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# Log-Periodic Oscillations in the Photo Response of Efimov Trimers

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**Abstract** The photoassociation of Efimov trimer, composed of three identical bosons, is studied using the long wavelength approximation. It is shown that for identical particles the leading contribution comes from the  $r^2$   $s$ -mode operator and from the quadrupole  $d$ -mode operator. Log-periodic oscillations are found in the photoassociation response function, both near the energy threshold for the leading  $s$ -wave reaction, and in the high frequency tail for all partial waves.

## 1 Introduction

The implementation of photo-association techniques in ultracold atomic traps [1] opened a new route for quantitative determination of universal properties in few-body systems [2]. In these experiments, radio-frequency (rf) induced molecule formation leads to enhanced atom loss rate from the trap. The variation in the atom loss rate with changing rf field frequency is a reflection of the photoresponse function of the atomic system, encoding the molecular structure and spectra.

Recently, trimer formation through rf association was discovered in both fermionic  ${}^6\text{Li}$  [3,4] and bosonic  ${}^7\text{Li}$  [5] systems. The three body system attracts special attention since Efimov's prediction from 1970 [6] that in the limit of a resonant 2-body interaction of a zero energy bound state, the 3-body system contains infinitely many bound states. These states exhibit universal properties: the spectrum is a geometric series where  $\epsilon_{n+1}/\epsilon_n = e^{-2\pi/s_0} \approx 1/515$  and the wave functions dilute as  $e^{-\pi/s_0} \approx 1/22.7$ . Efimov's prediction of these peculiar three body states, known as Efimov trimers, was experimentally verified in the last few years [7–10].

In a previous work [11], we have analyzed rf association process binding a molecule made of  $N$  identical bosons. Using the long wavelength approximation, we have shown that spin-flip and frozen-spin processes differ by their operator structure and by the reaction modes that dominate the process. Theoretical analysis of rf experiments based on the Franck–Condon factor, [12–16], is appropriate for describing spin-flip reactions. For frozen-spin reactions we have applied our results to study the dimer formation [11], and the quadrupole response of a bound bosonic trimer [17,18].

Here, we study trimer photoassociation using the hyperspherical adiabatic approximation. A zero-range potential is used to derive analytic results for the transition rates at the unitary point. Similarly to the dimer case [11], the  $s$ -mode and the  $d$ -mode are found to be the leading contributions. A new fingerprint of Efimov physics is observed, that is log-periodic oscillations in the trimer's photo response function.

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## 2 Multipole Expansion

The molecular photoassosiation rate is given by Fermi's golden rule,

$$r_{i \rightarrow f} = \frac{2\pi}{\hbar} \sum_i \sum_f |\langle f, \mathbf{k}\zeta | \hat{H}_I | i \rangle|^2 \delta(E_i - E_f - \hbar\omega_k), \quad (1)$$

where three particles in an initial continuum state with energy  $E_i$  form a bound state with energy  $E_f$  by emitting a photon with momentum  $\mathbf{k}$ , polarization  $\zeta$  and energy  $\hbar\omega_k = \hbar ck$ .  $\bar{\sum}_i$  is an average on the appropriate initial continuum states and  $\sum_f$  is a sum on the final bound states. The coupling between the neutral atoms and the radiation field takes the form  $\hat{H}_I = -e \int d\mathbf{x} \boldsymbol{\mu}(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{x})$ , where  $\mathbf{A}$  is the electromagnetic (EM) photon field, and  $\boldsymbol{\mu}(\mathbf{x})$  is the magnetization current. Here we consider only the one-body current  $\boldsymbol{\mu}(\mathbf{x}) = \mu_0 \sum_j \mathbf{S}_j \delta(\mathbf{x} - \mathbf{r}_j)$ , where  $\mu_0$  is the magnetic moment of a single particle, and  $\mathbf{S}_j$  and  $\mathbf{r}_j$  are the spin and position of particle  $j$ .

We assume that the initial and final atomic wave functions can be written as a product of symmetric spin  $|\chi\rangle$  and configuration space  $|\psi\rangle$  terms, and that the photon does not induce change in the spin structure of the system. In this case the transition matrix element can be written as

$$\langle f, \mathbf{k}\zeta | \hat{H}_I | i \rangle = -i\mu_0 \sqrt{\frac{\hbar c^2}{2\mathcal{V}\omega_k}} k \langle S_0 \rangle \langle \psi_{L'M'}^f | \sum_{j=1}^3 e^{i\mathbf{k}\cdot\mathbf{r}_j} | \psi_{LM}^i \rangle, \quad (2)$$

where  $\langle S_0 \rangle = \frac{1}{3} \sum_j \langle \chi_{M_F}^i | S_{j,0} | \chi_{M_F}^i \rangle$  is the average single particle magnetic moment, which plays the role of an effective charge. We normalize the EM field in a box of volume  $\mathcal{V}$ .

The wavelength of rf radiation is much larger than the typical dimension of the system  $R$ , thus  $kR \ll 1$ . Consequently, we can utilize the long wavelength approximation and expand the exponent in (2) to get

$$\sum_{j=1}^3 e^{i\mathbf{k}\cdot\mathbf{r}_j} \approx 3 + i \sum_{j=1}^3 \mathbf{k} \cdot \mathbf{r}_j - \frac{1}{6} \sum_{j=1}^3 k^2 r_j^2 - \frac{4\pi}{15} \sum_{j=1}^3 k^2 r_j^2 \sum_m Y_{2-m}(\hat{\mathbf{k}}) Y_{2m}(\hat{\mathbf{r}}_j), \quad (3)$$

where  $Y_{lm}$  are the spherical harmonics. Each order in this expansion has clear physical meaning. The zeroth order operator stands for elastic interaction. The first order operator is the dipole, which for identical particles is proportional to the center of mass and hence does not affect the relative motion of the atoms. At second order two operators appear: the  $r^2$  operator, corresponding to  $s$ -mode reaction, and the quadrupole terms, corresponding to  $d$ -mode reaction. For identical particles this is the leading term in low energy frozen spin reactions, and the transition probability scales as  $k^5$ . Summing over the initial and final magnetic numbers  $M, M'$ , the transition matrix element reads

$$\sum_{M, M'} |\langle f, \mathbf{k}\zeta | \hat{H}_I | i \rangle|^2 = \frac{4\pi \hbar c k^5 \mu_0^2}{2\Omega} |\langle S_0 \rangle|^2 \left( \frac{1}{6^2} |\langle \psi_{L'}^f | \sum_{j=1}^3 r_j^2 Y_0 \| \psi_{L'}^i \rangle|^2 + \frac{1}{15^2} |\langle \psi_{L'}^f | \sum_{j=1}^3 r_j^2 Y_2(\hat{\mathbf{r}}_j) \| \psi_{L'}^i \rangle|^2 \right). \quad (4)$$

## 3 The Three Body Problem

The dynamics of a quantum 3 particle system is governed by the Schroedinger equation

$$(T + W) \psi = E \psi \quad (5)$$

where  $T$  is the center of mass kinetic energy operator and  $W$  is the potential. In this study we shall limit our attention to short range 2-body forces, thus  $W = \sum_{i < j} V(|\mathbf{r}_i - \mathbf{r}_j|)$ . To eliminate the center of mass motion, we define the Jacobi coordinates,  $\mathbf{x} = \sqrt{\frac{1}{2}}(\mathbf{r}_2 - \mathbf{r}_1)$ , and  $\mathbf{y} = \sqrt{\frac{2}{3}}(-\mathbf{r}_3 + \frac{\mathbf{r}_1 + \mathbf{r}_2}{2})$ , which we transform into the hyperspherical coordinates  $\rho^2 = x^2 + y^2$ , and  $\Omega = (\alpha, \hat{x}, \hat{y})$ , where  $\tan \alpha = x/y$ .

In the limit of infinite scattering length,  $|a| \rightarrow \infty$ , the spatial wave function can be written as [19]  $\psi(\rho, \Omega) = \rho^{-5/2} \mathcal{R}(\rho) \Phi(\Omega)$ . The hyperspherical functions  $\Phi(\Omega)$  and the corresponding eigenvalue  $\nu^2$ , are the solutions of the hyperangular equation,

$$\left( \hat{K}^2 + \frac{2m\rho^2}{\hbar^2} \sum_i V(\sqrt{2}\rho \sin \alpha_i) + 4 \right) \Phi = \nu^2 \Phi, \quad (6)$$

where  $\hat{K}^2 = -\frac{1}{\sin 2\alpha} \frac{\partial^2}{\partial \alpha^2} \sin 2\alpha + \frac{\hat{l}_x^2}{\sin^2 \alpha} + \frac{\hat{l}_y^2}{\cos^2 \alpha} - 4$ . For low energy physics, when the extension of the wave function is much larger than the range of the potential, one can utilize the zero range approximation. In this approximation the lateral extension of the potential is neglected all together, and the action of the potential is represented through the appropriate boundary conditions. For a two-particle system the low energy interaction is dominated by the  $s$ -wave scattering length  $a$  and the wave function fulfills the boundary condition  $[u'/u]_{r=0} = -1/a$ . The corresponding 3-body condition is

$$\left[ \frac{1}{2\alpha\Phi} \frac{\partial}{\partial \alpha} 2\alpha\Phi \right]_{\alpha=0} = -\sqrt{2} \frac{\rho}{a}. \quad (7)$$

Plugging the solution of Eq. (6) into Eq. (7), one gets transcendental equations for  $\nu$ . For  $L = 0$  the resulting equation is [20]

$$\nu \cos(\nu\pi/2) - \frac{8}{\sqrt{3}} \sin(\nu\pi/6) = \frac{\sqrt{2}\rho}{a} \sin(\nu\pi/2). \quad (8)$$

For  $|a| = \infty$  the solution with lowest  $\nu^2$  is  $\nu_0 \approx 1.00624i$ , corresponding to the Efimov trimer.

For  $L = 2$  the corresponding equation is,

$$\nu(4 - \nu^2) \cos(\nu\pi/2) + 24\nu \cos(\nu\pi/6) + \frac{8}{\sqrt{3}}(\nu^2 - 10) \sin(\nu\pi/6) = -\frac{\rho}{a}(\nu^2 - 1) \sin(\nu\pi/2) \quad (9)$$

For  $|a| = \infty$  the lowest non-trivial solution is  $\nu_2 \approx 2.82334$ .

In the limit of  $|a| \rightarrow \infty$ ,  $\nu_L(\rho) = \nu_L$  and the hyperradial equation for  $\mathcal{R}(\rho)$  is similar to the Bessel equation,

$$-\mathcal{R}''(\rho) + \frac{\nu_L^2 - 1/4}{\rho^2} \mathcal{R}(\rho) = \epsilon \mathcal{R}(\rho). \quad (10)$$

where  $\epsilon = 2mE/\hbar^2$ .

We seek the solution for the bound ( $\epsilon < 0$ ) and continuum ( $\epsilon > 0$ ) cases:

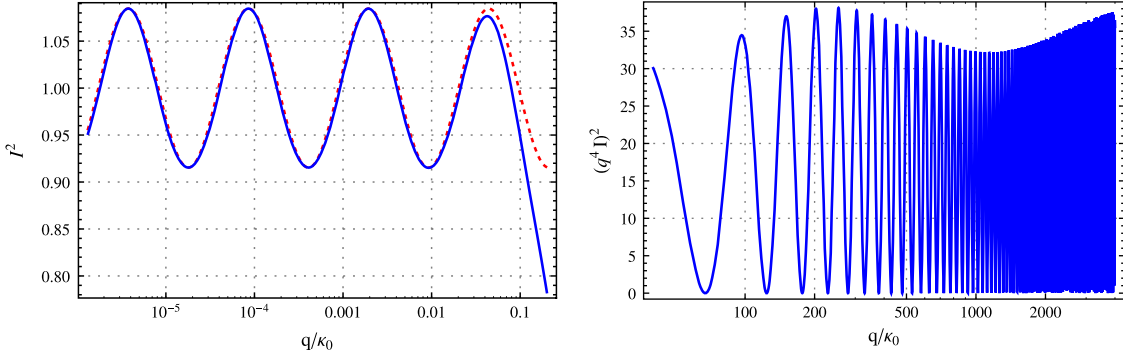
I. A *bound state* exists only for  $L = 0$ , and  $\nu_0 \approx 1.00624i$ . In this case the relevant solution is  $\sqrt{\rho} K_{\nu_0}(\kappa\rho)$ , where  $\kappa = \sqrt{-\epsilon}$ . Near the origin, this solution behaves like  $\sin(\nu \ln(\kappa\rho/2) + 0.301)$ , therefore regularization is needed to avoid collapse, e.g. setting  $\mathcal{R}(\rho \leq \rho_0) = 0$  for some finite  $\rho_0$ . The result is the discrete Efimov spectrum,  $\epsilon_n/\epsilon_0 = e^{-2\pi n/|\nu_0|} \approx 515^{-n}$ . The normalized wave functions are

$$\mathcal{R}_B^{(n)}(\rho) = \sqrt{2 \sin \nu_0 \pi / \nu_0 \pi \kappa_n} \sqrt{\rho} K_{\nu_0}(\kappa_n \rho) \quad (11)$$

II. For *continuum state*, the solution is

$$\mathcal{R}_L(\rho) = \sqrt{\frac{q\rho N_s}{2R}} [\sin \delta \text{Re}[J_{\nu_L}(q\rho)] + \cos \delta \text{Re}[Y_{\nu_L}(q\rho)]] \quad (12)$$

where  $q = \sqrt{\epsilon}$ ,  $N_s = 1/2(\pi)$  for imaginary (real)  $\nu$ , and we assume normalization in a sphere of radius  $R$ . The phase shift  $\delta$  is to be found from the boundary condition,  $\mathcal{R}_L(\rho_0) = 0$ .



**Fig. 1** The log periodic oscillations in the  $r^2$  transition matrix element as a function of the three-body wave number  $q$ , given in terms of the ground state wave number  $\kappa_0$ . The matrix element is presented in two different regimes:  $|I_s|^2$  near the reaction threshold (*left panel*) and  $(q^4 |I_s|^2)^2$  at the high frequency tail (*right panel*)

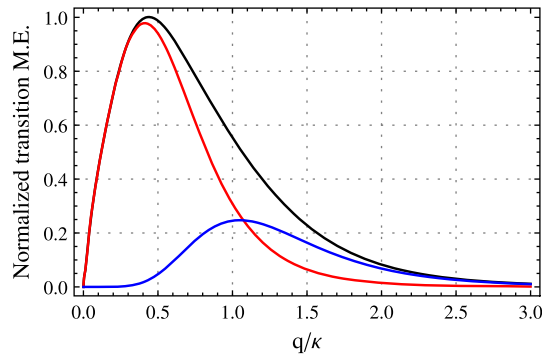
#### 4 The Transition Matrix Elements

Now that we have obtained the initial and final wave functions we are in position to evaluate the transition matrix elements, Eq. (4),  $I_\lambda = \langle \psi_{L'}^f | \sum_{j=1}^N r_j^2 Y_\lambda | \psi_L^i \rangle$ , for  $\lambda = s(d)$  corresponding to the  $r^2$  (quadrupole) operator, respectively.

*The  $r^2$  matrix element* — The  $r^2$  operator connects the  $L = 0$  bound state to an  $L = 0$  scattering state. We note that  $\sum_j r_j^2 = \rho^2 + 3R_{CM}^2$  where  $R_{CM}$  is the center of mass radius. As we neglect the center of mass excitations, the matrix element is reduced into the hyperradial integral  $I_s(\kappa, q) = \int_0^\infty d\rho \mathcal{R}_B^*(\rho) \rho^2 \mathcal{R}_s(\rho)$ . Evaluating this integral, the resulting response function reveals log periodic oscillations [21]. At threshold, the matrix element gets a particularly simple form which can be well approximated by

$$I_s(\kappa, q) \approx C \kappa^{-3} \sqrt{q} \left( 1 + \frac{B_3}{2} \cos(2s_0 \ln \frac{q}{\kappa}) \right), \quad (13)$$

where  $C$  is a constant that contains the normalization factors, and  $B_3 \approx 8.475\%$  is the normalized amplitude of the oscillations. The oscillations modulate the matrix element all the way to the high energy tail. In Fig. 1 the transition matrix element squared of the  $r^2$  operator is presented as a function of the continuum state wave number  $q$  normalized to the bound state wave number  $\kappa_0$ . The log periodic oscillations are clearly visible both near the reaction threshold and at the high frequency tail, where they are strongly modulated by the linear phase shift oscillations.



**Fig. 2** The normalized three-body transition matrix element, Eq. (4), as a function of the relative momentum  $q/\kappa$ . The sum (black),  $r^2$  (red, peaked at  $q = \kappa/2$ ), and quadrupole (blue, peaked at  $q = \kappa$ ) terms are given for the unitary point,  $|a| = \infty$

*The Quadrupole matrix element* — The quadrupole operator  $\sum_j r_j^2 Y_2^M(\hat{r}_j)$  connects the  $L = 0$  bound state with  $L = 2$  scattering states. In this case the reduced matrix element takes the form

$$I_d(\kappa, q) = \frac{3}{2} \sqrt{5} \int_0^\infty d\rho \mathcal{R}_B^*(\rho) \rho^2 \mathcal{R}_d(\rho) \int d\Omega \Phi_f^*(\Omega) \sum_j \cos^2 \alpha_j Y_{20}(\hat{y}_j) \Phi_i(\Omega). \quad (14)$$

In this case we find no log-periodic oscillations near threshold. Such oscillations, however, modulate the high energy tail of the response function, attenuated by  $q^{-4}$  and masked by the linear phase shift variation. These high energy oscillations appear not only in these cases but in all partial waves [21].

The relative contribution of the  $s$ ,  $d$  modes to the trimer formation is displayed in Fig. 2, where the last term in parenthesis on the rhs of Eq. (4) is presented normalized, along with the  $s$  and  $d$  components. Similarly to our findings for the dimer formation case [11], the  $s$ -wave association peaked around  $q = \kappa/2$  is dominant near threshold, while the  $d$ -wave association peaked at  $q = \kappa$  becomes dominant at higher energies.

## 5 Conclusion

In this work we have studied the photoassociation of universal trimers. Using the long wavelength approximation we have identified the two leading reaction mechanisms, the  $s$  and  $d$  modes, and evaluated their contributions. We have found that for identical bosons the trimer photoassociation cross section scales as  $k^5$  with the photon wave number. As expected the  $s$ -mode is dominant at low energy whereas the  $d$ -mode takes over when  $q/\kappa \geq 1$ .

We predict that the photoreaction cross-section exhibits log periodic oscillations throughout the energy spectrum. Near threshold these oscillations only appear in the leading  $s$ -mode. At the high frequency tail we find log periodic oscillations in all partial waves. These oscillations are the fingerprints of Efimov's physics.

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