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Abstract

We derive an integral equation describing N two-dimensional bosons with zero-range interactions and solve it for the ground state energy B_N by applying a stochastic diffusion Monte Carlo scheme for up to 26 particles. We confirm and go beyond the scaling $B_N \propto 8.567^N$ predicted by Hammer and Son (2004 *Phys. Rev. Lett.* **93** 250408) in the large- N limit.

1. Introduction

Non-relativistic two-dimensional bosons with zero-range interactions is one of the simplest and therefore fundamentally important models in modern science. It has essentially only one parameter—the number of particles N . The other parameter, the dimer binding energy B_2 , can always be set to unity by rescaling the coordinates. This model neighbors the model of one-dimensional attractive bosons, which is exactly solvable [1], and the model of three-dimensional bosons, the zero-range formulation of which is ill defined since its ground-state energy is not bound from below because of the Thomas collapse [2].

Bruch and Tjon [3] have shown that no Thomas collapse occurs in two-dimensions. They have found two bound states (ground and first excited) of three two-dimensional bosons. The energies of these states have been calculated over the years by various few-body methods with increasing accuracy [3–9]. In particular, the ground state trimer energy is known to be $B_3 = 16.5226874B_2$ [7, 8].

Calculations for the four-boson system have been done using finite-range potentials, but significant range corrections make the convergence to the zero-range limit problematic [10–12]. The current best estimate of the tetramer energy $B_4 = 197.3(1)B_2$ is obtained in the zero-range limit by using the effective field theory [13]. Note the fast increase of the binding energy with N .

Hammer and Son [7] have studied N -body droplets in the limit of large N and found that their energies increase exponentially with N ,

$$B_N/B_2 \propto r^N = 8.567^N, \quad (1)$$

and sizes decrease proportionally to $r^{-N/2} = 0.3417^N$. This result relies on the idea of asymptotic freedom and can be obtained in the mean-field approximation by introducing a logarithmic running of the coupling constant as a function of the droplet size. The proportionality coefficient in equation (1) is currently unknown and there is no theory which systematically accounts for finite- N corrections starting from the large- N limit. On the other hand, numerical efforts to approach the large- N regime from the few-body side have been impeded by the rapid growth of the configurational space and the necessity of an extrapolating procedure to remove finite-range effects. In particular, finite-range calculations of Blume [14] carried out for $N \leq 7$ were not conclusive enough to confirm or disprove equation (1). Lee [15] calculated energies of up to 10 particles by using lattice effective field theory and observed convergence towards $B_N/B_{N-1} = r$ although with a significant error bar, $r = 8.3(6)$.

In this paper we derive a many-body integral equation to describe the system directly in the zero-range limit and solve it by using our recently developed stochastic algorithm [16]. We tabulate B_N for all $N \leq 26$ and claim the values $c_1 = -2.06(4)$ and $c_2 = -8(2)$ for the first terms in the large- N expansion

$$\ln(B_N r^{-N}/B_2) = c_1 + c_2/N + \dots \quad (2)$$

2. Integral equation for N bosons

Consider the system of N two-dimensional bosons interacting with each other via a zero-range potential characterized by the two-body binding energy $B_2 = \kappa_0^2$ (we set $\hbar = m = 1$). The N -body wave function at the energy E satisfies the Schrödinger equation

$$\left(-\sum_{i=1}^N \nabla_i^2/2 - E \right) \Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = 0 \quad (3)$$

supplied with the Bethe–Peierls boundary conditions for each pair i, j in the limit $\mathbf{r}_i \rightarrow \mathbf{r}_j$

$$\Psi \propto \ln(\kappa_0 |\mathbf{r}_i - \mathbf{r}_j| e^\gamma/2). \quad (4)$$

The condition (4) substitutes the interaction potential by relating the logarithmically diverging part of the wave function and its regular (constant) part, terms proportional to higher powers of $|\mathbf{r}_i - \mathbf{r}_j|$ being omitted.

We then introduce the decomposition

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} \phi(\mathbf{r}_1, \dots, \mathbf{r}_{i-1}, \mathbf{r}_{N-1}, \mathbf{r}_{i+1}, \dots, \mathbf{r}_{j-1}, \mathbf{r}_N, \mathbf{r}_{j+1}, \dots, \mathbf{r}_{N-2}, \mathbf{r}_i, \mathbf{r}_j), \quad (5)$$

where $\phi(\mathbf{r}_1, \dots, \mathbf{r}_{N-2}; \mathbf{r}_{N-1}, \mathbf{r}_N)$ satisfies equation (3), but, in contrast to Ψ , is singular only for $\mathbf{r}_{N-1} \rightarrow \mathbf{r}_N$. So far we do not impose any particular boundary condition on ϕ in this limit, but require that $\phi(\mathbf{r}_1, \dots, \mathbf{r}_{N-2}; \mathbf{r}_{N-1}, \mathbf{r}_N)$ be symmetric in its first $N - 2$ arguments. For $E < 0$ (in this paper we only consider this case) the general form of ϕ satisfying the above constraints is (we introduce $M = N - 2$)

$$\begin{aligned} \phi(\mathbf{r}_1, \dots, \mathbf{r}_M; \mathbf{r}_{N-1}, \mathbf{r}_N) &= \int \frac{d^2 q_1 \dots d^2 q_M}{(2\pi)^{2M}} F(\mathbf{q}_1, \dots, \mathbf{q}_M) e^{i \sum_{j=1}^M \mathbf{q}_j \mathbf{r}_j - i (\sum_{j=1}^M \mathbf{q}_j) (\mathbf{r}_{N-1} + \mathbf{r}_N)/2} \\ &\times \frac{K_0[\kappa(\mathbf{q}_1, \dots, \mathbf{q}_M) |\mathbf{r}_{N-1} - \mathbf{r}_N|]}{2\pi}, \end{aligned} \quad (6)$$

where F is totally symmetric, K_0 is the decaying Bessel function, and $\kappa(\mathbf{q}_1, \dots, \mathbf{q}_M) = \sqrt{-E + \sum_{j=1}^M q_j^2/2 + (\sum_{j=1}^M \mathbf{q}_j)^2/4}$. Equation (6) corresponds to a linear combination of states of M non-interacting bosons with momenta $\mathbf{q}_1, \dots, \mathbf{q}_M$ and a molecule (described by the wave function K_0) with the binding energy κ^2 . We have chosen to work in the center-of-mass frame where the molecule momentum equals $-\sum_{j=1}^M \mathbf{q}_j$.

The wave function (5) with ϕ defined by equation (6) is symmetric and satisfies equation (3) for arbitrary F . This freedom is removed by the boundary condition (4) which, because of the symmetry, can be applied for a single pair, say, $N - 1, N$. Namely, by considering the limit $\mathbf{r}_N \rightarrow \mathbf{r}_{N-1}$ we observe that the logarithmic divergence in the sum of equation (5) comes only from the component with $i = N - 1, j = N$, given explicitly by equation (6). The corresponding logarithmic and regular contributions are obtained by substituting in equation (6) the asymptotic form $K_0(x) = -\ln(xe^\gamma/2)$. All other components in equation (5) contribute only to the regular part of Ψ in the limit $\mathbf{r}_{N-1} \rightarrow \mathbf{r}_N$. Denoting this latter contribution by $\mathcal{F}(\mathbf{r}_1, \dots, \mathbf{r}_{N-1})$ the boundary condition (4) leads to the equation

$$\int \frac{d^2 q_1 \dots d^2 q_M}{(2\pi)^{2M}} F(\mathbf{q}_1, \dots, \mathbf{q}_M) e^{i \sum_{j=1}^M \mathbf{q}_j \mathbf{r}_j - i (\sum_{j=1}^M \mathbf{q}_j) \mathbf{r}_{N-1}} \frac{1}{2\pi} \ln \frac{\kappa(\mathbf{q}_1, \dots, \mathbf{q}_M)}{\kappa_0} = \mathcal{F}(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}), \quad (7)$$

which, after Fourier transforming, explicitly reads

$$\begin{aligned} &\frac{F(\mathbf{q}_1, \dots, \mathbf{q}_M)}{2\pi} \ln \frac{\kappa(\mathbf{q}_1, \dots, \mathbf{q}_M)}{\kappa_0} \\ &= 2 \sum_{i=1}^M \int \frac{d^2 p_i}{(2\pi)^2} \frac{F(\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{p}_i, \mathbf{q}_{i+1}, \dots, \mathbf{q}_M)}{\kappa^2(\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{p}_i, \mathbf{q}_{i+1}, \dots, \mathbf{q}_M) + \left(\mathbf{q}_i + \frac{\mathbf{q}_1 + \dots + \mathbf{q}_{i-1} + \mathbf{p}_i + \mathbf{q}_{i+1} + \dots + \mathbf{q}_M}{2} \right)^2} \\ &+ \sum_{i < j} \int \frac{d^2 p_i d^2 p_j}{(2\pi)^4} \frac{F(\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{p}_i, \mathbf{q}_{i+1}, \dots, \mathbf{q}_{j-1}, \mathbf{p}_j, \mathbf{q}_{j+1}, \dots, \mathbf{q}_M)}{\kappa^2(\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{p}_i, \mathbf{q}_{i+1}, \dots, \mathbf{q}_{j-1}, \mathbf{p}_j, \mathbf{q}_{j+1}, \dots, \mathbf{q}_M) + (\mathbf{q}_i - \mathbf{q}_j)^2/4} (2\pi)^2 \\ &\times \delta \left(\mathbf{p}_i + \mathbf{p}_j + \sum_{k=1}^M \mathbf{q}_k \right). \end{aligned} \quad (8)$$

Equation (8) is the two-dimensional N -body analog of the three-dimensional three-body equation derived by Skorniakov and Ter-Martirosian [17]. The formulation of the N -body problem based on equation (8) provides a few advantages compared to the one based on equations (3) and (4). Very important for us is the fact that equation (8) incorporates the zero-range boundary condition and thus requires no zero-range extrapolation procedure. Another

feature is that this formulation reduces the configurational space of the problem by one set of single-particle coordinates giving obvious advantages for small N (3 or 4) where equation (8) can be solved by deterministic methods. However, this point is not essential for us in this work since we aim at significantly larger N . In section 3 we develop and apply to equation (8) a stochastic method based on the diffusion Monte Carlo (DMC) technique.

3. Stochastic method

In order to solve equation (8) we adopt the stochastic method developed by us for calculating binding energies and other characteristics of clusters consisting of up to four identical fermions interacting resonantly with another atom of different mass [16]. The idea of the method is particularly transparent in the currently discussed bosonic case where the function $F(\mathbf{q}_1, \dots, \mathbf{q}_M)$ is symmetric and can be assumed positive for the ground state. The solution procedure is then based on a stochastic process (diffusion of walkers) in the $2M$ -dimensional space $\{\mathbf{q}_1, \dots, \mathbf{q}_M\}$ for which equation (8) is the detailed-balance condition and F is (proportional to) the probability density distribution. We now briefly describe the procedure in the bosonic case (for more technical details applicable in general see [16] and its supplemental material).

We start with introducing a new function

$$f(\mathbf{q}_1, \dots, \mathbf{q}_M) = \frac{F(\mathbf{q}_1, \dots, \mathbf{q}_M)}{2\pi \prod_{i=1}^M (D + q_i^2)} \ln \frac{\kappa(\mathbf{q}_1, \dots, \mathbf{q}_M)}{\kappa_0}, \quad (9)$$

which will actually be the probability density distribution of walkers. The proportionality factor relating F and f in equation (9) is chosen such that f be normalizable, as this is not necessarily the case for F . Another requirement for the form of this proportionality factor is to make the sampling and branching (see below) tasks analytical and, therefore, fast. The parameter D shifts the typical momentum of the distribution and influences the convergence rate, but not the final result. Rewriting equation (8) in terms of f we obtain the integral equation

$$f = \sum_i \int d^2 p_i K(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i) f(\mathbf{q}_1, \dots, \mathbf{p}_i, \dots, \mathbf{q}_M) + \sum_{i < j} \int d^2 p_i d^2 p_j L(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i, \mathbf{p}_j) f(\mathbf{q}_1, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, \mathbf{q}_M), \quad (10)$$

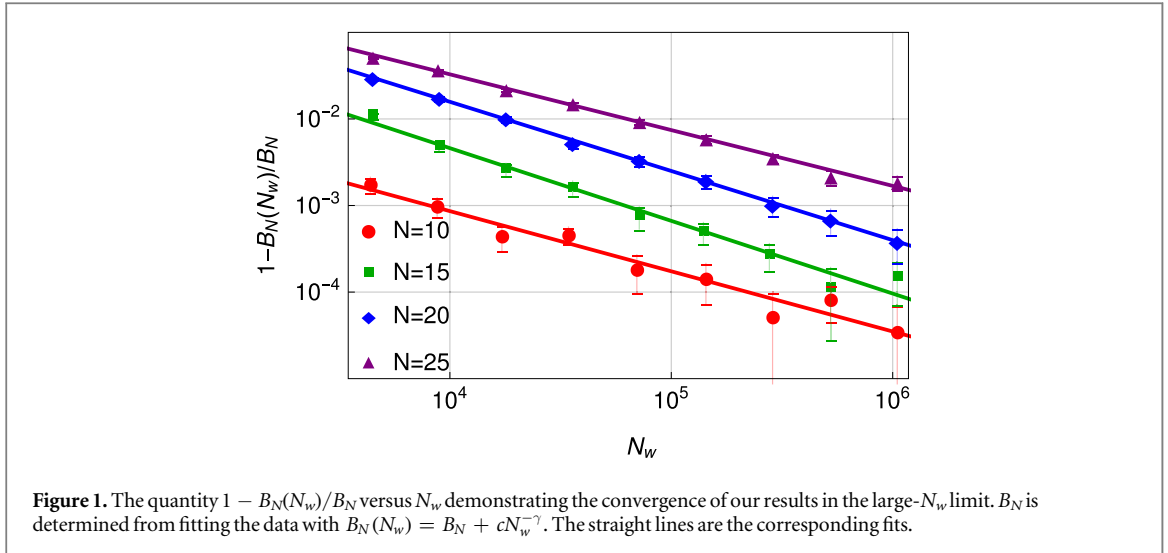
where K and L are abbreviations for bulky but straightforward expressions which we do not write explicitly; K and L correspond, respectively, to the first and second integral on the right-hand side of equation (8).

We then create an initial distribution of N_w walkers in the space $\{\mathbf{q}_1, \dots, \mathbf{q}_M\}$ and organize an iterative stochastic process of branching and moving them. Namely, at each iteration walkers are subject to two types of operations:

- *Single-particle moves*: A walker with coordinates $\mathbf{q}_1, \dots, \mathbf{p}_i, \dots, \mathbf{q}_M$ is branched on average $W_i = \int d^2 q_i K(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i)$ times and the i th coordinate of each of the descendants is moved from \mathbf{p}_i to \mathbf{q}_i according to the normalized distribution density function $K(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i)/W_i$. Single-particle moves of this type are similar to the ones we dealt with in the fermionic case [16]. Here we have to calculate the normalization integral and sample a product of two two-dimensional Laplacians, $K \propto (D + q_i^2)^{-1} [\kappa^2 + (\mathbf{v} + \mathbf{q}_i)^2]^{-1}$, where \mathbf{v} is half of the walker's total initial momentum. Both tasks are analytical and very fast. This move is repeated over all walkers and over all coordinates.
- *Pair moves*: A walker with coordinates $\mathbf{q}_1, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, \mathbf{q}_M$ is branched $W_{i,j} = \int d^2 q_i d^2 q_j L(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i, \mathbf{p}_j)$ times and coordinates i and j are moved, respectively, from \mathbf{p}_i and \mathbf{p}_j to \mathbf{q}_i and \mathbf{q}_j where the latter are drawn from $L(\mathbf{q}_1, \dots, \mathbf{q}_M; \mathbf{p}_i, \mathbf{p}_j)/W_{i,j}$. In fact, due to the delta function in the second term on the right-hand side of equation (8) and, therefore, in the function L , the value $\mathbf{q}_i + \mathbf{q}_j = -\mathbf{q}_1 - \dots - \mathbf{p}_i - \dots - \mathbf{p}_j - \dots - \mathbf{q}_M$ and only the difference $\mathbf{q} = \mathbf{q}_i - \mathbf{q}_j$ has to be sampled. Explicitly, the distribution of \mathbf{q} is governed by a product of three Laplacians, $L \propto [D + (\mathbf{v} + \mathbf{q})^2]^{-1} [D + (\mathbf{v} - \mathbf{q})^2]^{-1} (\kappa^2 + q^2)^{-1}$. The normalization integral is analytical and one can find a very fast rejection-based sampling procedure. This move is repeated over all walkers and over all pairs of coordinates.

We emphasize that fast branching and sampling is essential for the efficient implementation of this method for large $N \sim M$, particularly, since the number of pair moves grows proportionally to M^2 . In this context we note that the problem of several ideal bosons resonantly interacting with an impurity is technically easier as there are no pair terms in the corresponding integral equation.

Up to the noise related to the probabilistic branching procedure the walker population at a next iteration is given by $\sum_{i=1}^M W_i + \sum_{i < j} W_{i,j}$ summed over all walkers. If this number is systematically larger or smaller than the population at the previous iteration, we can correct the value of E , which implicitly enters in the weights W through the kernels K and L . Alternatively, and this is what we do in practice, we fix $E = -1$ and use κ_0 as the



parameter that controls the walker population and keeps it close to the initial N_w . More precisely, we adjust this parameter during every iteration in such a way that the sum of all branching weights of all walkers equals N_w .

Assuming that a steady equilibrium distribution of walkers is reached and that κ_0 is fixed at its exact value, it is easy to show that the detailed-balance equation for the above iterative scheme is exactly equation (10). With the population correction implemented, we do reach a steady state, but κ_0 fluctuates around a certain average value $\kappa_0(N_w)$. The amplitude of these fluctuations decreases with N_w and $\kappa_0(N_w)$ converges to the exact value in the limit $N_w \rightarrow \infty$ (see section 4).

4. Results

The free parameters of our algorithm are D and N_w . We find phenomenologically that $2 \lesssim D \lesssim 6$ maximizes the convergence rate. Results presented below are obtained with $D = 4$.

For a given walker population N_w , after thermalization and accumulation of statistics, our algorithm gives the quantity $\kappa_0(N_w)$ at fixed $E = -1$ with a certain statistical uncertainty. We then translate it into the N -body energy estimate $B_N(N_w)/B_2 = 1/\kappa_0^2(N_w)$. In order to verify the convergence of $B_N(N_w)$ towards B_N for a given N we run our code with various values of N_w and fit the obtained function $B_N(N_w)$ to the form $B_N(N_w) = B_N + cN_w^{-\gamma}$, where B_N , c , and γ are fitting parameters. The best-fit values of the exponent γ for different N all belong to the interval $(0.5, 1)$. In figure 1 we show the quantity $1 - B_N(N_w)/B_N$ for a few representative values of N together with the corresponding fits. The energies obtained by using this extrapolating procedure are reported in table 1. The error bars represent the sum of the statistical uncertainty and systematic uncertainty which we define as $|B_N(N_{w,\max}) - B_N|$, i.e., the deviation of the energy calculated with the largest walker pool from the extrapolated value. We should note here that the intrinsic time scale of our stochastic process is related only to the dimension of the configurational space and we benefit from relatively short thermalization and correlation times, at most a few hundreds of iterations for largest N . Thus, at approximately the same CPU cost we have a large room for increasing N_w and decreasing the number of iterations, which minimizes the finite- N_w systematic error while keeping the statistical uncertainty constant.

We can benchmark our results for $N = 3$ and 4 against the most accurate previous calculations. For the trimer we have $B_3/B_2 = 16.5225(2)$, in good agreement with $B_3/B_2 = 16.522688(1)$ [7] and $B_3/B_2 = 16.5226874$ [8]. Our tetramer result $B_4/B_2 = 197.20(1)$ is more precise than the previously known value $B_4/B_2 = 197.3(1)$ [13].

Our data for larger N are consistent with the value $r = 8.567$ [7]⁵ and with the assumption that the function $\ln(B_N r^{-N}/B_2)$ can be expanded in powers of $1/N$. In figure 2 we plot this function together with linear, quadratic, and cubic polynomial fits $\sum_i c_i N^{1-i}$ based, respectively, on the data for $N \geq 22$, $N \geq 16$, and $N \geq 10$. The fitting parameters $\{c_1, c_2, \dots\}$ ⁶ are, respectively, $\{-2.1, -6.01\}$, $\{-2.06, -7.88, 20.45\}$, and $\{-2.06, -7.94, 27.2, -77\}$. Based on these findings we claim $c_1 = -2.06(4)$ and $c_2 = -8(2)$.

⁵ We agree with $r = 8.567$ within the uncertainty of 0.3%, which equals the relative error of B_N/B_{N-1} for a few largest values of N in table 1.

⁶ Here $r = 8.567$ is not a fitting parameter. By fixing r we determine c_i 's with a significantly higher precision.

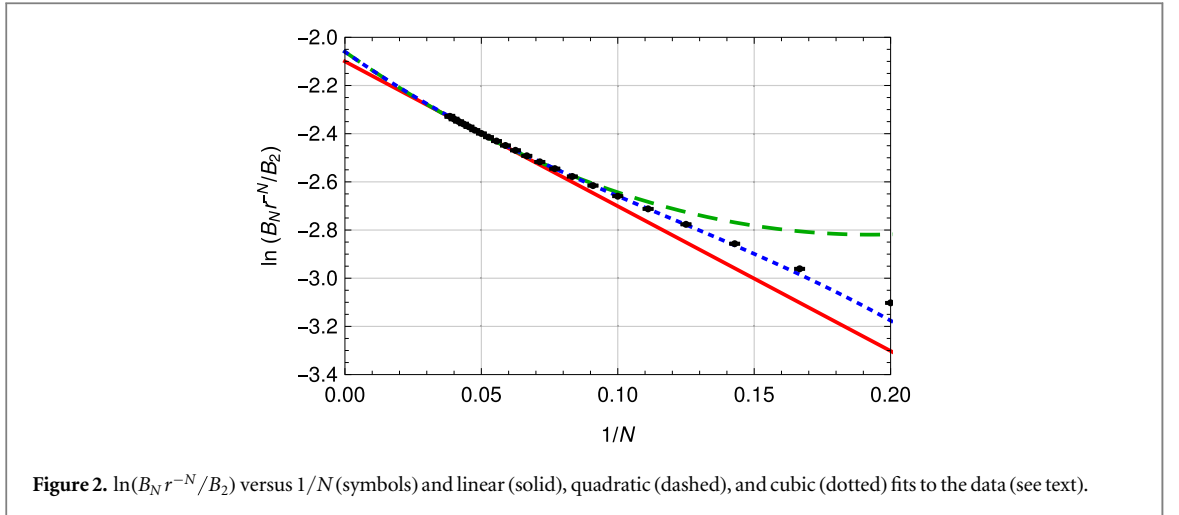


Table 1. The ground-state energy of the N boson droplet in units of the dimer energy. Previous calculations for $N = 3, 4$ give $B_3/B_2 = 1.6522688(1) \times 10^1$ [7] and $B_4/B_2 = 1.973(1) \times 10^2$ [13].

N	B_N/B_2	N	B_N/B_2
3	$1.65225(2) \times 10^1$	15	$8.135(2) \times 10^{12}$
4	$1.9720(1) \times 10^2$	16	$7.129(4) \times 10^{13}$
5	$2.0745(1) \times 10^3$	17	$6.232(2) \times 10^{14}$
6	$2.0471(1) \times 10^4$	18	$5.438(3) \times 10^{15}$
7	$1.9462(1) \times 10^5$	19	$4.734(2) \times 10^{16}$
8	$1.8070(1) \times 10^6$	20	$4.119(2) \times 10^{17}$
9	$1.6508(4) \times 10^7$	21	$3.577(2) \times 10^{18}$
10	$1.4905(2) \times 10^8$	22	$3.108(4) \times 10^{19}$
11	$1.3345(2) \times 10^9$	23	$2.694(5) \times 10^{20}$
12	$1.1873(4) \times 10^{10}$	24	$2.332(4) \times 10^{21}$
13	$1.0508(3) \times 10^{11}$	25	$2.018(4) \times 10^{22}$
14	$9.2596(9) \times 10^{11}$	26	$1.748(4) \times 10^{23}$

5. Summary and outlook

In this paper we have calculated the energies of two-dimensional bosonic droplets with strictly zero-range interactions for $N \leq 26$. Our results are consistent with the proportionality factor $r = 8.567$ in equation (1) and we estimate the leading finite- N corrections; we claim $B_N/B_2 \approx r^N \exp(c_1 + c_2/N)$ where $c_1 = -2.06(4)$ and $c_2 = -8(2)$. That c_2 turns out to be rather large indicates significant finite- N effects for N as large as 10. Our results should be useful as a reference point for developing a theory beyond [7] expected, in particular, to describe the droplet dynamics and excitations. The calculation of excited states with our current method is challenging since the solution in this case is a sign-changing function.

The exponential scaling of the size and energy raises obvious questions concerning the feasibility of observing such droplets. In the quasi-two-dimensional geometry for small ratio of the scattering length $a < 0$ to the confinement oscillator length l the dimer size is exponentially large, $\propto l \exp(\sqrt{\pi/2} l/|a|)$ [18]. The N -body droplet size, in this case proportional to $l \exp(\sqrt{\pi/2} l/|a| - N \ln \sqrt{r})$, should be much larger than l and smaller than the waist of the laser beams used for the quasi-two-dimensional confinement. This defines the window of parameters where the droplet is observable; one can see that N is not necessarily very small. Given extremely low values of a routinely produced in experiments (see, for example, [19–21]), it is realistic to observe droplets of a few tens of particles. The preparation of a droplet can be realized, for example, by simultaneously switching off of the radial confinement and sweeping the scattering length from a positive to negative value near a zero crossing.

Our method can be generalized to include the leading-order effective-range correction to the N -body energy (as has been done in three-dimensions [16]). One can then estimate the energy correction due to finite-range effects associated with the finite width of the cloud in the quasi-two-dimensional case. In addition, it should provide a link to few-body calculations based on realistic finite-range potentials. Finally, we mention that our method can be extended to bosonic mixtures with different masses.

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References

- [1] McGuire J B 1964 Study of exactly soluble one-dimensional N -body problems *J. Math. Phys.* **5** 622
- [2] Thomas L H 1935 The interaction between a neutron and a proton and the structure of H^3 *Phys. Rev.* **47** 903
- [3] Bruch L W and Tjon J A 1979 Binding of three identical bosons in two dimensions *Phys. Rev. A* **19** 425
- [4] Adhikari S K, Delfino A, Frederico T, Goldman I D and Tomio L 1988 Efimov and Thomas effects and the model dependence of three-particle observables in two and three dimensions *Phys. Rev. A* **37** 3666
- [5] Nielsen E, Fedorov D V and Jensen A S 1997 Three-body halos in two dimensions *Phys. Rev. A* **56** 3287
- [6] Nielsen E, Fedorov D V and Jensen A S 1999 Structure and occurrence of three-body halos in two dimensions *Few-Body Syst.* **27** 15
- [7] Hammer H-W and Son D T 2004 Universal properties of two-dimensional boson droplets *Phys. Rev. Lett.* **93** 250408
- [8] Kartavtsev O I and Malykh A V 2006 Universal low-energy properties of three two-dimensional bosons *Phys. Rev. A* **74** 042506
- [9] Brodsky I V, Kagan M Y, Klaptsov A V, Combescot R and Leyronas X 2006 Exact diagrammatic approach for dimer–dimer scattering and bound states of three and four resonantly interacting particles *Phys. Rev. A* **73** 032724
- [10] Tjon J A 1980 Bound states of four identical bosons in two dimensions *Phys. Rev. A* **21** 1334
- [11] Lim T K, Nakaichi S, Akaishi Y and Tanaka H 1980 Near-threshold behavior of the ground-state binding energies of the few-atom systems of ^4He and other bosons in two and three dimensions *Phys. Rev. A* **22** 28
- [12] Vranješ L and Kilić S 2002 Helium trimers and tetramers in two dimensions and quasi-two-dimensions *Phys. Rev. A* **65** 042506
- [13] Platter L, Hammer H-W and Meißner U-G 2004 Universal properties of the four-boson system in two dimensions *Few-Body Syst.* **35** 169
- [14] Blume D 2005 Threshold behavior of bosonic two-dimensional few-body systems *Phys. Rev. B* **72** 094510
- [15] Lee D 2006 Large- N droplets in two dimensions *Phys. Rev. A* **73** 063204
- [16] Bazak B and Petrov D S 2017 Five-body Efimov effect and universal pentamer in fermionic mixtures *Phys. Rev. Lett.* **118** 083002
- [17] Skorniakov G V and Ter-Martirosian K A 1956 Three body problem for short range forces: I. Scattering of low energy neutrons by deuterons *Zh. Eksp. Teor. Fiz.* **31** 775
Skorniakov G V and Ter-Martirosian K A 1957 *Sov. Phys. -JETP* **4** 648 (www.jetp.ac.ru/cgi-bin/e/index/e/4/5/p648?a=list)
- [18] Petrov D S and Shlyapnikov G V 2001 Interatomic collisions in a tightly confined Bose gas *Phys. Rev. A* **64** 012706
- [19] Fattori M, D’Errico C, Roati G, Zaccanti M, Jona-Lasinio M, Modugno M, Inguscio M and Modugno G 2008 Atom interferometry with a weakly interacting Bose–Einstein condensate *Phys. Rev. Lett.* **100** 080405
- [20] Nguyen J H V, Dyke P, Luo D, Malomed B A and Hulet R G 2014 Collisions of matter-wave solitons *Nat. Phys.* **10** 918
- [21] Lepoutre S, Fouché L, Boissé A, Berthet G, Salomon G, Aspect A and Bourdel T 2016 Production of strongly bound ^{39}K bright solitons *Phys. Rev. A* **94** 053626