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The Quadrupole Response of Borromean Bosonic Trimers

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Abstract The inelastic response of a Borromean trimer composed of three identical bosons is explored. To this end we use the quadrupole approximation as the excitation mechanism of the bosonic system from its ground state. We utilize the hyperspherical-harmonics expansion to solve the Schroedinger equation and the Lorentz integral transform method to calculate the reaction. It is found that the magnitude of the response function and corresponding sum rules increase as a power law when approaching the three-body threshold. It is also found that this increase is governed by unnatural exponents.

1 Introduction

Cold atoms physics is a rich research field for studying the properties, the structure, and the spectra of few-body systems [1]. In cold atoms experiments, bosonic atoms (such as ^{7}Li [2] and ^{85}Rb [3]) or fermionic atoms (such as ^{6}Li [4]) are trapped and cooled down to temperatures of μK, while strong magnetic field is applied to induce a controllable inter-particle interaction through a Feshbach resonance. The dynamic response of such systems is then studied applying an external radio frequency (rf) field to generate molecules by photon emission, stimulated by the external rf field. This process is the time reversal of the photodissociation of the molecule. This reaction can thus be seen as the cold-atoms analog of the photonuclear photoabsorption reaction often used to study the structure and the dynamics of few-nucleon systems and as input for describing astrophysical processes.

The response of a physical system to an external field is closely related to the static moments of that system. When the wavelength of the external field is much larger then the size of the system, the lowest moments are expected to dominate the reaction cross-section. For a bosonic trimer composed of structureless identical particles, the leading moments of the system are the quadrupole and the second order monopole moments. The dipole operator $\boldsymbol{D} \approx (\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3)$ is proportional to the center of mass coordinate and therefore cannot induce any internal excitations. Following this reasoning, we focus in this study, as in [5], on the quadrupole response of the trimer.

A Borromean trimer is a three-body system in which the trimer state is bound although none of the two-body subsystems are. The ^{6}He nucleus which can be regarded as an α-particle core plus two neutrons is a natural example of such system. In cold-atoms Boromean trimer are produced when the Feshbach resonance is tuned so that the scattering length is negative but the interaction is strong enough to bound the three-body system. In this study we limit our attention to the Boromean region.

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2 The Hamiltonian

Working in configuration space, the dynamics of the system are governed by the Schroedinger equation

$$\left(\sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \boldsymbol{r}_i^2} + \sum_{i\neq j} V_0 U(r_{ij}/r_0)\right)\Psi = E\Psi \ , \tag{1}$$

where for the interaction we assume a simple short-range attractive two-body force of a Yukawa $U = -\exp(-x)/x$, Gauss $U = -\exp(-x^2)$, or square well $U = -\Theta(1-x)$ form with a characteristic depth V_0 and range r_0. We use the hyperspherical-harmonics (HH) expansion [6–10]

$$\Psi(\rho,\Omega) = \sum_{K\leq K_{max},n} C_{n,[K]} R_n(\rho) \mathcal{Y}^{sym}_{[K]}(\Omega) \tag{2}$$

to solve the Schroedinger equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} - \frac{\hat{K}^2}{\rho^2}\right) + \sum_{i\neq j} g U(x_{ij})\right]\Psi = \epsilon\Psi \tag{3}$$

in the internal (Jacobi) coordinates. Here $x = r/r_0$, $g = V_0/(\hbar^2/mr_0^2)$, and $\epsilon = E/(\hbar^2/mr_0^2)$ are the dimensionless length, coupling constant and energy, respectively. $\rho = \sqrt{\eta_1^2+\eta_2^2}$, Ω are the hyperspherical coordinates associated with the Jacobi coordinates

$$\begin{aligned}\boldsymbol{\eta}_1 &= \sqrt{\frac{1}{2}}\,(\boldsymbol{x}_2 - \boldsymbol{x}_1)\\ \boldsymbol{\eta}_2 &= \sqrt{\frac{2}{3}}\left(\boldsymbol{x}_3 - \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}\right)\ ,\end{aligned} \tag{4}$$

and $\hat{K}$ is the hyperangular momentum operator [9,10]. $\mathcal{Y}^{sym}_{[K]}(\Omega)$ are the HH functions, symmetrized with respect to particle permutations, and for the hyper-radial expansion we use the basis functions

$$R_n(\rho) = \sqrt{\frac{n!}{(n+\nu)!}} b^{-3(A-1)/2}\left(\frac{\rho}{b}\right)^{(\nu-3A+4)/2} L_n^\nu(\rho/b)\exp[-\rho/(2b)]\,, \tag{5}$$

where $A = 3$ is the number of particles, $L_n^\nu(x)$ are the associated Laguerre polynomials, b is a range parameter, and the Laguerre parameter ν is taken to be $\nu = 5$. In practice we use b as a variational parameter.

3 The Response

The quadrupole response of the trimer is given by

$$R(\omega) = \int d\Psi_f |\langle\Psi_f|\hat{Q}|\Psi_0\rangle|^2 \delta(E_f - E_0 - \omega)\ , \tag{6}$$

Where $|\Psi_0\rangle$, E_0 ($|\Psi_f\rangle$, E_f) are the initial (final) state wave function and energy, $\int d\Psi_f$ is a shorthand notation for a sum over all possible final states, and for the transition operator

$$\hat{Q} = \alpha \sum_i r_i^2 Y_{20}(\hat{r}_i) \tag{7}$$

we take the $j_z = 0$ component of the quadrupole operator. The pre factor α in Eq. (7) includes the coupling constant and the kinematics of the reaction at hand and may also depend on the energy transfer ω.

The direct approach for evaluating the response function (6) is to calculate the final states $|\Psi_f\rangle$ over a dense grid in the continuum and to sum all the possible configurations that contribute to $R(\omega)$. An alternative

approach is to use the Lorentz integral transform (LIT) method [11,12] and to evaluate instead an integral transform

$$L(\sigma) = \int d\omega \frac{R(\omega)}{(\omega - \sigma)^2 + \Gamma^2} \ , \tag{8}$$

with a Lorentzian kernel defined by the resolution parameter Γ. The basic idea of considering the LIT function $L(\sigma)$ lies in the fact that it can be evaluated from the norm of a localized function $L(\sigma) = \langle \widetilde{\Psi}(\sigma) | \widetilde{\Psi}(\sigma) \rangle$, which is the unique solution of the inhomogeneous equation

$$(\hat{H} - E_0 - \sigma - i\Gamma) | \widetilde{\Psi}(\sigma) \rangle = \hat{Q} | \Psi_0 \rangle . \tag{9}$$

Due to the presence of the imaginary part Γ in (9) and the fact that its right-hand side is localized, one has a bound-state like asymptotic boundary condition. Thus, one can apply bound-state techniques for its solution. The response function $R(\omega)$ is finally obtained by inverting the LIT (8). The inversion process can be carried out in various ways [13–15], in particular we have used the analytic inversion method of [15].

Using the HH expansion, Eq. (2), we have calculated the ground state and the LIT function at few values of the coupling constant g in the Boromean region. In these calculations we used all the symmetrized HH basis states with $K \leq K_{max}$ for $K_{max} = 50 - 70$ and about $n = 150$ hyper-radial basis states.

For an attractive single Gaussian potential the unitary point, where the scattering length a diverge, appears at $g_\infty = g(a^{-1} = 0) = 2.684$ resulting a trimer ground-state binding energy of $E_{T\infty} = E_T(a^{-1} = 0) = 0.238\ \hbar^2/mr_0^2$ and rms matter radius $r_T(a^{-1} = 0) = 1.21\ r_0$. For the Yukawa potential $g_\infty = 1.68$ and the corresponding ground state binding energy and rms matter radius are $E_{T\infty} = 0.172\ \hbar^2/mr_0^2$ and $r_{T\infty} = 1.40\ r_0$. For the square well potential $g_\infty = \pi^2/4$ and the corresponding ground state binding energy and rms matter radius are $E_{T\infty} = 0.45\ \hbar^2/mr_0^2$ and $r_{T\infty} = 0.9\ r_0$, respectively.

In Fig. 1 we plot [5] the response functions corresponding to the Gauss potential with $g = 2.60, 2.30, 2.15$ or equivalently $E_T = 0.19, 0.047, 0.0034\ \hbar^2/mr_0^2$. From the plot it is seen that as the binding energy vanishes (a) the response function seems to diverge, and (b) the peak position is shifted to lower and lower energies. This two features can be understood as E_T is the relevant energy scale for the process [16] and that the response is proportional to the radius of the system that grows as $E_T \longrightarrow 0$.

4 Sum-Rules

In this work as in [5] we focus on the effect of the atom–atom interaction strength on the response function. A convenient way to study the evolution of the response function with the trimer's binding energy is to explore

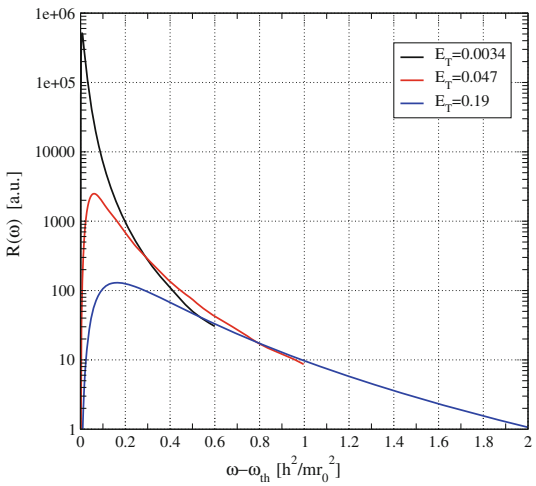

Fig. 1 (color online) The quadrupole response function of a bosonic trimer with Gauss interaction for $g = 2.60, 2.30, 2.15$ or $E_T = 0.19, 0.047, 0.0034\ \hbar^2/mr_0^2$

the dependence of the sum-rules

$$S_n \equiv \int_{\omega_{th}}^{\infty} d\omega\, \omega^n\, R(\omega) \tag{10}$$

on E_T. The sum rule S_n is the n'th moment of $R(\omega)$. For $n > 0$ it exists if $R(\omega) \longrightarrow 0$ faster than ω^{-n-1} as $\omega \longrightarrow \infty$. Due to the threshold law $R(\omega) \propto (\omega - \omega_{th})^2$ [17,18] the sum-rules are only defined for $n \geq -2$. The sum-rules can be expressed as an initial state observable utilizing the closure of the eigenstates of H. In particular we shall focus on the following sum-rules

$$\begin{aligned} S_1 &= \langle\Psi_0|[\hat{Q},[H,\hat{Q}]]|\Psi_0\rangle = \langle\Psi_0|\hat{Q}\,(H-E_0)\,\hat{Q}|\Psi_0\rangle \\ S_0 &= \langle\Psi_0|\hat{Q}\,\hat{Q}|\Psi_0\rangle \\ S_{-1} &= \langle\Psi_0|\hat{Q}\,\frac{1}{H-E_0}\,\hat{Q}|\Psi_0\rangle \end{aligned} \tag{11}$$

which can be easily calculated using the Lanczos method [19,20].

Naive scaling arguments leads us to expect that the energy dependence of the trimer radius follows the rule $r_T \sim 1/\sqrt{E_T}$. The quadrupole operator behaves as r^2 so $R(\omega) \sim r_T^4/E_T \sim 1/E_T^3$. It follows that the sum rules should fulfill the power law $S_n \sim 1/E_T^{2-n}$.

In Fig. 2 we present the calculated quadrupole sum-rules S_{-1}, S_0, S_1 for three different interaction models: a Gauss interaction, an attractive Yukawa force, and a square well. As expected, the sum-rules grow as $E_T \longrightarrow 0$, with the exception of the S_1 for the square well potential. It is seen that over a large energy range the sum-rules tends to follow a simple power law that exhibits a mild dependence on the specifics of the interaction. Fitting the Gauss interaction sum-rules to a simple power law $S_n = A_n E_T^{-q_n}$ we have found that $q_{-1} \approx 2.1$, $q_0 \approx 1.3$, and $q_1 \approx 0.5$. These results are in contrast with our naive expectations of $q_{-1} = 3$, $q_0 = 2$, and $q_1 = 1$. All these scaling laws are particularities of the 3-body system that (a) seems to depend only weakly on the potential model, and (b) differs dramatically from the corresponding two-body system.

5 Conclusions

Summing up, in this work we have studied the response of shallow trimers to a quadrupole excitation that takes the system from its ground state into the continuum. Using the HH expansion and the LIT methods we have calculated the response function and the sum-rules of the reaction. Over the energy range $0.003\ \hbar^2/mr_0^2 \leq E_T \leq 0.3\ \hbar^2/mr_0^2$ we have found, with one exception, that the sum-rules S_{-1}, S_0, S_1 grow with diminishing trimer binding energy. Repeating our calculations with three different potential models we have seen that as the

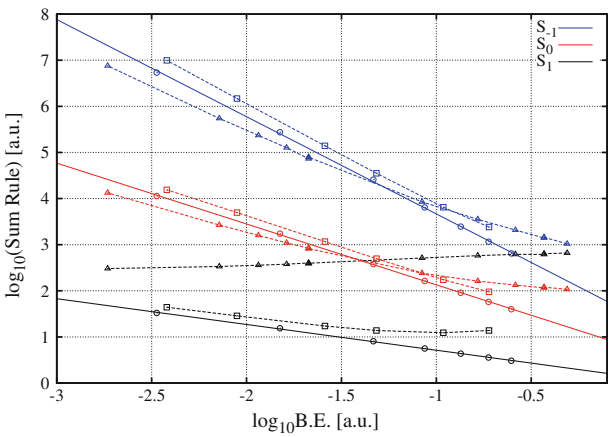

Fig. 2 (color online) Evolution of the sum rules S_{-1}, S_0, S_1 as a function of the trimer binding energy. The *upper/middle/lower* lines correspond to the $S_{-1}/S_0/S_1$ sum rules. For clarity, the sum-rules S_1 (S_{-1}) were divided (multiplied) by 10. The *circles* correspond to Gauss potential, the *squares* to Yukawa potential, and the *triangles* to square well potential

trimer becomes shallower the dependence of the sum-rules on E_T takes a path independent of the particular potential model used. Moreover, the exponents governing the sum-rules behavior in this region assume an unexpected values.

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