$

BB and Petrov, New J. Phys. **20**, 023045 (2018)

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- For excited states and more complex symmetries, we solve the $N+1$-body Schrödinger equation in a trap
- We use correlated Gaussian basis,

$$\psi^{\pi}_{LM}(\eta_1,\eta_2 ... \eta_{A-1}) = \sum_i c_i \hat{\mathcal{A}} \exp(-\frac{1}{2}\eta^T A_i \eta) \theta^{\pi}_{LM}(\mathbf{v}_i)$$

$\eta =$ Jacobi coordinates, A_i is $N \times N$ matrix, and $v_i = u_i^T \eta$ is a global vector,

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Y. Suzuki and K. Varga,

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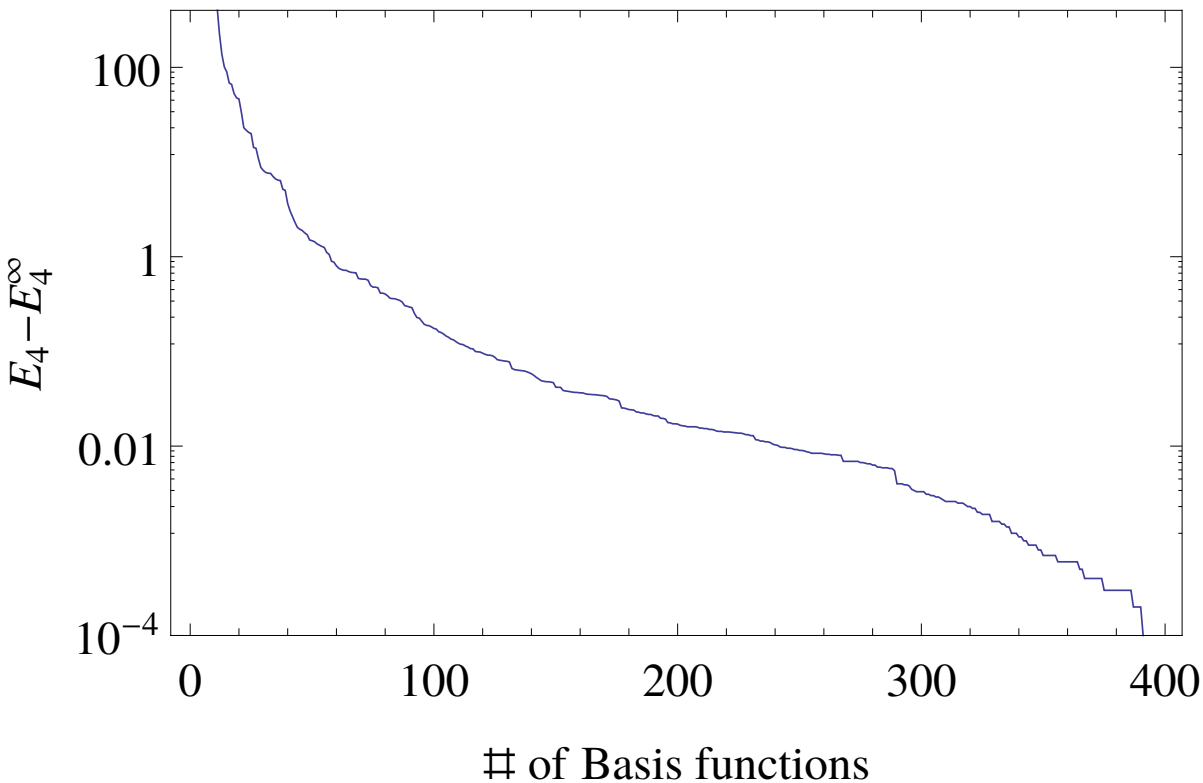

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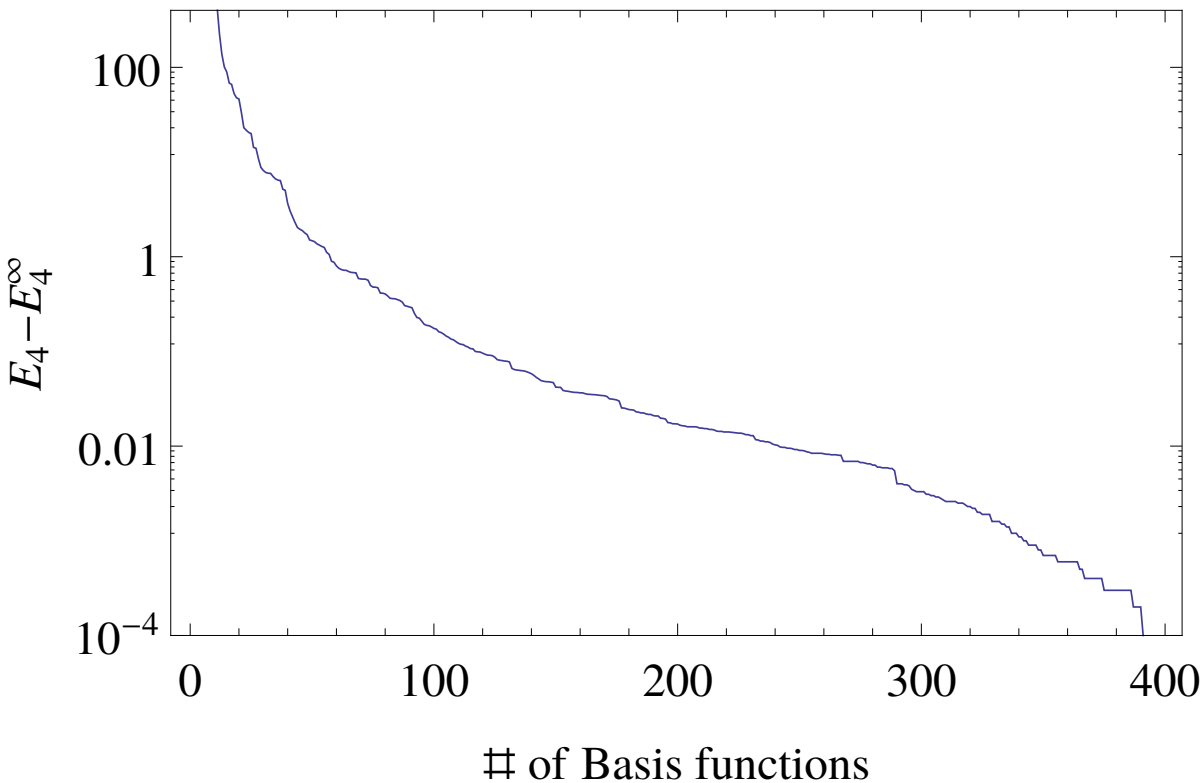

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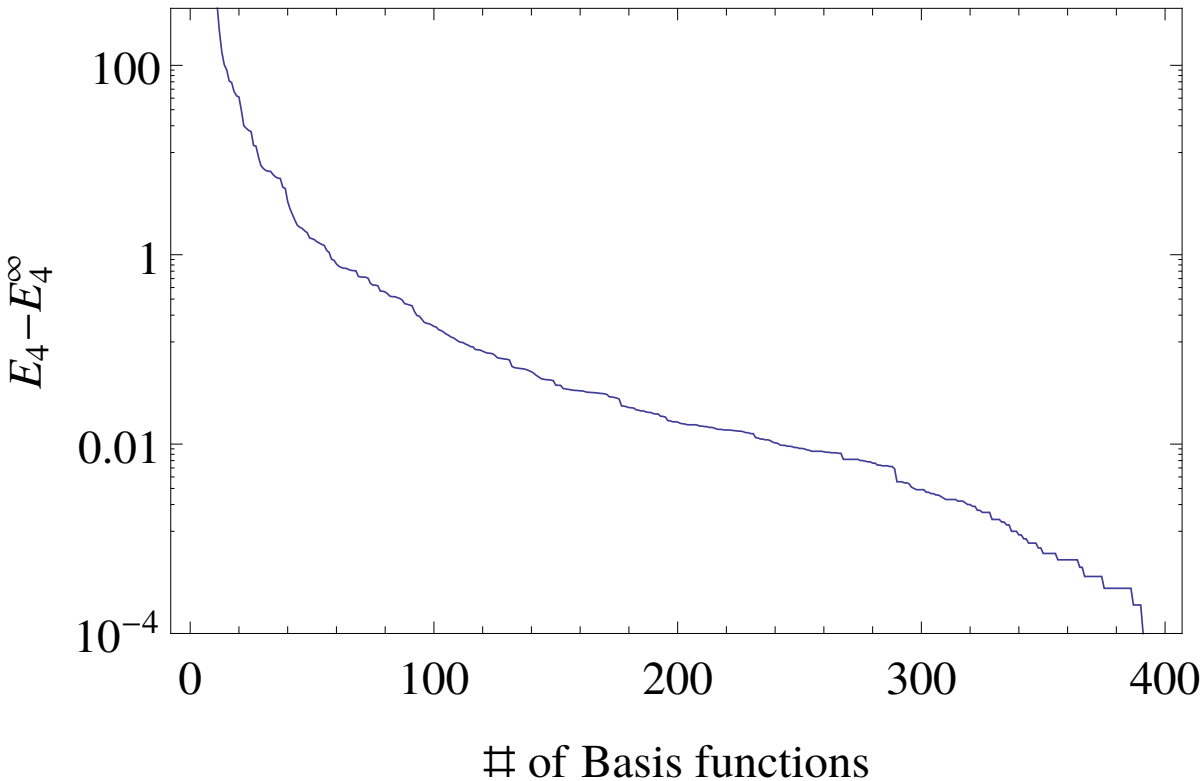

Conclusion

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BB, Eliyahu and van Kolck, Phys. Rev. A **94**, 052502 (2016).
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