Quadrupole Response of a Weakly Bound Bosonic Trimer

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The inelastic response of a bosonic trimer is explored in the confines of the Borromean region. To this end we model the interaction between the external field and the bosonic system as a photoabsorptionlike process and study the response of the trimer in the quadrupole approximation. We utilize the hyperspherical-harmonics expansion to solve the Schrödinger equation and the Lorentz integral transform method to calculate the reaction. It is found that the magnitude of the response function and corresponding sum rules increase exponentially when approaching the 3-body threshold. It is also found that this increase is governed by unnatural exponents. The connection between our results and radio-frequency experiments in ultracold atom systems is made.

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Inelastic reactions, where the system is perturbed from its initial position by an external probe, are an invaluable tool for studying the structure and the dynamics of few-nucleon systems and as input for describing astrophysical processes. In particular, perturbation induced reactions such as photonuclear reactions or electron scattering provide an important test for the nuclear force model and for exposing the underlying degrees of freedom, manifested as many-body or meson exchange currents. Studying these reactions is instrumental for understanding the nuclear dynamics, the complicated nuclear continuum, and the onset of collective behavior typical to large nuclei. Moreover, the study of such reactions is necessary for predicting cross sections of unmeasurable processes of astrophysical importance.

The similarity between atomic nuclei and cold-atom physics [1] calls for utilizing the properties of the latter systems for studying the universal properties and the reaction mechanisms manifested in nuclear systems. As a step in this direction, we consider in the current study the dissociation of a bound trimer composed of identical bosons by a photonlike perturbation. This reaction is up to kinematical factors the time reversal of the radio-frequency (rf) association experiments being carried out in cold-atom systems. In these experiments, bosonic atoms (^{7}Li [2] and ^{85}Rb [3]) are trapped and cooled down to degeneracy, while a strong magnetic field is applied to induce a controllable interparticle interaction through a Feshbach resonance. Then, molecules are generated by photon emission, stimulated by an external rf field. This process is the atomic analog of the nuclear radiative capture process.

In this Letter we limit our attention to Borromean 3-body states in which the trimer state is bound although none of the 2-body subsystems are bound. In these systems the 3-body physics is dominant and unmasked by 2-body effects.

The dynamics of the system are governed by the Schrödinger equation

$$\left(\sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \boldsymbol{r}_i^2} + \sum_{i\neq j} V_0 U(r_{ij}/r_0)\right)\Psi = E\Psi, \qquad (1)$$

where for the interaction we assume a simple attractive two-body force of a Yukawa potential $U = -\exp(-x)/x$ or Gauss potential $U = -\exp(-x^2)$ with a characteristic depth V_0 and range r_0. We use the hyperspherical-harmonics (HH) expansion [4–7] $\Psi(\rho,\Omega) = \sum_{K\leq K_{\max}} R_{[K]}(\rho)\mathcal{Y}_{[K]}(\Omega)$ to solve the Schrödinger equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} - \frac{\hat{K}^2}{\rho^2}\right) + \sum_{i\neq j} g U(x_{ij})\right]\Psi = \epsilon\Psi \qquad (2)$$

in the internal (Jacobi) coordinates. Here $x = r/r_0$, $g = V_0/(\hbar^2/mr_0^2)$, and $\epsilon = E/(\hbar^2/mr_0^2)$ are the dimensionless length, coupling constant, and energy, respectively. $\rho = \sqrt{\eta_1^2 + \eta_2^2}$ Ω are the hyperspherical coordinates associated with the Jacobi coordinates $\boldsymbol{\eta}_1 = \sqrt{1/2}(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ and $\boldsymbol{\eta}_2 = \sqrt{2/3}[\boldsymbol{x}_3 - (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2]$, and $\hat{K}$ is the hyperangular momentum operator [7].

The response of a physical system to an external field is closely related to the static moments of that system. When the wavelength of the external field is much larger then the size of the system, the lowest moments are expected to dominate the reaction cross section. For a bosonic trimer composed of identical particles, the leading moments of the system are the quadrupole and the 2nd order monopole moments as the dipole operator is proportional to the center of mass coordinate and therefore cannot induce any internal excitations. Following this reasoning, we focus in this study on the quadrupole contribution to the trimer's response function given by

$$R(\omega) = \int d\Psi_f |\langle\Psi_f|\hat{Q}|\Psi_0\rangle|^2 \delta(E_f - E_0 - \omega). \qquad (3)$$

Here $|\Psi_0\rangle$, E_0 ($|\Psi_f\rangle$, E_f) are the initial (final) state wave function and energy, and $\int d\Psi_f$ is a shorthand notation for a sum over all possible final states. The transition operator

$$\hat{Q} = \alpha \sum_i r_i^2 Y_{20}(\hat{r}_i) \tag{4}$$

is the $j_z = 0$ projection of the quadrupole operator. The prefactor α in Eq. (4) includes the coupling constant and the kinematics of the reaction at hand and may also depend on the energy transfer ω.

For an electromagnetic (EM) excitation, the interaction Hamiltonian $H_I = \mathbf{J}(\boldsymbol{x}) \cdot \mathbf{A}(\boldsymbol{x})$ between the radiation field and a system of neutral particles is dominated by the magnetization current $\mathbf{J}(\boldsymbol{x}) \approx \nabla \times \boldsymbol{\mu}(\boldsymbol{x})$, and the interaction Hamiltonian is proportional to the magnetic field $H_I \approx \boldsymbol{\mu}(\boldsymbol{x}) \cdot \mathbf{B}(\boldsymbol{x})$. The magnetization density $\boldsymbol{\mu}(\boldsymbol{x}) = \sum_i \mu_A \mathbf{S}_i \delta(\boldsymbol{x} - \boldsymbol{x}_i)$ is the local sum of the spins $\mathbf{S}_i$ times the magnetic moment of the individual atom μ_A. If all the constituents possess the same magnetization state and this state is frozen during the reaction process, then the magnetization current is proportional to the density operator $\rho(\boldsymbol{x}) = \sum_i \delta(\boldsymbol{x} - \boldsymbol{x}_i)$. This situation is valid when all the spins are polarized and the EM field does not induce any internal spin transitions. Under these conditions the reaction of the system to the external photon field is very similar to the reaction of a system of charged particles. Similarly, it can be seen that the quadrupole operator (3) and the 2nd order monopole operator $\sum_i r_i^2$ are the leading contributions to the reaction cross section [8]. In this case, the leading magnetic contribution $M_{3,0}$ to the photoabsorption cross section is given by $\sigma(\omega) = 2\pi^2 \frac{e^2}{\hbar c \omega} R(\omega)$, where $R(\omega)$ is given by Eq. (3), $\alpha = \mu_A k^3 \sqrt{12/35}/15$ and k is the photon wave number.

We note that the assumptions outlined above are valid for rf experiments in atomic traps where MHz photons are used [2,3]. The difference between the response function (3) of the photodisintegration process and the photoassociation reaction is the statistical weight function that accounts for the probability of finding three particles in a continuum initial state.

The direct approach for evaluating the response function (3) is to calculate the final states $|\Psi_f\rangle$ over a dense grid in the continuum and to sum all the possible configurations that contribute to $R(\omega)$. An alternative approach is to use the Lorentz integral transform (LIT) method [9,10] and to evaluate instead an integral transform

$$L(\sigma) = \int d\omega \frac{R(\omega)}{(\omega - \sigma)^2 + \Gamma^2}, \tag{5}$$

with a Lorentzian kernel defined by the resolution parameter Γ. The basic idea of considering the LIT function $L(\sigma)$ lies in the fact that it can be evaluated from the norm of a localized function $L(\sigma) = \langle \tilde{\Psi}(\sigma) | \tilde{\Psi}(\sigma) \rangle$, which is the unique solution of the inhomogeneous equation

$$(\hat{H} - E_0 - \sigma - i\Gamma)|\tilde{\Psi}(\sigma)\rangle = \hat{Q}|\Psi_0\rangle. \tag{6}$$

Because of the presence of the imaginary part Γ in (6) and the fact that its right-hand side is localized, one has a bound-state-like asymptotic boundary condition. Thus, one can apply bound-state techniques for its solution. The response function $R(\omega)$ is finally obtained by inverting the LIT (5). The inversion process can be carried out in various ways [11–13], in particular, we have used the analytic inversion method of [13].

Working with an attractive single Gaussian potential, the unitary point, where the scattering length a diverges, appears at $g_\infty = g(a^{-1} = 0) \approx 2.684$ resulting in a trimer ground-state binding energy of $E_{T\infty} = E_T(a^{-1} = 0) \approx 0.238\hbar^2/mr_0^2$ and rms matter radius $r_T(a^{-1} = 0) = 1.21 r_0$. Using the HH expansion we have calculated the LIT function at few values of $g \leq g_\infty$. In the calculations we typically used $K_{\max} = 70$ and about 150 radial basis states. The LIT function for $g = 2.30$ is presented in Fig. 1 for $\Gamma = 0.1\hbar^2/mr_0^2$ over a large range of σ values. The scattering length for this value of g is $a \approx -6.78 r_0$ and the trimer's binding energy is $E_T = 0.0466\hbar^2/mr_0^2$. It is expected [14] that the reaction cross section will reach its peak at energies of the order of E_T, consequently we expect the same for the LIT function $L(\sigma)$.

At threshold $\omega_{\text{th}} = E_T$, the response function is expected to have a quadratic dependence $(\omega - \omega_{\text{th}})^2$ on the energy transfer [15]. In the inversion process [13] we use this threshold law and expand the response using two sets of basis functions,

$$R(\omega) = (\omega - \omega_{\text{th}})^2 \sum_{n=0}^{N} C_n (\omega - \omega_{\text{th}})^n e^{-\beta\omega}, \tag{7}$$

and,

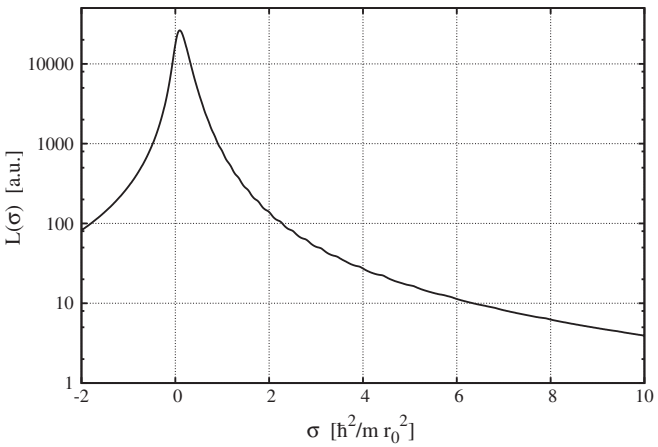

FIG. 1. The LIT function for the quadrupole response of a bosonic trimer. The width of the Lorentzian is $\Gamma = 0.1\hbar^2/mr_0^2$ and the interaction strength is $g = 2.30$.

$$R(\omega) = (\omega - \omega_{\rm th})^2 \sum_{n=1}^{N} C_n e^{-\beta\omega/n}, \qquad (8)$$

where β and $\{C_n\}$ are free parameters used for fitting the LIT function. It should be noted that (a) the exponential fall of our trial functions is correct for a harmonic oscillator potential, but merely serves to localize the response function for any other potential, and (b) the inversion process is an ill-posed process and therefore N serves as a high frequency cutoff.

In Fig. 2 we present the response function obtained with these two basis sets, (7) and (8). From the figure it is evident that the two assumptions lead to the same response function. Moreover, as we improve the inversion by increasing the number of basis functions the tail of the response is slowly established for $\omega - \omega_{th} \leq 1.5\hbar^2/mr_0^2$. In [16] the behavior of the tail was derived to be $1/\omega^2$ for $\omega < \hbar^2/mr_0^2$. Analyzing the behavior of the response in the tail region from the peak to $\omega \approx 1.5\hbar^2/mr_0^2$ we have found that $d\ln R/d\ln\omega$ is an almost linearly monotonically decreasing function. Integrating this function we get the odd looking approximation

$$R(\omega) \sim \omega^{-a-b\ln(\omega)}; \qquad \omega_{\rm peak} < \omega < 1.5\hbar^2/mr_0^2 \qquad (9)$$

for the response function. Fitting this form to the tail of the calculated responses we get an excellent agreement not only for the example at hand but also for different values of g and for different potential models. The values of a, b depend on the potential form and strength. For a Gauss potential with $g = 2.60$ we found $a \approx 2.8$ and $b \approx 0.7$, for $g = 2.30$ we found $a \approx 3.5$ and $b \approx 0.5$.

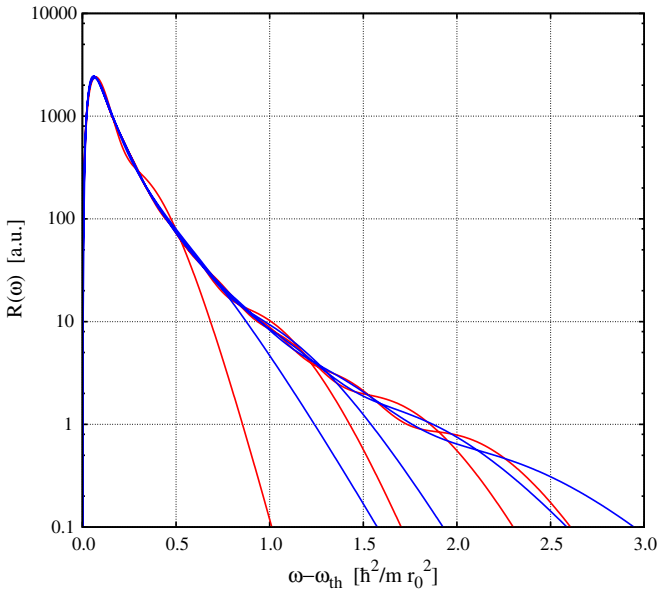

FIG. 2 (color online). The convergence of the response function as a function of the number of basis states. The red (light gray) lines correspond to the polynomial expansion (7), the blue (dark gray) lines to the exponential expansion (8). In both cases $N = 4, 8, 12, 16$ increases from left to right. The LIT function is taken from Fig. 1.

We would like to focus now on the evolution of the response function with the trimer's binding energy. To this end we have plotted in Fig. 3 the response functions corresponding to the cases $g = 2.60, 2.30, 2.15$ or $E_T = 0.19, 0.047, 0.0034\hbar^2/mr_0^2$. From the plot it is seen that as the binding energy vanishes (a) the response function seems to diverge and (b) the peak position is shifted to lower and lower energies. To understand these features we note that E_T is the relevant energy scale of the process. As $E_T \to 0$ the pick position shifts with it, and the response function being proportional to the radius of system grows.

A convenient way to study the evolution of the response function with the trimer's binding energy is to explore the dependence of the sum rules

$$S_n \equiv \int_{\omega_{\rm th}}^{\infty} d\omega\, \omega^n R(\omega) \qquad (10)$$

on the trimer's properties. The sum rule S_n is the nth moment of $R(\omega)$. For $n > 0$ it exists if $R(\omega) \to 0$ faster than ω^{-n-1} as $\omega \to \infty$. Because of the threshold law the sum rules are only defined for $n \geq -2$. The sum rules can be expressed as an initial state observable utilizing the closure of the eigenstates of H. In particular we shall focus on the following sum rules:

$$S_1 = \langle\Psi_0|[\hat{Q},[H,\hat{Q}]]|\Psi_0\rangle = \langle\Psi_0|\hat{Q}(H - E_0)\hat{Q}|\Psi_0\rangle,$$
$$S_0 = \langle\Psi_0|\hat{Q}\,\hat{Q}\,|\Psi_0\rangle,$$
$$S_{-1} = \langle\Psi_0|\hat{Q}\frac{1}{H - E_0}\hat{Q}|\Psi_0\rangle, \qquad (11)$$

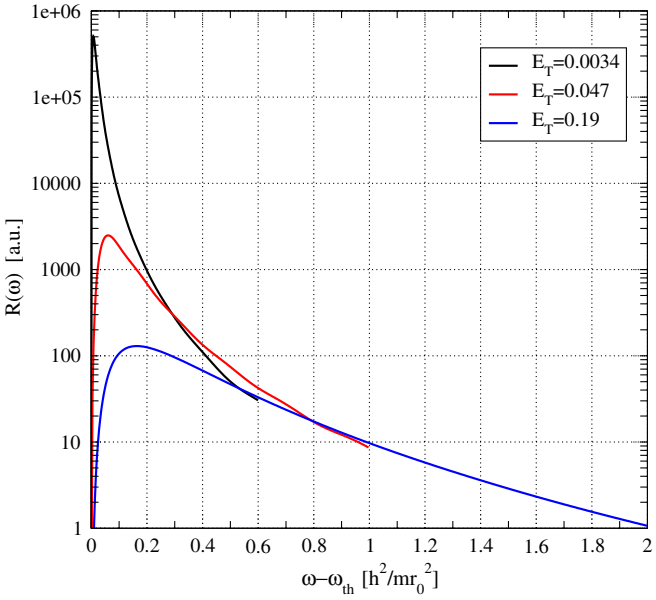

FIG. 3 (color online). The quadrupole response function of a bosonic trimer for $g = 2.60, 2.30, 2.15$ or $E_T = 0.19, 0.047, 0.0034\hbar^2/mr_0^2$.

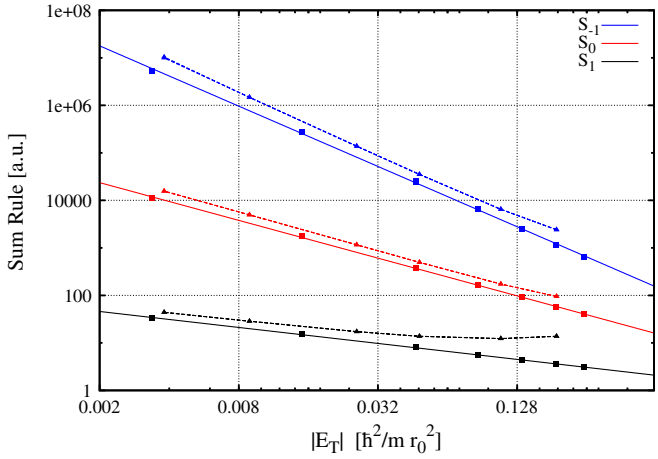

FIG. 4 (color online). Evolution of the sum rules S_{-1}, S_0, S_1 as a function of the trimer binding energy. The upper, middle, and lower lines correspond to the S_{-1}, S_0, and S_1 sum rules. or clarity, the sum rules S_1 (S_{-1}) were divided (multiplied) by 10. The squares correspond to Gauss potential and the triangles to Yukawa potential.

which can be easily calculated using the Lanczos method [17,18].

Naive scaling arguments leads us to expect that the energy dependence of the trimer radius follows the rule $r_T \sim 1/\sqrt{E_T}$. The quadrupole operator behaves as r^2 so $R(\omega) \sim r_T^4/E_T \sim 1/E_T^3$. It follows that the sum rules should fulfill the power law $S_n \sim 1/E_T^{2-n}$.

In Fig. 4 we present the calculated quadrupole sum rules S_{-1}, S_0, S_1 for two different interaction models: a Gauss interaction and an attractive Yukawa force. As expected, the sum rules grow as $E_T \to 0$. It is seen that over a large energy range the sum rules follow a simple power law that exhibits a mild dependence on the specifics of the interaction. Fitting the Gauss interaction sum rules to a simple power law $S_n = A_n E_T^{-q_n}$ we have found that $q_{-1} \approx 2.1$, $q_0 \approx 1.3$, and $q_1 \approx 0.5$. These results are in contrast with our naive expectations of $q_{-1} = 3$, $q_0 = 2$, and $q_1 = 1$. This contradiction can be partially resolved since in our calculations $r_T \sim E_T^{-0.28}$ instead of following the square-root law. All these scaling laws are particularities of the 3-body system that (a) seems to depend only weekly on the potential model, and (b) differs dramatically from the corresponding 2-body system.

Summing up, we have studied the response of shallow trimers to a photodisintegration type reaction that does not alter the constituents themselves. Using the HH expansion and the LIT methods we have calculated the response function and the sum rules for the quadrupole excitation. Over the energy range $0.003\hbar^2/mr_0^2 \leq E_T \leq 0.3\hbar^2/mr_0^2$ we have found that the peak of the response function is increased by 4 orders of magnitude. We have also found that the sum rules S_{-1}, S_0, S_1 grow roughly exponentially with diminishing trimer binding energy. As the trimer becomes shallower the dependence of the sum rules on E_T takes a path independent of the particular potential model used. Moreover, the exponents governing the behavior of the sum rules in this region assume unexpected values.

In view of the strong dependence of the photoabsorption cross section on the trimer's properties it is not inconceivable that the reverse photoassociation process will be accessible experimentally in cold-atom systems.

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