Kruskal Space and the Uniformly Accelerated Frame*

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(Received 6 June 1966)

The striking formal similarities between the diagram of Kruskal space in general relativity and that of the uniformly accelerated rigid rod in special relativity are shown to be the result of certain physical similarities.

ANYONE who has contemplated the Minkowski diagram for the "uniformly accelerated rod" and the Kruskal diagram for "extended" Schwarzschild space must have been struck by their formal similarities. Actually, these formal similarities correspond to *physical* similarities, and since the accelerated rod is much more easily visualized than Kruskal space, the former can be used to understand certain aspects of the latter. The purpose of the present paper is to exhibit this analogy. At least one part of the analogy was already noted implicitly by Einstein and Rosen,[1] and explicitly by Bergmann,[2] namely that the pseudosingularity at the Schwarzschild radius $r=2m$ resembles the "cutoff" of the accelerated rod. The other major similarity which we discuss here is that a static gravitational field in one region of space–time involves a preferred instant in the extended space–time: the instant when $r=2m$ changes from a collapsing to an expanding light front in Kruskal space, and the instant when $X=0$ changes from a negatively to a positively moving light front in Minkowski space. (See Fig. 1.)

We must necessarily begin by summarizing, without proof, some of the well-known properties of Kruskal space and of the accelerated rod. In its original coordinates, the Schwarzschild metric of an isolated point mass m is

$$ds^2=(1-2m/r)dt^2-(1-2m/r)^{-1}dr^2-r^2d\omega^2, \quad (1)$$

with

$$d\omega^2=d\theta^2+\sin^2\theta d\phi^2, \quad (2)$$

where, for simplicity, the units are chosen so as to make both the speed of light and the constant of gravitation equal to unity.[3] This metric suffers from two blemishes: (i) it has an apparent (coordinate-dependent) singularity at the "Schwarzschild radius" $r=2m$, and (ii) it is extensible—i.e., there are free paths (timelike geodesics) which, when produced indefinitely in their own proper time, lead outside the region covered by the chosen coordinates without encountering a singularity. For example, a radially outgoing free particle is found to have crossed $r=2m$ at a finite instant by its own proper time reckoning, but at Schwarzschild coordinate time $t=-\infty$; again, an infalling free particle crosses $r=2m$ at a finite proper time, but at $t=+\infty$.

Kruskal[4] discovered a metric which represents a "maximal analytic extension" of Schwarzschild's metric. His coordinates u and v take the place of Schwarzschild's r and t, respectively, and his metric is

$$ds^2=f^2(dv^2-du^2)-r^2d\omega^2,$$
$$f^2=(32m^3/r)\exp(-r/2m), \quad (3)$$

where $d\omega^2$ has the same significance as in (2), and $r=r(u,v)$ is defined uniquely by

$$(r/2m-1)\exp(r/2m)=u^2-v^2, \quad r>0. \quad (4)$$

With (4), and the additional relation

$$t=4m \operatorname{arctanh}(v/u), \quad (5)$$

it can be shown that the "quadrant" $u>|v|$

* This work was supported, in part, by NASA Grant No. NsG 269-62 and by the U. S. Air Force, Office of Aerospace Research: AFOSR under Grant No. AF-AFOSR 903-65.

[1] A. Einstein and N. Rosen, Phys. Rev. **48**, 73–77 (1935). See especially their remark at the top of p. 75: "The hypersurface $u=0$ (or, in the original variables, $r=2m$) plays here the same role as the hypersurface $x_1=0$ in the previous example." Nonetheless, they apparently still believed that $r=2m$ was an intrinsic singularity, whereas they knew that $x_1=0$ (our $X=0$) was not.

[2] P. G. Bergmann, Phys. Rev. Letters **12**, 139 (1964).

[3] See, for example, R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, (Oxford University Press, Oxford, England, 1934), pp. 202–205.

[4] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960). In the present paper ds^2 is chosen opposite in sign to Kruskal's, and t and m are written for his T and m^*.

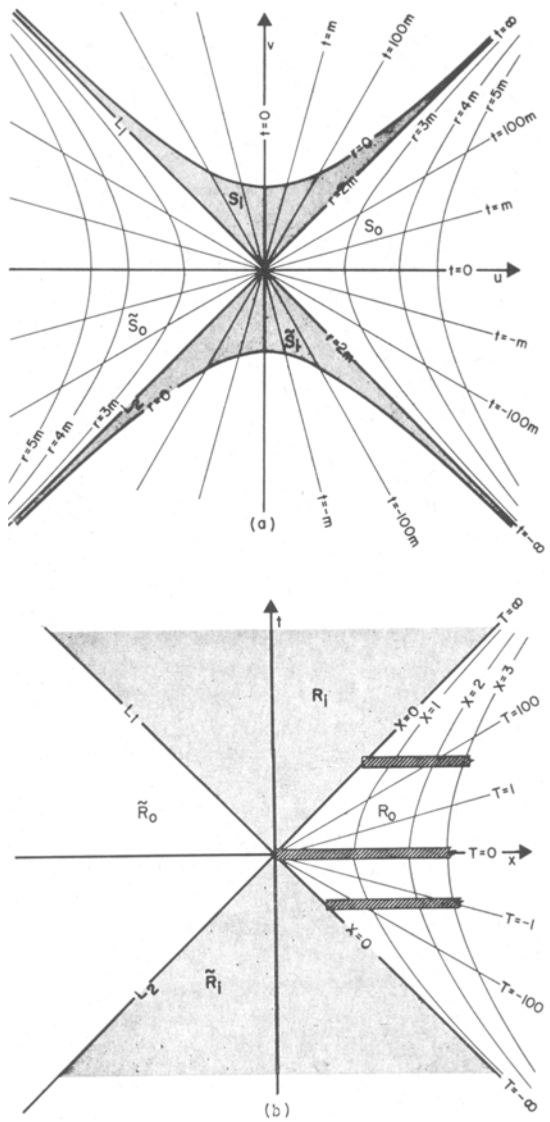

FIG. 1. (a) The Kruskal diagram and (b) the uniformly accelerated rod.

[marked S_0 in Fig. 1(a)] of Kruskal space corresponds to (i.e., transforms into) the whole of "outer" Schwarzschild space, characterized by (1) with $-\infty<t<+\infty$ and $r>2m$. On the other hand, the metric (3), subject to (4), is analytic throughout the region $v^2-u^2<1$, and it is inextensible. At $v^2-u^2=1$, (i.e., $r=0$) the space becomes intrinsically singular, in the sense that the curvature becomes infinite. As a matter of fact, the quadrant $u<-|v|$ (marked $\tilde{S}_0$) corresponds to another copy of outer Schwarzschild space, while each of the regions $|u|<v<(1+u^2)^{\frac{1}{2}}$, $-(1+u^2)^{\frac{1}{2}}<v<-|u|$ (marked S_i and $\tilde{S}_i$, respectively) corresponds to a space having metric (1) but $r<2m$, i.e., an "inner" Schwarzschild vacuum metric, which also satisfies the Einstein vacuum-field equations and spherical symmetry; in that region, however, r is timelike and t spacelike.

The definition of time in general relativity is very largely arbitrary. Any one-parameter family of spacelike hypersurfaces drawn through space–time, such that each event lies on exactly one of them, and such that the (real, continuous) parameter λ is in one–one correspondence with the hypersurfaces, provides an acceptable definition of time, viz., λ. (Even this kind of time cannot always be globally defined.) Evidently Kruskal's v is an acceptable and convenient time coordinate. The hypersurfaces $v=$constant *before* time $v=-1$ consist of two disconnected branches, each quasi-Euclidean for large $|u|$, but having a spherically symmetric cuspidal singularity at $r=0$: the three-dimensional analog of the two-dimensional singularity obtained by sticking a sharp pencil into a stretched rubber sheet. At time $v=-1$ these two branches spring a connection at their cusps, which develops into a smooth bridge or "wormhole," reaching its maximum radius $2m$ at $v=0$; thereafter the bridge shrinks and finally breaks off at $v=1$, whereupon the branches separate again. To avoid such splitting of the spatial universe by a mass point (although it may well be objected that a *real* mass point corresponds to a more sophisticated metric), I have elsewhere[5] suggested a topological identification of the Kruskal event pairs (u,v,θ,ϕ) and $(-u, -v, \theta, \phi)$.

It is important to observe that in Kruskal's metric there is no singularity whatever at $r=2m$, though what goes on there is nevertheless of considerable interest. Radial light-rays (null geodesics) in that metric correspond precisely to the lines with slope ± 1 in the diagram (e.g., $v=\pm u$). The hypersurface $r=2m$ is null, i.e., a potential light front, and the paths $r=2m$, $\theta=$constant, $\phi=$constant, are null geodesics. At each v instant, the surface $r=2m$ is simply a

[5] W. Rindler, Phys. Rev. Letters **15**, 1001 (1965). I learned recently that G. Szekeres had already proposed this in Publ. Math. Debrecen **7**, 285 (1960), and that he, in fact, had independently obtained the Kruskal metric.

2-sphere of radius $2m$. Note, incidentally, that *each* point in the Kruskal diagram represents a 2-sphere having for its radius the r corresponding to that point. Now suppose an observer A remains at fixed r, θ, ϕ in the region S_0. It is easily verified[6] that A is then in "hyperbolic motion" with proper acceleration $m/r^2(1-2m/r)^{\frac{1}{2}}$, and thus he experiences a constant gravitational field of precisely that intensity. Moreover, his world line is infinitely extensible in his own proper time into past and future. It is therefore somewhat surprising that the cause of this time-infinite static field in S_0 is nonstatic: what goes on *inside* $r=2m$ has a definite time development, and, in fact, a preferred time origin, $v=0$. At that instant the light front $v=-u$, which we call L_1 and which bounds S_0 below, suddenly peels off and disappears into "another space," while *from* that other space a second light front L_2 ($v=u$) has come to replace L_1.

It is clear from the diagram that every radially outgoing light ray in S_0 can be produced backwards, and has crossed L_1 at some finite (negative) v instant. The same can be shown to be the case for radially outgoing free particles. Observers like A can therefore have knowledge of events *inside* $r=2m$, though not of all such events: precisely of those in the region $\tilde{S}_i$. Events of that region are seen by A at *all* times. Events occurring above the line $v=u$ in the diagram are totally unknowable to observers confined to S_0: thus L_2 is an "event horizon"[7] for all such observers. Of course, both light and particles can be sent from S_0 *into* $r=2m$, but, again, not to all events inside: precisely to those in the region S_i.

The observer A might draw a space–time diagram such as that in Fig. 2(a) of his region of interest. He has realized the deficiencies of the Schwarzschild time coordinate t, and adopts Kruskal's v. He straightens the "kink" in the lower bound ($v<0$: L_1, $v>0$: L_2) of S_0, and he also straightens his own world line. Except for the hypertubular region $r\leqslant 2m$, his space–time is reasonably "ordinary," and becomes Minkowskian for large r. But the tube has that strange origin $v=0$, at which instant it changes the direction of its penetrability to light rays

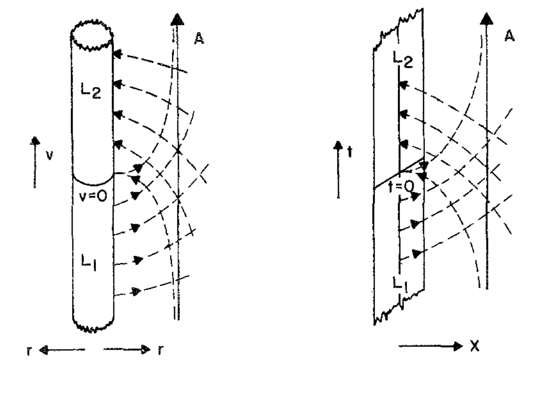

FIG. 2. Space–time diagrams of (a) outer Schwarzschild space and (b) the parallel gravitational field.

(indicated by the dotted lines) and matter. And, of course, it is a gateway, or wormhole, to another whole universe similar to, but distinct from, A's (unless some such identification as suggested in Ref. 5 above is adopted).

Now for the uniformly accelerated frame. Consider a rod of arbitrary length resting along the x axis of Minkowski space V_4. At time $t=0$, we wish to give one point of the rod a certain positive constant proper acceleration, and we want the rod as a whole to "move rigidly,"[8] i.e., in such a way that the proper length of each of its infinitesimal elements is preserved. It turns out that each point of the rod must then move with a different though also constant proper acceleration, the necessary acceleration increasing in the negative x direction and becoming infinite at a well-defined point of the rod; the rod can evidently not be extended beyond or even quite up to that point, since an infinite proper acceleration corresponds to motion at the speed of light. If we arrange things so that this "cutoff" point lies originally (i.e., at $t=0$) at the origin, the equation of motion of the point originally at $x=X$ is

$$x^2-t^2=X^2 \tag{6}$$

(we recall that the units are chosen so as to make $c=1$). We take X as a convenient spatial coordinate *on* the rod. We can, of course, continue the motion backward in time, i.e., assume that what we called the "original" position of

[6] See W. Rindler, Phys. Rev. **119**, 2082 (1960).

[7] See W. Rindler, Monthly Notices Roy. Astron. Soc. **116**, 662 (1956).

[8] For a discussion of this situation, see, for example, W. Rindler, *Special Relativity* (John Wiley & Sons, Inc., New York, 1966), 2nd ed., p. 42.

the rod at $t=0$ was merely a position of instantaneous rest, and that the various points of the rod are subject to constant proper acceleration for all $t>-\infty$. Figure 1(b) shows the world lines $X=$constant (hyperbolas) of some fixed points of the rod, and "snapshots" of the rod at $t=0$ and two other instants of Minkowski time. The lines $t=\pm x$ divide the plane into four regions, viz., R_0 (that occupied by the rod) and then, counterclockwise, R_i, $\tilde{R}_0$, and $\tilde{R}_i$.

It can be seen from (6) that the proper acceleration of the point X of the rod is $1/X$. Hence an observer at X feels a constant gravitational field of intensity $1/X$. The observers on the rod can so synchronize their clocks that each sees all the other clocks neither gain nor lose relative to his own; each observer must simply speed up his proper clock by a factor equal to the reciprocal of his coordinate X. Let T denote this new time. Then the relation between x, t (Minkowski coordinates) and X, T (rod coordinates) is given by

$$x=X\cosh T,\quad t=X\sinh T, \tag{7}$$

whence

$$t/x=\tanh T, \tag{8}$$

and also

$$ds^2=dt^2-dx^2=X^2dT^2-dX^2. \tag{9}$$

The "velocity reversal" event of each fixed point on the rod is, of course, not absolute: it depends on the inertial frame from which the rod is viewed. A Lorentz transformation applied to the Minkowski coordinates,

$$\begin{aligned} x'&=x\cosh\psi-t\sinh\psi,\\ t'&=-x\sinh\psi+t\cosh\psi, \end{aligned} \tag{10}$$

where

$$\exp 2\psi=(1+v)/(1-v)$$

induces the transformation

$$X'=X,\quad T'=T-\psi \tag{11}$$

on the rod coordinates (X' and T' being defined similarly to X and T in terms of x' and t'). Thus the hyperbolas $X=$constant go over into themselves and each $T=$constant line goes over into another. The velocity reversal for all rod points now occurs ψ units earlier by rod time. But the velocity reversal of the cutoff point $X=0$ *is* a unique event! The lines $t=\pm x$ (marked L_1 and L_2 in the diagram) are potential light paths. At $t=0$ the photon L_1 at the end of the rod peels off and disappears into the distance. Another photon L_2 has come from distant space and taken its place. Thus, A *knows theoretically* that his extended space has, relative to him, a preferred time origin, though in his own region he cannot find a preferred moment.

The analogy with Kruskal space should now be clear:

(i) Each radius vector ($\theta=$constant, $\phi=$constant, $r\geqslant 2m$) in outer Schwarzschild space S_0 corresponds *qualitatively* to a uniformly accelerating rod. Of course, the force laws are different: in S_0 the force varies inversely as the squared distance, while on the rod it varies inversely as the distance.

(iii) Minkowski and rod coordinates correspond to Kruskal and Schwarzschild coordinates, respectively. This is seen by comparing (4) and (5) with (6) and (8), or simply from the diagrams in Fig. 1.

(iii) Just as the accelerated rod cannot be continued beyond $X=0$, so a radial rod, each of whose points is fixed at constant r in S_0, cannot be extended beyond $r=2m$. The changeover of the photons at the lower "ends" of the rods is analogous in S_0 and R_0. L_2 is the event horizon for observers in R_0, who can have knowledge of events in $\tilde{R}_i$ but not in R_i or $\tilde{R}_0$; they can send information only to events in R_i. Events of $\tilde{R}_i$ are seen by observers on the rod at all times.

(iv) A pseudo-Lorentz transformation (10) applied to u, v in place of x, t leaves invariant the form of the Kruskal metric, just as (10) leaves invariant the form of the Minkowski metric, and it induces a transformation

$$r'=r,\quad t'=t-4m\psi \tag{12}$$

of the Schwarzschild coordinates associated with the Kruskal coordinates in analogy with (11).

(v) If we consider not one rod but many, one along each line $y=$constant, $z=$constant, in V_4, and all moving according to (6), we have the standard model of a parallel gravitational field. Of course, there is a cutoff, namely the "plane" $X=0$, where the field becomes infinite. An observer A in that field might draw a space–time diagram of his region of interest such as is shown in Fig. 2(b), which is analogous to Fig. 2(a).

(vi) To illustrate the *formal* similarity between conditions at $r=2m$ and $X=0$, we adapt Bergmann's argument (Ref. 2) and then go one step further. By a suitable choice of the units of mass, time, and distance we can make not only the constant of gravitation and the speed of light equal to unity, but also $m=\frac{1}{4}$, whence $r=\frac{1}{2}$ at the Schwarzschild radius. Then, writing $X^2=1-2m/r$ and $T=t$, we have (suppressing the θ, ϕ terms) from (1)

$$\begin{aligned} ds^2 &= (1-2m/r)dt^2-(1-2m/r)^{-1}dr^2 \\ &= X^2dT^2-(1-X^2)^{-4}dX^2 \\ &= X^2dT^2-(1+4X^2+\cdots)dX^2 \end{aligned} \tag{13}$$

for small X. This shows an analogy between the metrics (1) and (9) near the "critical" loci. A further transformation of type (7)—say, $u=X\cosh T$, $v=X\sinh T$—leads to

$$ds^2=dv^2-du^2+\left\{1-\frac{1}{(1+v^2-u^2)^4}\right\}\frac{(udu-vdv)^2}{(u^2-v^2)}. \tag{14}$$

This metric does not reduce to dv^2-du^2 at the critical locus $u^2-v^2=0$, but it is nevertheless regular there. In fact, though much less elegant and convenient than Kruskal's, the metric (14) also represents a maximal analytic extension of outer Schwarzschild space. In it, the singularity $r=0$ has gone to the infinite part of the diagram ($v^2-u^2\to\infty$) while $r=\infty$ has come to the finite part: $u^2-v^2=1$.

It may be remarked that deSitter space, in the original time-independent metric, also represents a static gravitational field and as such possesses a cutoff, namely at the well-known horizon light front. In the maximal analytic extension there also exists a definite moment at which this horizon changes its character from an outgoing to an incoming light front. Similar sudden changeovers occur in the Reissner–Nordström and Kerr metrics, which each possess *two* horizons. This situation seems to be characteristic of static gravitational fields, and the uniformly accelerating frame illustrates the mechanism.