Reprinted by permission from Nature, Phys. Sci. 229 (1971) 177-179 and the authors

Extraction of Rotational Energy from a Black Hole

THERE has been considerable interest recently in the question of the gravitational collapse of a massive body and of the possible astrophysical consequences of the existence of the "black hole" which general relativity predicts should sometimes be the result of such a collapse. In particular, the question has arisen whether the mass–energy content of a black hole could, under suitable circumstances, be a source of available energy. We now consider the extraction of rotational energy from a black hole, not least because the rotational energy (defined appropriately) of a black hole should, in general, be comparable with its total mass–energy[1].

The extraction of rotational energy from neutron stars is important in the theory of pulsars (in which case the rotational energy is of the general order of only 10^{-5} of the mass–energy of the object), where the coupling between angular momentum of the star and the surrounding medium is presumed to be achieved by means of magnetic field lines in plasma. It is exceedingly unlikely that such a process could be made to work in the case of a black hole, for the magnetic dipole moment of a collapsing body should be effectively swallowed in a very short space of time by the resulting black hole, the external dipole field being radiated away (ref. 2 and unpublished work of W. Israel). The coupling we shall now describe between the angular momentum of the black hole and that of the external world depends on a somewhat contrived process, but once it has been established that such a process is theoretically possible, it becomes reasonable to ask whether processes of this general kind might occasionally even occur naturally. (An even more contrived process is described in ref. 3.)

The process we suggest depends on a property of Kerr's solution of Einstein's vacuum equations[4]. This solution describes a rotating black hole, and is interesting because there are strong indications that it may in fact represent the general limiting situation of a black hole, any additional asymmetries being rapidly eliminated by gravitational radiation (ref. 2 and unpublished work of B. Carter). Our process depends on the existence of a region between the two surfaces known as the stationary limit and the (absolute) event horizon. The stationary limit is a surface at which a particle would have to travel with the local light velocity in order to appear stationary to an observer at infinity; and just inside which no particle (its velocity having to be bounded by the local light velocity) can remain stationary as viewed from infinity. The event horizon is the effective boundary of the black hole inside which no information can escape to the outside world. The stationary limit lies outside the event horizon in the Kerr solution touching it only where they intersect the rotation axis. (The stationary limit and the event horizon coincide in the case of the Schwarzschild solution, which is the limiting case of the Kerr solution for zero angular momentum).

To define the stationary limit more precisely, we consider the Killing vector (say, $\partial/\partial t$) which generates the time-translation symmetry of the Kerr solution. This becomes null at the stationary limit and spacelike within it. The Killing vector $\partial/\partial t$ is uniquely defined by the property that it is timelike with unit norm at infinity. The mass–energy of a test particle with four-momentum p_a is defined to be its scalar product with $\partial/\partial t$, namely the t-component, p_0. This quantity is conserved in collisions, is conserved along the world line of a freely moving particle between collisions, and agrees with the ordinary definition of mass–energy at infinity, so unquestionably it does describe the energy of a test particle correctly (as measured from infinity). The significance of the region between the stationary limit and the event horizon is that because $\partial/\partial t$ is spacelike there, it is possible for the energy p_0 to be negative in this region even though the vector p_a may be timelike (or null) and future pointing (as it must be for a real particle). Inside the event horizon this can still be true, but is of little value to us because the effects cannot be observed from outside.

Our process is now as follows. A particle (0) with four-momentum $\overset{(0)}{p}_a$ is dropped into the relevant region. It then splits into two particles 1 and 2, with

$$\overset{(0)}{p}_a=\overset{(1)}{p}_a+\overset{(2)}{p}_a$$

so that the mass–energy (as measured from infinity) of one of them is negative

$$\overset{(1)}{p}_0<0$$

and such that the other escapes back to infinity. The first of these is then "swallowed" by the black hole (it crosses the event horizon, in other words) while the second carries more mass–energy back to infinity than the original particle possessed

$$\overset{(2)}{p_0} > \overset{(0)}{p_0}$$

Thus some of the energy of the black hole has been extracted in the process.

It will be necessary to go into some detail in order to show that such a process actually can be realized with physical particle orbits. We demonstrate this with an explicit example. It is convenient to choose coordinates for the Kerr metric so that it takes the form

$$ds^2 = dt^2 - dr^2 - d\theta^2 + 2a \sin^2 \theta \, dr \, d\varphi - (r^2 + a^2) \sin^2 \theta \, d\varphi^2$$
$$- \frac{2mr}{\Sigma} (dr - a \sin^2 \theta \, d\varphi + dt)^2$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ (ref. 5). We choose units so that $c = 1$; then $m = GM$ and $a = J/M$, where M is the gravitational mass, G is Newton's constant of gravitation and J is the angular momentum of the "hole", and thus a is the specific angular momentum of the hole. The coordinates have been chosen so that for $a > 0$, test particles with $\varphi > 0$ revolve in the same sense as the hole spins, and so that when $a = 0$ the metric reduces to the Eddington–Finkelstein form of the Schwarzschild solution[6,7], t being an advanced time parameter.

The stationary limit and the event horizon occur at

$$r = m + \sqrt{m^2 - a^2 \cos^2 \theta}$$

and

$$r = m + \sqrt{m^2 - a^2} = r_+$$

respectively. For a particle travelling on an equatorial geodesic $\theta = \pi/2$, and p^2 and p_2 are always zero (numbering the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$). Because both $\partial/\partial t$ and $\partial/\partial\varphi$ are Killing vector fields, not only the particle's energy $E = p_0$ but also its angular momentum $\Lambda = -p_3$ is a conserved quantity. In the equatorial plane the stationary limit is at $r = 2m$ and a physical (that is, non-spacelike and future-directed) test particle has negative energy E if and only if $\Lambda < 0$ and $r - 2m < -\Delta\mu^2 r/\Lambda^2$, where μ is the rest mass of the particle and $\Delta = r^2 - 2mr + a^2$. The least energy such a particle can have is

$$\frac{2ma\Lambda + \sqrt{\Delta r\{r\Lambda^2 + \mu^2 (r^3 + a^2 r + 2ma^2)\}}}{r^3 + a^2 (r + 2m)} \qquad (1)$$

Taking $a = 9m/10$, a specific angular momentum not very different from that of the Sun, then the event horizon occurs at $r_+ = m(1+\sqrt{19}/10) < 3m/2$. Let the particle (0) have unit rest mass and unit energy (so it drops from rest at infinity). Suppose that at $r = 1.75m$, (0) splits into particles (1) and (2)

$$1 = \overset{(1)}{E} + \overset{(2)}{E}, \qquad \overset{(0)}{\Lambda} = \overset{(1)}{\Lambda} + \overset{(2)}{\Lambda}$$

and also

$$\overset{(0)}{p^1} = \overset{(1)}{p^1} + \overset{(2)}{p^1} \tag{2}$$

From formula (1), the least energy a physical test particle can have at $r = 1.75m$, obtained by putting $\mu = 0$, is approximately $0.0872\overset{(1)}{\Lambda}/m$. Therefore a convenient permitted value for $\overset{(1)}{E}$ is

$$\overset{(1)}{E} = \overset{(1)}{\Lambda}/20m \tag{3}$$

Writing $\overset{(2)}{\Lambda} = \alpha m \overset{(2)}{E}$ and solving equations (2) and (3) for $\overset{(1)}{E}$ and $\overset{(2)}{E}$ we get

$$\overset{(1)}{E} = \frac{\overset{(0)}{\Lambda}/m - \alpha}{20-\alpha}, \qquad \overset{(2)}{E} = \frac{20 - \overset{(0)}{\Lambda}/m}{20-\alpha}$$

In order for (0) to get inside $r = 2m$, it is necessary that $\overset{(0)}{\Lambda} < 481m/180$. Therefore the numerator is positive, and to recover the largest possible amount of energy we must take the largest possible value of α less than 20. A physical test particle at $r = 1.75m$ which ultimately escapes to infinity must have Λ less than approximately $2.96Em$. Hence it is convenient to take $\alpha = 2{,}9$. In order to be able to satisfy $\overset{(0)}{p^1} = \overset{(1)}{p^1} + \overset{(2)}{p^1}$ at $r = 1.75m$, $\overset{(0)}{p^1}$ must be small. Thus we choose $\overset{(0)}{\Lambda}$ so that (0) has nearly reached an apsis at $r = 1.75m$. The choice $\overset{(0)}{\Lambda} = 2.63m$ will do. With these values for α and $\overset{(0)}{\Lambda}$, we have $\overset{(1)}{E} = -0.0158$ and $\overset{(2)}{E} = 1.0158$. The apsidal equation

$$\mu^2 r^3 \dot{r}^2 = (E^2 - \mu^2)r^3 + 2m\mu^2 r^2 + (a^2(E^2-\mu^2) - \Lambda^2)r + 2m(\Lambda - aE)^2$$

for (0) gives $\overset{(0)}{p}{}^1 \doteqdot -0.0341$. Choosing $\overset{(1)}{\mu}=0$ the apsidal equations for (1) and (2) give $\overset{(1)}{p}{}^1 \doteqdot -0.0411$, $\overset{(2)}{p}{}^1 \doteqdot 0.007$ and $\overset{(2)}{\mu} \doteqdot 0.306$. Thus (2) is moving outwards with more energy than (0) possessed and escapes to infinity. The particle (1) has negative energy and cannot escape outside $r=2m$; it finally falls to within the event horizon. The angular momentum of (2) exceeds that of (0) by $-\overset{(1)}{\Lambda}$, that is, approximately $0.316m$.

We do not know what upper limit for $\overset{(2)}{E}$ is attainable for processes of this kind. But if the efficiency of the process is to be gauged by the ratio of energy gained to angular momentum (because, after all, it is the rotational energy of the hole that we are proposing to extract), then we are constrained by formula (1) which gives us the inequality

$$2mr_+ \overset{(1)}{E} - a\overset{(1)}{\Lambda} > 0 \tag{4}$$

This also expresses the fact that the scalar product of $\overset{(1)}{p}{}^a$ with the null Killing vector at the event horizon must be positive when (1) crosses the horizon. Thus the energy gain $(-\overset{(1)}{E})$ cannot be greater than $a/2mr_+$ times the angular momentum gain $(-\overset{(1)}{\Lambda})$.

It is interesting to note the relation of this inequality to another phenomenon, namely the apparent fact that the size of the hole (as measured by the surface area of its event horizon) increases even though its mass M can decrease. The surface area of $r=r_+$ at $t=\text{constant}$ (or, indeed, of any other cross-section of the tube $r=r_+$) is

$$A = 8\pi m r_+$$

setting $\mathrm{d}m = -\overset{(1)}{E}$ and $\mathrm{d}(ma) = -\overset{(1)}{\Lambda}$ we retrieve the inequality (4) in the form $\mathrm{d}A>0$. In fact, from general considerations one may infer that there should be a natural tendency for the surface area of the event horizon of a black hole to increase with time whenever the situation is non-stationary. Thus the ideal of maximum efficiency would appear to be achieved whenever this surface area increase is as small as possible. The particular example described here is somewhat inefficient in this respect.

We thank R. P. Geroch for discussions.

R. Penrose

Department of Mathematics,
Birkbeck College,
London

R. M. Floyd

Willesden College of Technology and
Department of Mathematics,
Birkbeck College,
London

Received December 16, 1970.

[1] Bardeen, J. M., *Nature*, **226**, 65 (1970).
[2] Israel, W., *General Relativity and Gravitation*, **2**, No. 1 (Plenum, in the press).
[3] Penrose, R., *Rivista del Nuovo Cimento, Serie* 1, **1**, 252 (1969).
[4] Kerr, R. P., *Phys. Rev. Lett.*, **11**, 237 (1963).
[5] Boyer, R. H., and Lindquist, R. W., *J. Math. Phys.*, **8**, 265 (1967).
[6] Eddington, A. S., *Nature*, **113**, 192 (1924).
[7] Finkelstein, D., *Phys. Rev.*, **110**, 965 (1958).