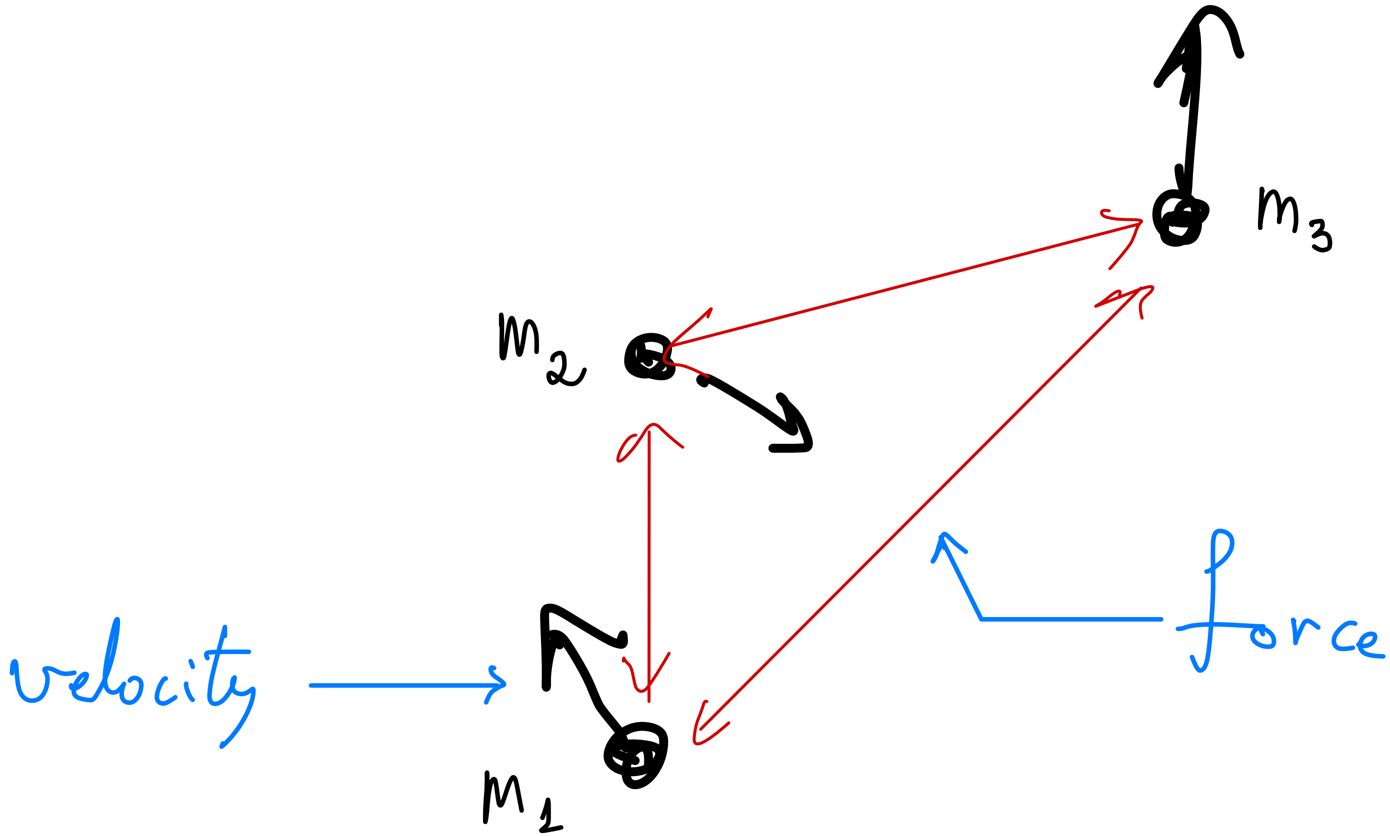
Natural dynamical reduction of the three-body problem

Barak Kol
Hebrew University of Jerusalem

Hebrew U. Seminar, 28 July 2021
based on arXiv:2107.12372

* The problem
* Coord. for center of mass frame – $\vec{w} \in \mathbb{C}^3$
* Orientation-geometry decomposition –
  formulation w. $\vec{J}$
* Applications
* Perspective

The three-body problem



e.g. $$F = \frac{G m_1 m_2}{r^2}$$

$$\mathcal{L} = \frac{1}{2}\sum m_a \dot{r}_a^{\,2} - \left(\frac{G m_1 m_2}{r_{12}} + \text{cyc.}\right)$$

General background

Newton explained planetary motion.

Universal law of gravitation

2-body problem.

What about 3-body ?

General background

The moon, perturbation theory, central configurations,
Poisson bracket, secular approximation,
chaos, modern topology,
computer simulations, statistical theory

Newton, d'Alembert, Clairaut, Euler, Lagrange, Laplace,
Gauss, Jacobi, Hill, Poincaré,
Levi-Civita, Lemaître; Hylleraas, Fock,
Agekian & Anosova, Valtonen, Monaghan, Heggie, Hut, McMillan,
Smale, Saari, C. Moore, Chenciner, Montgomery & Moeckel
Naoz, Peretz, Stone.

Dynamical reduction

* Simplify the dynamics through choice of variables
* Conserved quantities enable it
* Examples
  - Small oscillations w. $n>1$ d.o.f.
  - Central force & effective radial potential

My road

* Stone & Leigh (2019) - Approx. Stat. soln in closed form
* BK (2020) - Flux-based theory & to appear w. collab.*

Regularized phase space volume

$$\sigma(E_0, \vec{L}_0) = \int \prod_{a=1}^{3} \left(d^3 r_a \, d^3 p_a \right) \delta(E_0, \vec{L}_0, \vec{P}, \vec{R}_{cm})$$

* Dandekar
Mazumdar
Lederer

see also Ginat & Perets 2020

Coordinates for center of mass frame

$\vec{r}_1, \vec{r}_2, \vec{r}_3$ 3 vector d.o.f.

$\vec{0} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3$ 1 vector constraint

In 2-body

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

In 3-body relative positions (transl. invar.)

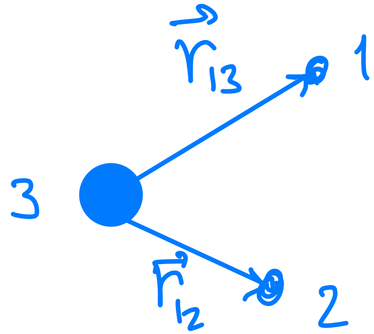
$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$: $\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{31}$

not independent : $\vec{r}_{12} + \vec{r}_{23} + \vec{r}_{31} = 0$

Popular choices of coord.

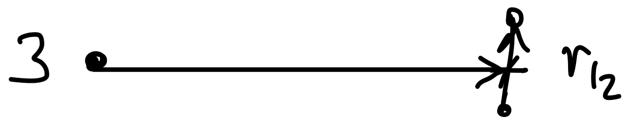
* Planetary $M_3 \gg m_1, m_2$

$$\vec{r}_{13}, \quad \vec{r}_{23}$$



* Lunar / hierarchical

$$\vec{r}_{12}, \quad \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} - \vec{r}_3$$



Inspired by Lagrange's solution to the cubic (1771)

$$ax^3 + bx^2 + cx + d = 0$$

Define

$$\vec{w} = \vec{r}_1 + \eta\,\vec{r}_2 + \bar{\eta}\,\vec{r}_3$$

$$\eta = \exp(2\pi j/3)$$

Translation invar.

Transforms nicely under perm.: $1 \to 2 \to 3 \to 1 \qquad w \to \bar{\eta}\,w$

$2 \leftrightarrow 3 \qquad w \to \bar{w}$

Toward formulation

(a) translations ✓ $\qquad \vec{w}$

(b) rotations $\qquad \vec{w} \to \vec{\omega}, w$

(c) rotations within plane, $\mathbb{C}^2/U(1)$, $\qquad w \to \vec{g}$

(d) Transform $\qquad \vec{\omega} \to \vec{J}$

Appendices

* Non coord. velocities & rigid body

Euler eq's from L, H

* Bi-complex numbers & isotropic (2d) oscillator

Orientation — Geometry decomposition

$\vec{J}^2 = \text{const}$

chaotic core

Geometry space — $\vec{g}$

$\leftrightarrow (r, \theta, \varphi)$

pipe joint

Formulation

$$\mathcal{L}_J(\vec{J}, \vec{g}) := \mathcal{L} - \vec{\omega}\cdot\vec{J} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \qquad \mathcal{O}(J^k)$$

$\uparrow$ geometry space

motion potential

$$\mathcal{L}_0 = \frac{1}{8I}\dot{I}^2 + \frac{3M_3}{8M}\frac{r^4}{I}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - \left(\frac{Gm_2m_3}{r_{23}} + \text{cyc.}\right)$$

$$I = \sum m_a r_a^2 \underset{\text{CM}}{=} \frac{1}{M}\left(m_2m_3 r_{23}^2 + \text{cyc.}\right)$$

$$M = m_1 + m_2 + m_3$$

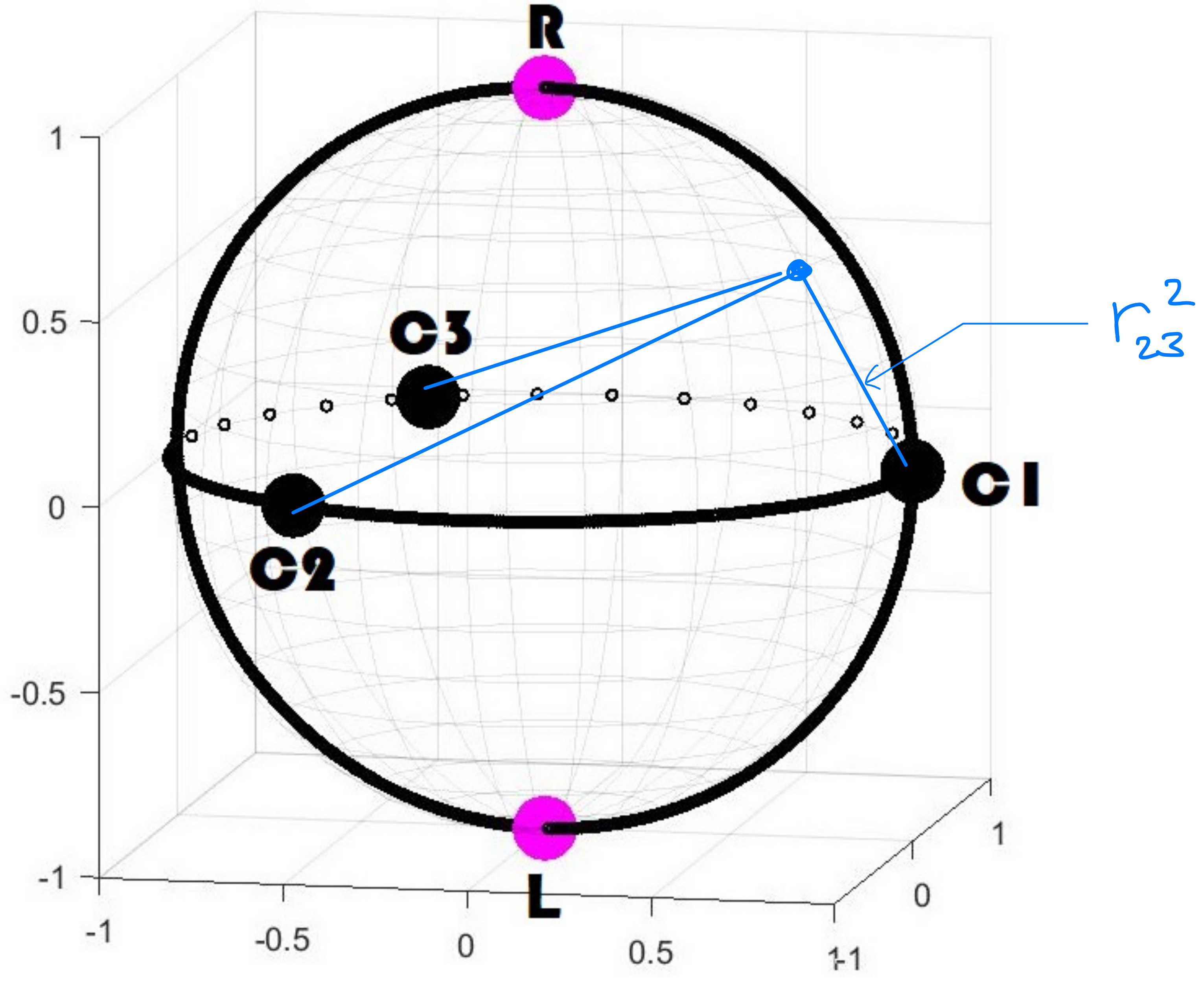
$$M_2 = m_1m_2 + m_2m_3 + m_3m_1$$

$$M_3 = m_1 m_2 m_3$$

$$r_{23}^2 = r^2(1 - \sin\theta\cos\varphi)$$

$$r_{31}^2 = \text{''}\left(\text{''} - \text{''}\cos\left(\varphi - \tfrac{2\pi}{3}\right)\right)$$

shape sphere metric conformal to round S^2.


R
C3
C1
C2
L
r_{23}^2
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0
1
-1

$$\mathcal{L}_1 = J_3 \frac{L_G}{I} \; ; \quad L_G = -\frac{r^2}{2M}\Big[m_2 m_3\big((1-\cos\theta - \sin\theta\cos\varphi)\dot\phi - \sin\phi\,\dot\theta\big) + \text{cyc.}\Big]$$

in "ψ_+-gauge"

Magnetic-like 2-form

$$B = \frac{3}{2} M m_3 J_3 \frac{\sin\theta \, d\varphi \, d\theta}{\left[m_2 m_3 (1-\sin\theta\cos\varphi) + \text{cyc.}\right]^2}$$

monopole charge at $\vec{g} = 0$.

$$-\mathcal{L}_2 = \frac{1}{2I} J_3^2 + \frac{M}{2M_3} \frac{1}{\Delta^2} I^{\alpha\beta} J_\alpha J_\beta \qquad \alpha, \beta = 1, 2$$

$$\Delta^2 = \frac{3}{4} r^4 \cos^2\theta$$

Δ is twice triangle area

$$I^{\alpha\beta} = \frac{1}{M}\left[m_2 m_3 \rho_{23}^{\alpha} \rho_{23}^{\beta} + \text{cyc.}\right] \qquad \sim I^{-1}$$

$$\rho_{23} = ir\left[\cos\frac{\theta}{2} - e^{-i\phi}\sin\frac{\theta}{2}\right]$$

in Υ_+ gauge

$$\{J_i, J_j\} = -\varepsilon_{ijk} J_k$$

centrifugal potential

Equations of motion

$$\frac{d}{dt} J_i = -\{J_i, \mathcal{L}_J\}$$

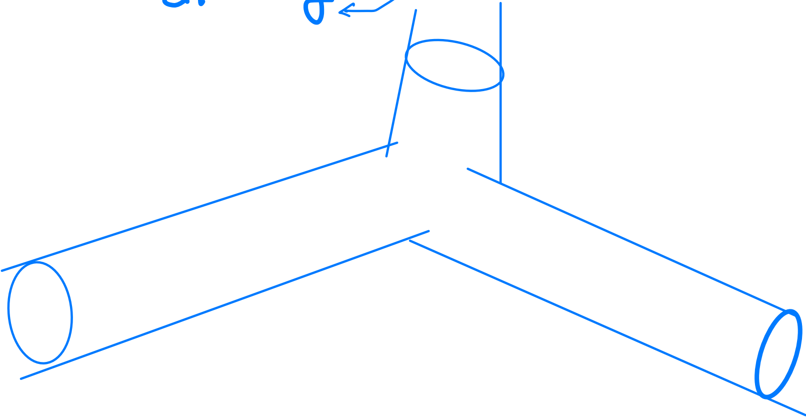
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_J}{\partial \dot{q}^s}\right) = \frac{\partial \mathcal{L}_J}{\partial q^s} \qquad q^s = (r, \theta, \varphi)$$

Applications

* to statistical soln
* to exact solutions

Regularized phase space volume

$d\Gamma \sim \frac{dF}{\sigma}$ ← reg. phase vol.

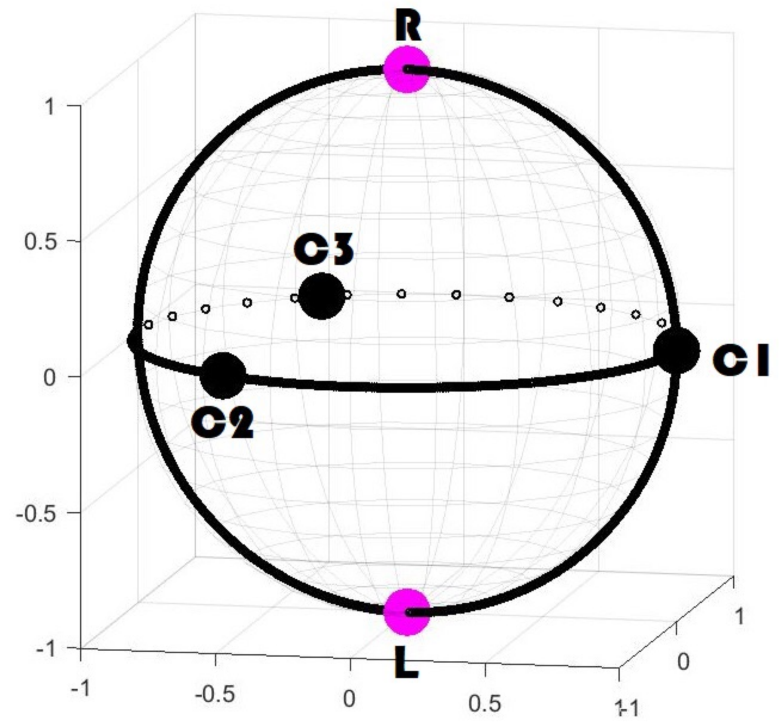


$\bar{\sigma} = \sigma_{bare} - \sigma_{ref}$

to appear,
w. Dandekar,
Mazumdar, Lederer

Novel derivation of Lagrange's equilateral soln & its stability

Equilibria in $\vec{g}$ $\equiv$ rigid rotation



$\vec{g}$ space

Perspective

Newton 1687 $\ddot{\vec{r}}_1 = -G\left(\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}}\right)$

Lagrange 1772 $\frac{d^2}{dt^2} r_{12}^2 = \dots$

...

Jacobi 1842 Elimin. of nodes

Levi-Civita 1915 $\vec{\omega}$, H

Murnaghan 1936, planar symmetric r_{ab}^2 (3 d.o.f)

Van Kampen & Wintner 1937 3d (4 d.o.f)

Lemaître 1952 symmetric & smooth at collinear

Moeckel & Montgomery 2013 planar, shape sphere, modernize

Helium spectrum

$m_1 = m_2 = m_e \quad M = \infty$

Hylleraas 1929 $\quad \vec{J}^2 = 0 \quad \sim \quad$ d.o.f : r_{ab}^2

Gronwall 1932 $\quad \sim \quad \vec{g} \in \mathbb{R}^3$

Fock 1954 $\quad \sim \quad w \in \mathbb{R}^4$

Tkachenko 1978 , $\mathcal{H}$ for general $\vec{J}$.

Novelties here

* Defn. of w
* Formulation in $\vec{J}$.

Recap.