

# Chapter 139

## Efimov Physics Beyond Three Particles



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**Abstract** Efimov physics originally refers to a system of three particles. Here we review recent theoretical progress seeking for manifestations of Efimov physics in systems composed of more than three particles. Clusters of more than three bosons are tied to each Efimov trimer, but no independent Efimov physics exists there beyond three bosons. The case of a few heavy fermions interacting with a lighter atom is also considered, where the mass ratio of the constituent particles plays a significant role. Following Efimov's study of the  $(2 + 1)$  system, the  $(3 + 1)$  system was shown to have its own critical mass ratio to become Efimovian. We show that the  $(4 + 1)$  system becomes Efimovian at a mass ratio which is smaller than its sub-systems thresholds, giving a pure five-body Efimov effect. The  $(5 + 1)$  and  $(6 + 1)$  systems are also discussed, and we show the absence of 6— and 7—body Efimov physics there.

### 139.1 Introduction

Universal aspects of few-body systems with large scattering length have attracted attention in recent years from both theory and experiment perspectives [1]. Universality occurs when there is a large separation between the scale of the underlying physics and the scale of the phenomena observed. For example, if the inter-particle interaction range is much shorter than the spatial extent of the wave function governed by the scattering length  $a$ , most of the time the particles will be out of the potential range and therefore not sensitive to its fine details.

A few examples of relevant systems come to mind. In low-energy nuclear physics the scattering length of the singlet and triplet channels are  $a_s \approx -23.4$  fm and  $a_t \approx 5.42$  fm, while the long-range part of the nucleonic interaction, determined by the pion mass, is  $R \approx \hbar/m_\pi c \approx 1.4$  fm.

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Larger scale separation can be found in  ${}^4\text{He}$  atoms. Here the He-He scattering length is  $a \approx 90 \text{ \AA}$ , while the Van Der Waals interaction range is  $r_{vdW} \approx 5.4 \text{ \AA}$ .

Another interesting case is ultracold atoms near Feshbach resonance. Here the scattering length can be tuned to arbitrary value using, for example, external magnetic field,

$$a(B) = a_{bg} \left( 1 + \frac{\Delta}{B - B_0} \right).$$

A fascinating effect was predicted by Efimov [2] for three identical bosons with resonating interaction: the existence of infinite tower of bound trimers. For a recent review see [3].

In this paper, we study Efimov physics beyond three particles. We start with a short review of universal features and Efimov physics in three identical bosons and in the  $(2 + 1)$  system, which is a mixture of two identical fermions and distinguishable particle. Then we go beyond three particles and discuss the  $N > 3$  identical bosons system, as well as the  $(N + 1)$  systems with  $N \leq 6$ .

## 139.2 Methods and Results

In order to study universality, one would like to neglect the system-specific details and concentrate on the universal features.

To do so one could use the zero-range limit, i.e. eliminate the spatial extent of the potential while applying the Bethe-Peierls boundary condition when two particles touch each other,

$$\frac{\partial \log(r_{ij}\psi)}{\partial r_{ij}} \xrightarrow{r_{ij} \rightarrow 0} -\frac{1}{a} \quad (139.1)$$

where  $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$  is the distance between any pair of interacting particles.

### 139.2.1 The Universal Dimer

A trivial example for universality is the existence of universal dimer composed of two identical bosons of mass  $m$  for  $a > 0$ . Working in the center-of-mass frame and taking the zero-range limit, one has to solve the free Schrödinger equation for the relative coordinate  $r$  and to apply the Bethe-Peierls boundary condition (139.1) at zero, giving for the bound state  $\psi(r) \propto \exp(-r/a)/r$  corresponds to an energy of  $-1/ma^2$ , where here and thereafter  $\hbar$  is set to 1.

This prediction is indeed valid for the three examples mentioned above. The deuteron binding energy, 2.22 MeV, is fairly close to the universal prediction  $1/ma_i^2 \approx 1.4 \text{ MeV}$ . The  ${}^4\text{He}$  atoms dimer binding energy was measured recently to be about 1.76(15) mK [4], where the universal prediction is  $1/ma^2 \approx 1.48 \text{ mK}$ .

Since the next correction is of order of  $r_0/a$ , where  $r_0$  is the effective range, one would expect the universal prediction to be even better with ultracold atoms, and indeed this is the case [5].

### 139.2.2 Efimov Physics in Three Identical Bosons

Adding another identical boson, the situation is changed dramatically, as Efimov has shown [2].

To see that one can start from the Faddeev equation for zero-range potential, and then transform to hyperspherical coordinates. In the unitary limit  $a \rightarrow \infty$ , the energy is then determined by one-dimensional equation for the hyper-radius,  $\rho^2 \propto r_{12}^2 + r_{23}^2 + r_{13}^2$ ,

$$\left( -\frac{d^2}{d\rho^2} + \frac{s^2 - 1/4}{\rho^2} \right) R(\rho) = E R(\rho), \quad (139.2)$$

where  $s$  is the eigenvalue of the corresponding hyper-angular equation. Equation (139.2) has two interesting features. First, the effective three-body potential has long range  $\propto \rho^{-2}$ , in contrast to the zero-range two-body interaction we start with. Second, it exhibits scale invariance, therefore if  $R(\rho)$  is a solution with the corresponding energy  $E$ ,  $R(\lambda\rho)$  is also solution with the energy  $\lambda^2 E$  for arbitrary  $\lambda$ .

At small  $\rho$ ,  $E$  can be neglected, and the solution for (139.2) is  $R_{\pm}(\rho) \propto \rho^{1/2 \pm s}$ . The solution behavior is therefore determined by  $s$ . For  $s^2 > 0$  the solution can be set to  $R_+(\rho)$ , while for  $s^2 < 0$  the solution is a combination of two oscillating functions, whose relative phase is still needed to be fixed.

In the latter case the effective potential is attractive and one faces fall of a particle to the center of  $\rho^{-2}$  potential, i.e. the energy here is not bound from below [6].

Introducing three-body potential barrier at some finite  $\rho_0$  saves us from this collapse by setting the system ground state. The scale invariance is now broken into discrete scale invariance, with  $\lambda_n = e^{-\pi n/|s|}$ , and therefore the energies are quantized, giving infinite series of bound states with geometric-series spectrum  $E_n = E_0 e^{-2\pi n/|s|}$ . Here  $\rho_0$  is a three-body parameter, which sets the ground state energy and also fixes the relative phase of  $R_{\pm}$ .  $s$  is the scale factor which governs the scaling characters of the energies and the wave functions. For three identical boson it has the value  $s = 1.00624 i$ .

This prediction had to wait about four decades before its verification in ultracold gases experiments, where particle loss from the trap is a three-body process, showing a significant signal when new Efimov state is formed. Studies of loss features of ultracold  $^{133}\text{Cs}$  [7],  $^{39}\text{K}$  [8], and  $^7\text{Li}$  [9, 10] gases gave the first experimental verification for Efimov physics.

The existence of Efimov trimers in  $^4\text{He}$  atoms was predicted long ago [11], where due to the finite scattering length only two trimers should exist. Only recently the

excited trimer was seen experimentally [12], giving another verification for Efimov's prediction.

### 139.2.3 Efimov Physics Beyond Three Identical Bosons

Shortly after Efimov original paper, Amado and Greenwood have claimed that there is no Efimov effect for four or more particles [13].

However, three-body Efimov physics has a footprint in the four body system, were two tetramers are tied to each Efimov trimer [14], a prediction which was verified in ultracold atoms experiments [15]. The tetramer and trimer energies are correlated, similar to the correlation between triton and alpha binding energies known as the Tjon line [16, 17].

Larger clusters of identical bosons also exist, and their energies are correlated to the trimer energy, therefore not showing independent  $N$ -boson Efimov physics [18]. See, however [19].

### 139.2.4 Mass Imbalanced Fermionic Mixtures

Another system relevant to Efimov physics is a mixture of identical fermions and distinguishable particle with different mass.

Consider two heavy atoms with mass  $M$  interacting with a light atom with mass  $m$ . First, we apply the Born-Oppenheimer approximation, valid in the limit of  $M \gg m$  [20, 21]. Here the motion of the light particle is first solved assuming the heavy particles position is fixed at  $\pm R/2$ , giving

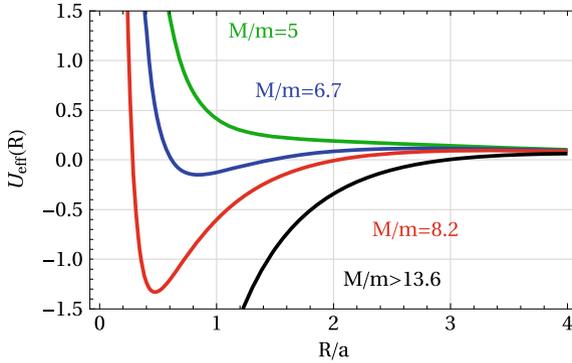
$$\psi_{\mathbf{R}}^{\pm}(\mathbf{r}) \propto \frac{e^{-\kappa_{\pm}(R)|\mathbf{r}-\mathbf{R}/2|}}{|\mathbf{r}-\mathbf{R}/2|} \pm \frac{e^{-\kappa_{\pm}(R)|\mathbf{r}+\mathbf{R}/2|}}{|\mathbf{r}+\mathbf{R}/2|} \quad (139.3)$$

where  $\mathbf{r}$  is the light particle position. Applying the boundary condition (139.1) gives

$$\kappa_{\pm}(R) \mp \frac{e^{-\kappa_{\pm}(R)R}}{R} = 1/a. \quad (139.4)$$

The energy of the light atom  $\epsilon_{\pm}(R) = -\kappa_{\pm}^2(R)/2m$  is then considered as an effective potential between the heavy atoms. The minus state corresponds to repulsive effective potential, while the plus state induces attractive potential,

$$\epsilon_{\pm}(R) \approx \begin{cases} -\frac{0.16}{mR^2} & R/a \ll 1 \\ -\frac{1}{2m} \left( \frac{1}{a^2} + \frac{\exp(-R/a)}{aR} \right) & R/a \gg 1 \end{cases} \quad (139.5)$$



**Fig. 139.1** The effective potential between heavy particles induced by the light particle as a function of their distance, for different mass ratios in the Born-Oppenheimer approximation. For a small mass ratio, the effective interaction is repulsive (green). As the mass ratio increases  $p$ -wave resonance occurs (blue), then the potential well becomes deep enough to support bound trimer (red), and finally, the attraction wins and Efimov states emerge (black)

The heavy-particles equation for  $R \ll a$  is therefore identical to (139.2), replacing  $\rho$  by  $R$ . Here  $s^2 = l(l + 1) - 0.16M/m + 1/4$  for angular momentum  $l$ .

In the bosonic case, the ground state has  $l = 0$ , giving purely attractive  $-1/R^2$  effective potential, and therefore Efimov physics.

In the fermionic case, the permutation symmetry dictates odd angular momentum and the ground state has  $l = 1$ . The centrifugal barrier  $l(l + 1)/MR^2$  therefore competes with the  $-1/mR^2$  attraction, where the competition is governed by the mass ratio. Figure 139.1 shows the effective potential for various mass ratios  $M/m$ .

This simple picture indeed catches the physics here. For a small mass ratio, the effective potential is repulsive, and no bound trimer exists. As the mass ratio increases, the potential becomes more attractive, and a  $p$ -wave resonance occurs. Indeed this resonance was found in ultracold  $^{40}\text{K}$ - $^6\text{Li}$  mixture [22]. For larger mass ratio the potential well is deep enough to support a universal  $1^-$  bound state [23]. For even larger mass ratio the system becomes Efimovian [24].

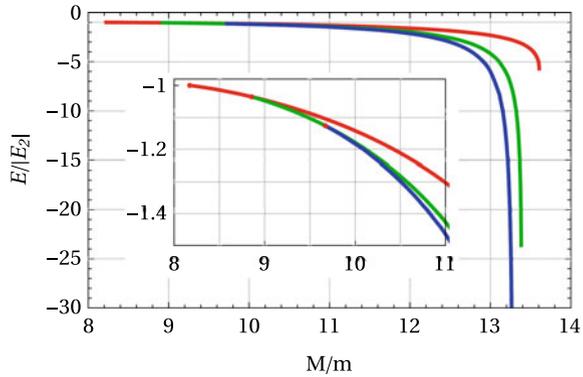
To proceed beyond this approximation, it is convenient to follow Skorniakov and Ter-Martirosian formalism [25, 26]. Here instead of solving the Schrödinger equation, one utilizes the zero-range potential to get an integral equation.

For the  $(N + 1)$  case, the STM equation in momentum space is [27],

$$\frac{1}{4\pi} \left( \frac{1}{a} - \kappa \right) F(\mathbf{q}_1, \dots, \mathbf{q}_{N-1}) = \int \frac{d^3 q_N}{(2\pi)^3} \frac{\sum_{i=1}^{N-1} F(\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{q}_N, \mathbf{q}_{i+1}, \dots, \mathbf{q}_{N-1})}{-2\mu E + \frac{\mu}{M} \sum_{i=1}^N q_i^2 + \frac{\mu}{m} \left( \sum_{i=1}^N \mathbf{q}_i \right)^2}, \tag{139.6}$$

where  $\mu = Mm/(M + m)$  is the reduced mass. The function  $F(\mathbf{q}_1, \dots, \mathbf{q}_{N-1})$  can be considered as the relative wave function of  $N - 1$  heavy atoms with momenta  $\mathbf{q}_1, \dots, \mathbf{q}_{N-1}$  and a heavy-light pair, the momentum of which equals  $-\sum_{i=1}^{N-1} \mathbf{q}_i$ .

**Fig. 139.2** The energies of the universal  $(N + 1)$  states in units of the dimer binding energy, as a function of the mass ratio. Shown are results for the  $1^- (2 + 1)$  state (red), the  $1^+ (3 + 1)$  state (green), and the  $0^- (4 + 1)$  state (blue). The inset shows zoom-in on the thresholds region. Adapted from [27]



Here

$$\kappa = \sqrt{-2\mu E + \frac{\mu}{M} \sum_{i=1}^{N-1} q_i^2 + \frac{\mu}{M+m} \left( \sum_{i=1}^{N-1} \mathbf{q}_i \right)^2}. \tag{139.7}$$

For a specific value of total angular momentum and parity  $L^\pi$ , the equation can be further simplified. For the  $(2 + 1)$  case the relevant symmetry for both universal trimer and Efimov state is  $1^-$ . The function  $F$  therefore takes the form  $F(\mathbf{q}) = f(q)\hat{\mathbf{z}} \cdot \hat{\mathbf{q}}$ , leaving one-dimensional equation which can be easily solved for  $E$ , as can be seen in Fig. 139.2. The universal trimer binding threshold could thus be obtained,  $M/m = 8.173$ , in agreement with the results obtained in [23] in the hyperspherical formalism.

To find the mass ratio where the system becomes Efimovian one would like to calculate the scale factor  $s$  which approaches zero at that point. For that, one can calculate the large- $q$  asymptote of  $f$ , which has the form

$$f(q) \propto q^{-2-s}. \tag{139.8}$$

Solving (139.6) for  $f$  and fitting the results to extract  $s$ , the Efimov threshold can be found at  $M/m = 13.607$ , in agreement with the result of [24].

An interesting alternative, which may be more suitable for larger systems, is to utilize the mapping between the free-space system with finite  $a$  and the trapped system at unitarity [28, 29], whose energy is

$$E = \hbar\omega(s + 2n + 1) \tag{139.9}$$

where  $\omega$  is the trap frequency,  $s$  is the same scale factor and  $n$  counts hyper-radial excitations. Hence, one can extract the scale factor  $s$  from the trapped energies.

Now that we have built our toolbox, we can face an interesting question: how many heavy fermions can be bound by a single light atom?

For the  $(3 + 1)$  case the relevant symmetry is  $1^+$ , and  $F$  takes the form  $F(\mathbf{q}_1, \mathbf{q}_2) = f(q_1, q_2, \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_2) \hat{\mathbf{z}} \cdot \hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_2$ , leaving a three-dimensional integral equation which can be solved in deterministic method. It was shown that a universal  $1^+$  tetramer exists for a mass ratio  $M/m \gtrsim 9.5$  [30]. Moreover, an  $1^+$  Efimov states exist above  $M/m > 13.384$  [31].

Adding another fermion, the relevant symmetry is  $0^-$ , therefore  $F(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = f(q_1, q_2, q_3, \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_2, \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_3, \hat{\mathbf{q}}_2 \cdot \hat{\mathbf{q}}_3) \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_3$ , but the resulting six-dimensional integral equation is too hard to be solved with conventional method. Hence a novel method, which we call the STM-DMC method, is introduced [27, 32], where  $f$  is treated as density probability function for so-called walkers, whose stochastic dynamics is governed in such a way that their detailed-balance condition is (139.6). Given  $E$ ,  $a$  is than changed in each iteration to keep the walkers' number constant.

Using this method, (139.6) can be solved for the  $(2 + 1)$ ,  $(3 + 1)$ , and  $(4 + 1)$  cases, getting both energies and scale factors. Figure 139.2 shows the energies of the universal states in these systems, its inset focuses on their thresholds. Known results are reproduced, i.e. the thresholds for the  $(2 + 1)$  universal trimer and Efimov states. Moreover, we can locate better the threshold of the  $(3 + 1)$  universal tetramer to be 8.862 [27], and confirm the threshold for four-body Efimov states [31].

We can now explore the *terra incognita*  $(4 + 1)$  system. Here we find a  $0^-$  universal pentamer as well as  $0^-$  Efimov states. In Fig. 139.2 we also plot the energies for the universal  $0^-$  pentamer, showing it is bound for mass ratio above 9.672. In Fig. 139.3 we show the scale factor for this system, showing Efimov  $0^-$  states emerge here above  $M/m = 13.279$  [27].

The different threshold for the  $(N + 1)$  states are summarized in Table 139.1.

We see that the  $(N + 1)$  systems with  $N = 2, 3$  and  $4$  exhibit similar behavior, showing pure  $N + 1$ -body Efimov physics. Does this pattern continue for  $N \geq 5$ ?

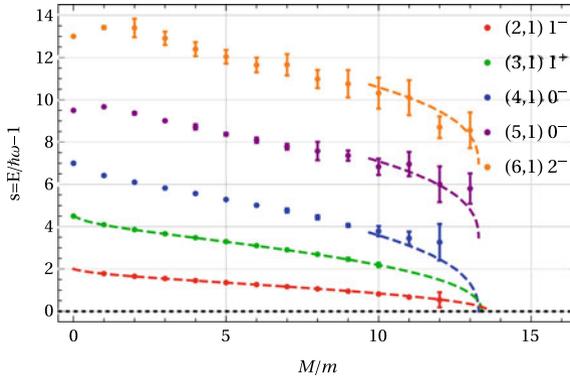
The relevant symmetry for the  $(5 + 1)$  ground state is  $0^-$ , signaling that the additional fermion populates an excited  $s$ -shell, which has a radial node. This causes the stochastic method to suffer from a sign problem. Hence we choose here another approach, which is to extract the scale factor from the energies in a harmonic trap.

These energies were calculated using a Gaussian potential with a finite range  $R_0$ ,

$$V(r) = -V_0 e^{-\frac{r^2}{2R_0^2}} \tag{139.10}$$

**Table 139.1** The thresholds for universal and Efimov states in the  $(N + 1)$  systems

System	$L^\pi$	Universal state	Efimov state
2+1	$1^-$	8.173	13.607
3+1	$1^+$	8.862	13.384
4+1	$0^-$	9.672	13.279
5+1	$0^-$	?	–
6+1	$2^-$	?	–



**Fig. 139.3** The scale factor as extracted from the energies of the  $(N + 1)$  state in a harmonic trap at unitarity, as a function of the mass ratio. The Efimov limit corresponds here to  $s = 0$ . Circles stand for the results extrapolated from finite-range gaussian potential to the zero-range limit. Dashed curves are for results acquired directly in the zero-range limit, by solving the STM equation on a grid (for  $(2 + 1)$  and  $(3 + 1)$  systems) or with stochastic method (for the  $(4 + 1)$  case). The Dashed curves for the  $(5 + 1)$  and  $(6 + 1)$  cases taken from the  $(4 + 1)$  case with appropriate shift, showing no Efimov effect exists in these cases

where the results are extrapolated to the zero-range limit  $R_0 \rightarrow 0$ . The  $(N + 1)$ -body Schrödinger equation is solved using a basis of correlated gaussians, chosen by the stochastic variational method [33]. Using (139.9), the scale factor is then extracted [34].

In Fig. 139.3 we show the  $(N + 1)$  system scale factor for  $N \leq 6$ . Efimov threshold here is signaled by  $s = 0$ . Results obtained with other methods are also shown. Indeed the scale factors for  $N = 2, 3$  and  $4$  hit zero at the Efimovian threshold. However, no sign for Efimov physics is found in the  $N = 5$  and  $6$  systems.

### 139.3 Conclusion

Efimov physics beyond three particles is studied here. For identical bosons, no independent Efimov effect exists beyond three particles, although bosonic clusters are tied to each Efimov trimer. For the  $(N + 1)$  case of  $N$  identical fermions interact with distinguishable particle, Efimov states occur for mass ratio exceeds the relevant threshold for the  $(2 + 1)$ ,  $(3 + 1)$ , and  $(4 + 1)$  systems. However, no Efimov state exists for the  $(5 + 1)$  and  $(6 + 1)$  systems.

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