Gapless Superfluid in $SU(2n + 1)$ Fermions Systems

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We investigate the generalized Hubbard model of $(2n + 1)$ Fermion species interacting via a symmetric contact attraction potential. We prove that the ground state of such system is a gapless superfluid, where a full Fermi surface coexists with a superfluid. Moreover, doing so we prove the existence of a free mode in a strongly interacting system, regardless of the potential strength. This proof holds at the mean field level. A Grassmannian Gaussian integration technique is used to deal with the problem. Our predictions may be relevant to future high-spin cold atoms experiments.

The smooth crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluidity, valid for weakly interacting Fermions, to a Bose-Einstein Condensation (BEC), valid for (composite) Bosons, was suggested theoretically about 30 years ago [1]. This prediction was verified experimentally in the last decade by several groups using trapped cold atoms. Two-component Fermionic atoms, like ^{40}K [2] or ^{6}Li [3], were trapped and the Feshbach resonance was used to control the interaction between them.

Recently, higher spin atoms were trapped, for which F, the hyper-fine spin of the atom, enables studying systems of $(2F + 1)$ Fermion species. For example, an $F = 1$ system was materialized with ^{6}Li atoms [4], an $F = 5/2$ with ^{173}Yb atoms [5], and $F = 9/2$ with ^{87}Sr atoms [6]. Note that the experimental challenge in these systems is not limited to trapping and cooling down all the species of the relevant multiplet, but doing so while conserving the $SU(N)$ symmetry. This goal is yet to be achieved in the BCS-BEC crossover regime. This is due to the fact that the external magnetic field used to control the interaction between the particles breaks the $SU(N)$ symmetry.

Experiments with $N = 2$ cold atoms are used as a playground for condensed matter physics [7]. Similarly, high-spin experiments may shed light on other fields of physics. The $SU(3)$ case is relevant for color-superconductivity in QCD, where the three quark colors generate an $SU(3)$ symmetry. The $SU(4)$ case is relevant for nuclear physics, where four types of Fermions exist (the proton and neutron with their spin up or down), and form an approximate $SU(4)$ symmetry. It is well known that pairing plays an important role in nuclear physics [8]. Therefore, the remarkable advance and flexibility of the cold atoms experiments gives the opportunity of making experiments relevant to other fields of physics which by themselves are hard to control and measure.

The generalization of superfluidity theory to the case of more then two Fermion species arose soon after the BCS paper was published [9], presenting the case of two band superconductivity [10]. Since then, various techniques were used to deal with the $SU(N)$ case, see for example Ref. [11]. Considerable theoretical attention was dedicated to the $SU(3)$ case [12–14] and the $SU(4)$ case [15]. Recently, guided by the experimental development, the $SU(6)$ case was also studied [16].

A gapless superfluid, where a superfluid with gapped excitation spectrum coexists with a gapless branch of excitation, is a well known phenomena that naturally emerge in the case of particle symmetry breaking. For example, when there is a difference in the density or the mass of the interacting species, the result is a mismatch in their Fermi surfaces and therefore a gapless superfluid emerges [17]. Here, in contrast, we study a gapless superfluid in the symmetric case, where the $SU(N)$ symmetry is unbroken and completely holds.

In this letter we focus on S-wave pairing in systems with odd number of species $N = 2n + 1$. Starting with the $N = 1$ case, i.e. one particle species, we note that pairing is forbidden and only a gapless excitation can exist. For the $N = 3$ case, it was proven that a gapless excitation coexists with gapped superfluid excitation [13]. For the general case it was claimed but not proven, that for any odd $N = 2n + 1$ the ground state of an $SU(N)$ symmetric system is a gapless superfluid [14]. Here we prove this hypothesis to be true in general.

Let us consider a gas of N species of Fermions, interacting via a contact $SU(N)$-symmetric pairing potential. This system can be described by the generalized Hubbard Hamiltonian,

$$H - \mu N =$$
$$\int d\mathbf{x} \left[\sum_\alpha c_\alpha^\dagger \left(-\frac{\nabla^2}{2m} - \mu \right) c_\alpha - g \sum_{\alpha < \beta} c_\alpha^\dagger c_\beta^\dagger c_\beta c_\alpha \right], \ (1)$$

where $c_\alpha^\dagger$ (c_α) is the creation (annihilation) operator for an α type Fermion, μ is the chemical potential, and $g > 0$ is the attractive contact pairing coefficient. Note that the contact interaction coupling constant g must be regularized. This is usually done through the relation $m/4\pi a_s = g^{-1} + \sum_k (2\epsilon_k)^{-1}$ between g and the scattering length a_s, which is the relevant physical parameter.

Using the Hubbard-Stratonovich transformation [18] to decouple the interaction in the Cooper channel, one

gets the partition function,

$$\mathcal{Z} = \int D(\bar{\Psi}, \Psi) \int \prod_{\alpha < \beta} D(\bar{\Delta}_{\alpha\beta}, \Delta_{\alpha\beta}) e^{-S}, \qquad (2)$$

where

$$S = \int_0^\beta d\tau \int d^3x \left[\frac{1}{g}|\Delta|^2 - (\bar{\Psi}\,\Psi)\mathcal{G}^{-1}\left(\begin{array}{c} \Psi^T \\ \bar{\Psi}^T \end{array} \right) \right]. \quad (3)$$

The field operator $\Psi = (\psi_1\,\psi_2\,...\psi_N)$ is a vector of coherent states, and $\Delta_{\alpha\beta} = -\Delta_{\beta\alpha}$ is the complex pairing field describing an S-wave Cooper pair consisting of Fermions of types α and β. The total magnitude of the pairing fields is given by $|\Delta|^2 = \sum_{\alpha < \beta} |\Delta_{\alpha\beta}|^2$, and $\mathcal{G}$ is the extended Nambu-Gor'kov propagator.

At this point let us assume that the pairing fields are fixed in time and space $\Delta_{\alpha\beta}(\mathbf{x}, t) = \Delta_{\alpha\beta}$. This simplifying assumption allows us to write $\mathcal{G}^{-1}$ in a block diagonal form in the momentum-frequency $(\mathbf{k}, i\omega)$ space. Each of these blocks is a $2N \times 2N$ matrix of the form

$$\mathcal{G}^{-1}(\mathbf{k}, i\omega) = \left(\begin{array}{cc} \mathbb{1}(i\omega - \xi_k) & D \\ D^\dagger & \mathbb{1}(i\omega + \xi_k) \end{array} \right), \qquad (4)$$

where $\mathbb{1}$ is the $N \times N$ identity matrix, $\xi_k = k^2/2m - \mu$, and D is the pairing matrix $D_{\alpha\beta} = \Delta_{\alpha\beta}$. The corresponding $2N$ spinors are $(\bar{\Psi}(\mathbf{k}, i\omega), \Psi(-\mathbf{k}, -i\omega))$, and the sum on ω is limited to positive Matsubara frequencies only.

Note that the vector $(\bar{\Psi}\,\Psi)$ is not a standard Nambu spinor. In order to consider all possible $N(N-1)$ pairing types the dimension of $\mathcal{G}^{-1}$ must be $2N \times 2N$, and therefore the spinors are $2N$-vectors. This raises the problem of performing the integration over the N-vectors $\Psi, \bar{\Psi}$, which in (3) seems not to be Gaussian anymore. However in the block diagonal form (4), the integration is still Gaussian, and hence can be done quite easily.

To show that, let us assume that $\left(\begin{array}{c} \mathbf{u}_i \\ \mathbf{v}_i \end{array} \right)$ is an eigenvector of $\mathcal{G}^{-1}$ with the corresponding eigenvalue $i\omega - \lambda_i$. One can easily verify that there exists a twin eigenvector $\left(\begin{array}{c} \mathbf{v}_i^* \\ \mathbf{u}_i^* \end{array} \right)$ with the corresponding eigenvalue $i\omega + \lambda_i$. Hence, the eigendecomposition of $\mathcal{G}^{-1}$ is given by $Q^\dagger \mathcal{G}^{-1} Q = \mathbb{1}i\omega - \left(\begin{array}{cc} \Lambda & \\ & -\Lambda \end{array} \right)$, where $Q = \left(\begin{array}{cccccc} \mathbf{u}_1 & ... & \mathbf{u}_N & \mathbf{v}_1^* & ... & \mathbf{v}_N^* \\ \mathbf{v}_1 & ... & \mathbf{v}_N & \mathbf{u}_1^* & ... & \mathbf{u}_N^* \end{array} \right)$, and Λ is a real diagonal matrix composed of the eigenvalues λ_i. Using this transformation the integration over the Fermionic fields reads

$$\int D(\bar{\Psi}, \Psi) e^{-(\bar{\Psi}\,\Psi)\mathcal{G}^{-1}\left(\begin{array}{c} \Psi^T \\ \bar{\Psi}^T \end{array} \right)} = \prod_i (i\omega - \lambda_i)(-i\omega - \lambda_i),$$

$$(5)$$

where $D(\bar{\Psi}, \Psi)$ stands for integration over the fields $\Psi(\mathbf{k}, i\omega), \bar{\Psi}(\mathbf{k}, i\omega), \Psi(-\mathbf{k}, -i\omega), \bar{\Psi}(-\mathbf{k}, -i\omega)$. Restoring

the momenta and frequencies summation one finally gets $\prod_{i, i\omega, \mathbf{k}} (i\omega - \lambda_i)$.

Now that we have established the Grassmannian integration, we shall prove that for odd $N \geq 1$, there is always a gapless branch of excitations, i.e. one of the eigenvalues of $\mathcal{G}^{-1}$ is simply $i\omega - \xi_k$. To show that, we recall Jacobi's theorem [19] that if M is a skew symmetric matrix of order $2n + 1$ then the determinant of M vanishes. Since $D^\dagger$ is such a matrix, one can conclude that there exists a vector $\tilde{\mathbf{v}}$ for which $D^\dagger \tilde{\mathbf{v}} = 0$, therefore,

$$\left(\begin{array}{cc} \mathbb{1}(i\omega - \xi_k) & D \\ D^\dagger & \mathbb{1}(i\omega + \xi_k) \end{array} \right) \left(\begin{array}{c} \tilde{\mathbf{v}} \\ 0 \end{array} \right) = (i\omega - \xi_k) \left(\begin{array}{c} \tilde{\mathbf{v}} \\ 0 \end{array} \right).$$

Consequently $\left(\begin{array}{c} \tilde{\mathbf{v}} \\ 0 \end{array} \right)$ and $\left(\begin{array}{c} 0 \\ \tilde{\mathbf{v}}^* \end{array} \right)$ are twin eigenvectors of $\mathcal{G}^{-1}$ with eigenvalues $i\omega - \xi_k$ and $i\omega + \xi_k$, respectively. These eigenvectors and eigenvalues correspond to a gapless Fermi liquid excitation spectrum, thus proving our claim.

As an example, we explicitly consider the $SU(3)$ and $SU(5)$ cases.

In the case of three Fermion species, the gapless branch of excitations, ξ_k, has the corresponding eigenvector $\left(\begin{array}{c} \tilde{\mathbf{v}} \\ 0 \end{array} \right)$, where $\tilde{\mathbf{v}} = \left(\begin{array}{c} \bar{\Delta}_{23} \\ -\bar{\Delta}_{13} \\ \bar{\Delta}_{12} \end{array} \right)$ and $D^\dagger \tilde{\mathbf{v}} = 0$. The spectra of the two gapped branches are $E_k = \pm\sqrt{\xi_k^2 + |\Delta|^2}$. The mean field values of $\Delta_{\alpha\beta}$ can be found from the saddle point approximation $\partial S/\partial\bar{\Delta}_{\alpha\beta} = 0$ yielding three identical gap equations,

$$\frac{1}{g} = \sum_k \frac{1}{2E_k} \tanh\frac{\beta E_k}{2}. \qquad (6)$$

Note that the same gap equation is valid for the $SU(2)$ case, resulting in the same magnitude of the (total) pairing gap and the same critical temperature. The density equation for the α Fermion species is,

$$\rho_\alpha = \sum_k \left[f_\alpha n_F(\xi_k) + (1 - f_\alpha)\frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\tanh\frac{\beta E_k}{2} \right) \right],$$

$$(7)$$

where $f_\alpha = |\Delta_{\beta\gamma}|^2/|\Delta|^2$. If the system is balanced, i.e. contains the same number of particles for each kind, we get $|\Delta_{12}| = |\Delta_{23}| = |\Delta_{13}|$ and $f_\alpha = 1/3$. That is to say that one third of the system remains gapless while the residual two thirds are superfluid. Note that since the different $\Delta_{\alpha\beta}$'s are equal, none of the species is unpaired, but instead there is a collective gapless excitation.

In the case of five Fermion species, the gapless branch of excitations has the corresponding eigenvector $\left(\begin{array}{c} \tilde{\mathbf{v}} \\ 0 \end{array} \right)$,

where $\tilde{\mathbf{v}} = \begin{pmatrix} \bar{\eta}_1 \\ -\bar{\eta}_2 \\ \bar{\eta}_3 \\ -\bar{\eta}_4 \\ \bar{\eta}_5 \end{pmatrix}$,

$$\eta_\alpha = \Delta_{\beta\gamma}\Delta_{\delta\epsilon} - \Delta_{\beta\delta}\Delta_{\gamma\epsilon} + \Delta_{\beta\epsilon}\Delta_{\gamma\delta},$$

and $D^\dagger \tilde{\mathbf{v}} = 0$. The spectra of the four gapped branches are $E_k = \pm\sqrt{\xi_k^2 + \frac{1}{2}|\Delta|^2 \pm \frac{1}{2}\sqrt{|\Delta|^4 - 4|\eta|^2}}$, where $|\eta|^2 = \sum_\alpha |\eta_\alpha|^2$. Note that the cases where $|\Delta|^4 = 4|\eta|^2$ and $|\eta|^2 = 0$ will be of specific interest, as the former has two doubly degenerated gapped branches and the latter has three gapless branches and two gapped branches.

To conclude, we have studied a system of N-Fermion species with an attractive, on-site, $SU(N)$ symmetric interaction. We have proved that for odd N, the ground state of the system is a gapless superfluid. Doing so, we have used a generalized Grassmannian Gaussian integration. The key point in our proof is the fact that the determinant of the skew-symmetric pairing matrix vanishes if its dimension is odd. As a result a gapless free Fermi mode appears. The cases $N = 3$ and $N = 5$ were solved analytically at the mean field level to demonstrate our general claim. We hope that future cold atoms experiments would confirm our predictions.

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[1] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, edited by A. Pekalski and R. Przystawa (Springer-Verlag, Berlin, 1980); P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).

[2] M. Greiner, C. A. Regal, and D. S. Jin, Nature **426**, 537 (2003); C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92** 040403 (2004).

[3] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004); M. W. Zwierlein *et al*, Phys. Rev. Lett. **92**, 120403 (2004); M. W. Zwierlein *et al*, Phys. Rev. Lett. **94**, 180401 (2005); M. W. Zwierlein *et al*, Nature **435**, 1047 (2005); C. Chin et al, Science **305**, 1128 (2004).

[4] T. B. Ottenstein, T. Lompe, M. Kohnen, A. N. Wenz, and S. Jochim, Phys. Rev. Lett. **101**, 203202 (2008); J. H. Huckans, J. R. Williams, E. L. Hazlett, R. W. Stites, and K. M. O'Hara, Phys. Rev. Lett. **102**, 165302 (2009).

[5] T. Fukuhara, Y. Takasu, M. Kumakura, and Y. Takahashi, Phys. Rev. Lett. **98**, 030401 (2007); S. Taie, Y. Takasu, S. Sugawa, R. Yamazaki, T. Tsujimoto, R. Murakami, and Y. Takahashi, Phys. Rev. Lett. **105**, 190401 (2010).

[6] B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, and T. C. Killian, Phys. Rev. Lett. **105**, 030402 (2010).

[7] I. Bloch, J. Dalibar, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).

[8] D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. **75**, 607 (2003), D. M. Brink and R. A. Broglia, *Nuclear Superfluidity: Pairing in Finite Systems* (Cambridge University Press, Cambridge, 2005).

[9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[10] H. Suhl, B. T. Matthias, and L. R. Walker, Phys. Rev. Lett. **3**, 552 (1959).

[11] T. Ho and S. Yip, Phys. Rev. Lett. **82**, 247 (1999); S. Capponi, G. Roux, P. Lecheminant, P. Azaria, E. Boulat, and S. R. White, Phys. Rev. A **77**, 013624 (2008); P. Lecheminant, P. Azaria, and E. Boulat, Nucl. Phys. B **798**, 443 (2008).

[12] T. Paananen, J.-P. Martikainen, and P. Törmä, Phys. Rev. A **73**, 053606 (2006); A. Rapp, G. Zarand, C. Honerkamp, and W. Hofstetter, Phys. Rev. Lett. **98**, 160405 (2007); T. N. De Silva, Phys. Rev. A **80**, 013620 (2009).

[13] A. G. K. Modawi and A. J. Leggett, J. Low Temp. Phys. **109**, 625 (1997); L. He, M. Jin, and P. Zhuang, Phys. Rev. A **74**, 033604 (2006); P. F. Bedaque and J. P. D'Incao, Ann. Phys. (N.Y.) **324**, 1763 (2009).

[14] C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. **92**, 170403 (2004); Phys. Rev. B **70**, 094521 (2004).

[15] J. A. Evans, G. G. Dussel, E. E. Maqueda, and R. P. J. Perazzo, Nucl. Phys. A **367**, 77 (1981); C. Wu, J. P. Hu, and S. C. Zhang, Phys. Rev. Lett. **91**, 186402 (2003); C. Wu, Mod. Phys. Lett. B **20**, 1707 (2006); D. Controzzi and A. M. Tsvelik, Phys. Rev. Lett. **96**, 097205 (2006); T. Sogo, G. Röpke, and P. Schuck, Phys. Rev. C **81**, 064310 (2010); Y. Nishida and D. T. Son, Phys. Rev. A **82**, 043606 (2010); G. F. Bertsch, J. Dukelsky, B. Errea, and C. Esebbag, Ann. Phys. (N.Y.) **325**, 1340 (2010), S. Mao, X. Huang, and P. Zhuang, Phys. Rev. C **79**, 034304 (2009).

[16] M. A. Cazalilla, A. F. Ho, and M. Ueda, New J. Phys. **11**, 103033 (2009).

[17] W. V. Liu and F. Wilczek, Phys. Rev. Lett. **90**, 047002 (2003).

[18] J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959); R. L. Stratonovich, Doklady **2**, 416 (1958).

[19] H. Eves, *Elementary Matrix Theory* (Dover Publications, New York, 1966).