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Uw referentie(s): SL/101

Artikelomschrijving bij aanvraagnummer: 649381

Artikel: On the energy of the gravitational...
Auteur: Nordstrom
Tijdschrift: PROCEEDINGS OF THE SECTION OF SCIENCES OF THE KONINKLIJKE
Jaar: 1918 **Vol.** 20 **Aflevering:**
Pagina(s): 1238-1245 **Plaatsnr.:** 539 A

Met ingang van 1 april 2003 zullen de prijzen voor fotokopie levering buitenland stijgen met € 0,05 per pagina

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Physics. — "*On the Energy of the Gravitation Field in* EINSTEIN'S *Theory.*" By Dr. GUNNAR NORDSTRÖM. (Communicated by Prof. H. A. LORENTZ.)

(Communicated in the meeting of January 26, 1918.)

In a preceding paper we considered some general theorems derived from EINSTEIN'S gravitation theory, and especially a field with spherical symmetry.[1]) Referring to this paper, which in the following will be denoted by a Roman I, the energy of the gravitation field will now be calculated, according to EINSTEIN'S conception viz. characterized by the quantity t_4^4 [2]) from formula (10) I. In order to obtain a result that holds for an electric field too, I shall first calculate the gravitation field of an electric centre.

§ 1. *The field of an electric centre.*

The gravitation field of an electric body can be calculated by the aid of the variation principle in the form (1) I or (1*a*) I, if only we keep in mind that the electro-magnetic field gives an additive contribution to HAMILTON'S function $\mathfrak{M}$.

We put

$$\mathfrak{M} = \mathfrak{M}^{(e)} + \mathfrak{M}^{(m)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\mathfrak{M}^{(e)}$ refers to the electro-magnetic field, $\mathfrak{M}^{(m)}$ to the matter (in a limited sense). For $\mathfrak{M}^{(e)}$ we have [3])

$$\mathfrak{M}^{(e)} = \frac{1}{8\pi} \sqrt{-g} \sum_{\alpha,\beta,\mu,\nu} g^{\alpha\mu} g^{\beta\nu} \left(\frac{\partial \varphi_\beta}{\partial x_\alpha} - \frac{\partial \varphi_\alpha}{\partial x_\beta}\right) \left(\frac{\partial \varphi_\nu}{\partial x_\mu} - \frac{\partial \varphi_\mu}{\partial x_\nu}\right) + 2 \sum_\sigma \varphi_\sigma \mathfrak{w}^\sigma, \quad (2)$$

where φ indicates the components of the 4 dimensional potential, $\mathfrak{w}_\sigma$ the components of the 4 dimensional electric current.

When the field is stationary and all electric charges at rest, we have

$$\varphi_1 = \varphi_2 = \varphi_3 = 0, \qquad \mathfrak{w}^1 = \mathfrak{w}^2 = \mathfrak{w}^3 = 0.$$

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[1]) G. NORDSTRÖM, "On the mass of a material system according to the gravitation theory of EINSTEIN. These Proceedings, XX, 1917, p. 1076.

[2]) It will be known that a different conception of the gravitation theory has been enunciated by H. A. LORENTZ.

[3]) J. TRESLING, These Proceedings XIX, p. 892.
A. D. FOKKER, These Proceedings XIX, p. 968.

We put

$$\varphi_4 = \varphi, \qquad \mathfrak{w}^4 = \varrho.$$

φ indicates then the electro-static potential, ϱ the density of the electricity. Further we assume, that the field possesses spherical symmetry and choose the time coordinate so that $g_{r4}=0$. We then have $\sqrt{-g}=uwp^2$, and for $\mathfrak{M}^{(e)}$ we find the following expression, the validity of which is seen most simply for a point on one of the axes of coordinates, but which must be generally valid as $\mathfrak{M}^{(e)}$ depends on r only and not on the direction from the centre,

$$\mathfrak{M}^{(e)} = -\frac{p^2}{4\pi\, uw}\left(\frac{d\varphi}{dr}\right)^2 + 2\,\varphi\varrho \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the integral of $\mathfrak{M}^{(e)}$ over a 4 dimensional extension of a fitly chosen form we obtain

$$\iiiint \mathfrak{M}^{(e)}\, dx_1\, dx_2\, dx_3\, dx_4 = 4\pi\,(t_2-t_1)\int_{r_1}^{r_2}\mathfrak{M}^{(e)}\, r^2\, dr, \quad . \quad . \quad . \quad (4)$$

where

$$4\pi\int_{r_1}^{r_2}\mathfrak{M}^{(e)}\, r^2\, dr = \int_{r_1}^{r_2}\left\{-\frac{r^2 p^2}{uw}\left(\frac{d\varphi}{dr}\right)^2 + 8\,\pi\,\varphi\,\varrho\, r^2\right\} dr. \quad . \quad (4a)$$

The laws for the electric field we found by variation of φ, while u, w, p, ϱ were kept constant. As the expressions for $\mathfrak{G}^*$ and $\mathfrak{M}^{(m)}$ do not contain φ, we obtain by this variation

$$2\frac{d}{dr}\left(\frac{r^2p^2}{uw}\frac{d\varphi}{dr}\right) + 8\,\pi\,\varrho\, r^2 = 0$$

The integration gives

$$-\frac{r^2p^2}{uw}\frac{d\varphi}{dr} = e\,(r), \qquad \frac{d\varphi}{dr} = -\frac{uw}{p^2}\frac{e\,(r)}{r^2}. \quad . \quad . \quad . \quad . \quad (5)$$

where $e(r)$ denotes the total charge in a sphere with radius r. Outside the body $e(r)$ is constant and equal to the total charge of the body.

As in the infinite w has the value c (see I p. 1079), we easily see that e and ϱ are its charge and density in *electro-magnetic units*, φ on the contrary the potential in *electro-static units*.

Now we shall calculate the gravitation field. This calculation can be made with the aid of formulae (38) I. Then we must first calculate the stress-energy-tensor for the electro-magnetic field by replacing $\mathfrak{M}$ in formula (2) I by $\mathfrak{M}^{(e)}$ and by introducing the expression (2).

We shall however shorten the calculation by application of the variation principle in the form (28) I. In these variations we must keep φ and ϱ constant and vary u, w and $v = rp$. On the right-hand side of (28) I we have of course to introduce $\mathfrak{M} = \mathfrak{M}^{(e)} = \mathfrak{M}^{(m)}$, so that this form is split up into two parts which we shall consider separately. First we consider the part containing $\mathfrak{M}^{(e)}$. We then obtain, attending to (4a) and (5),

$$4\pi\,\delta\int_{r_1}^{r_2}\mathfrak{M}^{(e)}\,r^2\,dr = \int_{r_1}^{r_2}\left\{-\left(\frac{d\varphi}{dr}\right)^2\delta\left(\frac{v^2}{uw}\right)\right\}dr = -\int_{r_1}^{r_2}\frac{u^2w^2e^2}{v^4}\,\delta\left(\frac{v^2}{uw}\right)dr,$$

$$\delta\int_{r_1}^{r_2}\mathfrak{M}^{(e)}\,r^2\,dr = \int_{r_1}^{r_2}\frac{uw\,e^2}{4\,\pi v^2}\left(\frac{\delta u}{u} - 2\,\frac{\delta v}{v} + \frac{\delta w}{w}\right)dr. \quad . \quad . \quad . \quad (6)$$

As to the part containing $\mathfrak{M}^{(m)}$ we still have our former formula (36) I, if only we keep in mind that, $\mathfrak{M}$ being replaced by $\mathfrak{M}^{(m)}$, $\mathfrak{T}$ gets the significance of a stress-energy-tensor for the matter, if the electric field is considered as *not* belonging to the matter. In this § $\mathfrak{T}$ will keep this altered significance.

Considering the equation (31) I and (6) together with (36) I, in which $\mathfrak{M}^{(m)}$ has been introduced instead of $\mathfrak{M}$, we obtain for the variation formula:

$$\left.\begin{aligned} 2\,\delta\int_{r_1}^{r_2}\left\{\frac{w\,v'^2 + 2v\,v'\,w'}{u} + uw\right\}dr = \varkappa\int_{r_1}^{r_2}\left\{\left(\frac{uw\,e^2}{4\pi v^2} + 2r^2\,\mathfrak{T}_r^r\right)\frac{\delta u}{u} + \right. \\ \left. + 2\left(-\frac{uw\,e^2}{4\pi\,v^2} + 2r^2\,\mathfrak{T}_p^p\right)\frac{\delta v}{v} + \left(\frac{uw\,e^2}{4\pi\,v^2} + 2r^2\,\mathfrak{T}_4^4\right)\frac{\delta w}{w}\right\}dr. \end{aligned}\right\} \quad . \quad (7)$$

Executing the variations we find the following set of equations, which take the place of the set (38) I

$$\left.\begin{aligned} &-\frac{wv'^2 + 2\,vv'w'}{u^2} + w = \frac{\varkappa}{u}\left(\frac{uw\,e^2}{8\pi\,v^2} + r^2\,\mathfrak{T}_r^r\right), \\ &-\frac{wv'' + v'w' + vw''}{u} + (vw' + wv')\frac{u'}{u^2} = \frac{\varkappa}{v}\left(\frac{uw\,e^2}{8\pi\,v^2} + r^2\,\mathfrak{T}_p^p\right), \\ &-\frac{2v\,v'' + v'^2}{u} + u + 2v\,v'\,\frac{u'}{u^2} = \frac{\varkappa}{w}\left(\frac{uw\,e^2}{8\pi\,v^2} + r^2\,\mathfrak{T}_4^4\right). \end{aligned}\right\} \quad . \quad (8)$$

These equations determine the gravitation field, when the tensor $\mathfrak{T}$ for the matter and the distribution of the electricity are given [1]).

[1]) Comparing this set of equations with (38) I we see from the right-hand sides of (8), how the components of the stress-energy-tensor for the electro-magnetic

We shall calculate the field *outside the electric body* where we have

$$\mathfrak{T}_r^r = \mathfrak{T}_p^p = \mathfrak{T}_4^4 = 0,$$
$$e = \text{constant}.$$

In order to execute the calculation of the field we must fix the system of coordinates and we do this by putting the condition

$$p = 1 \quad \text{viz.} \quad v = r \quad \ldots \ldots \ldots \ldots \quad (9)$$

Introducing this in (8), the last equation divided by $-u$ gives

$$\frac{1}{u^2} - 2r\frac{u'}{u^3} = 1 - \frac{\varkappa}{8\pi}\frac{e^2}{r^2},$$

$$\frac{d}{dr}\left(\frac{r}{u^2}\right) = 1 - \frac{\varkappa}{8\pi}\frac{e^2}{r^2}.$$

By integration we obtain

$$\frac{r}{u^2} = r - \alpha + \frac{\varkappa}{8\pi}\frac{e^2}{r},$$

where α is an integration constant. To simplify the formulae we put

$$\frac{\varkappa e^2}{8\pi} = \varepsilon^2 \quad \ldots \ldots \ldots \ldots \quad (10)$$

and find

$$\frac{1}{u^2} = 1 - \frac{\alpha}{r} + \frac{\varepsilon^2}{r^2} \quad \ldots \ldots \ldots \ldots \quad (11)$$

This formula expresses u as a function of r. An expression for w gives us the first formula (8). If in that formula we introduce the expression found for $\frac{1}{u^2}$ and reverse the sign, we obtain

$$(w + 2r\,w')\left(1 - \frac{\alpha}{r} + \frac{\varepsilon^2}{r^2}\right) = w - w\frac{\varepsilon^2}{r^2}.$$

A simple calculation gives

$$2\frac{w'}{w} = \frac{\frac{\alpha}{r^2} - 2\frac{\varepsilon^2}{r^3}}{1 - \frac{\alpha}{r} + \frac{\varepsilon^2}{r^2}}.$$

On the right-hand side the numerator is just the derivative of the

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field can be expressed by e, u, w, p, and the coordinates. We find that in the electro-magnetic field $+\frac{uw}{8\pi p^2}\frac{e^2}{r^4}$ corresponds to $\mathfrak{T}_r^r$ and $\mathfrak{T}_4^4$, $-\frac{uw}{8\pi p^2}\frac{e^2}{r^4}$ to $\mathfrak{T}_p^p$. For the diagonal sum of the components we find identically zero.

denominator, so that integrating, and choosing the integration constant in such a way that in the infinite w^2 has the value c^2, we find

$$w^2 = c^2\left(1 - \frac{\alpha}{r} + \frac{\varepsilon^2}{r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

So we have calculated the gravitation field. Comparing (11) and (12) we obtain

$$u\,w = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and further we find (comp. I note on p. 1087)

$$\sqrt{-g} = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Because of (10) ε determines the electric charge of the body. The constant α determines the mass of the body. Formula (50) I gives namely

$$m = \frac{4\pi\,\alpha}{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The same expression for m has been deduced in I p. 1087 for the case $c = 1$. Now we see that this expression also holds when the unit of time has been chosen in another way.

We obtained the relation (15) by application of formula (50) I, where we assumed, that the matter has a finite extension. If the electro-magnetic field is reckoned to the matter as is done in I, the mâtter has, strictly speaking, an infinite extension. This has however no influence on the validity of formula (50) I in the case in question. The quantities of the electric field approach namely sufficiently to zero, when the distance from the centre increases.

We may ask on what conditions $g_{44} = w^2$ can become zero and negative. According to (12) u^2 becomes then infinite resp. negative. The expression (12) shows that outside the body

$$w^2 = 0 \quad \text{for} \quad r = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \varepsilon^2}.$$

For smaller values of r w^2 is negative.

Only when

$$\frac{\alpha^2}{4} > \varepsilon^2$$

w^2 becomes therefore zero and negative for real values of r. If on the contrary $\varepsilon^2 > \frac{\alpha^2}{4}$, then w^2 and u^2 are everywhere finite and positive. Then we may very well assume the mass and the charge to be concentrated in a mathematical point.

§ 2. *The energy of the gravitation field.*

In this § we shall calculate the distribution of the energy in a gravitation field with spherical symmetry viz. we shall calculate the quantity $t_4{}^4$ in such a field. The body which excites the field may also be electric. Our calculation be based upon the formulae (17) I and (13) I, which give

$$\varkappa t^4{}_4 = -\tfrac{1}{2}\varkappa \sum_\alpha \mathfrak{T}_\alpha^\alpha + \tfrac{1}{2}\sum_\tau \frac{\partial \mathfrak{A}_\tau}{\partial x_\tau} . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We suppose $\sum_\alpha \mathfrak{T}_\alpha^\alpha$ to be given as a function of r. When we can calculate the vector $\mathfrak{A}$ we find from this formula also $t_4{}^4$. We shall deduce an expression for $\mathfrak{A}_1$ which holds at a point of the X_1 axis ($x_2 = x_3 = 0$). According to (5) I we have:

$$\mathfrak{A}_1 = \tfrac{1}{2}\sqrt{-g}\sum_{\mu\nu\sigma}(g^{\mu 1}g^{\nu\sigma} - g^{\sigma 1}g^{\mu\nu})\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right).$$

As in I § 3 we shall choose the time-coordinate, so that $g_{r4} = 0$. At a point of the X_1 we have because of (33) I and (25) I:

$$\left.\begin{aligned} g^{11} &= -\frac{1}{u^2}, \quad g^{22} = g^{33} = \frac{1}{p^2}, \quad g^{12} = g^{13} = g^{23} = 0, \\ \frac{\partial g_{12}}{\partial x_2} &= \frac{\partial g_{13}}{\partial x_3} = -\frac{1}{r}(u^2 - p^2), \quad \frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{33}}{\partial x_1} = -2pp'. \end{aligned}\right\} \quad . \quad (17)$$

As to the calculation of $\mathfrak{A}_1$ we first remark that the terms of the summation for which $\sigma = \mu$ give a contribution zero. In order that a term may give a contribution different from zero, either σ or μ has to be equal to 1, while the other two of the three indices σ, μ, ν must be equal.

$\mu = 1$, $\sigma = \nu \neq 1$ gives for $\mathfrak{A}_1$ the contribution:

$$\tfrac{1}{2}\sqrt{-g}\,.\,g^{11}\left(g^{22}\frac{\partial g_{22}}{\partial x_1} + g^{33}\frac{\partial g_{33}}{\partial x_1} + g^{44}\frac{\partial g_{44}}{\partial x_1}\right),$$

$\sigma = 1$, $\mu = \nu \neq 1$ gives the contribution:

$$-\tfrac{1}{2}\sqrt{-g}\;g^{11}\left\{g^{22}\left(2\frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right) + g^{33}\left(2\frac{\partial g_{13}}{\partial x_3} - \frac{\partial g_{33}}{\partial x_1}\right) - g^{44}\frac{\partial g_{44}}{\partial x_1}\right\}.$$

The two contributions together give $\mathfrak{A}_1$. Introducing the expressions (17) we obtain for $\mathfrak{A}_1 = \mathfrak{A}_r$

$$\mathfrak{A}_r = -4\frac{wpp'}{u} + \frac{2}{r}uw\left(1 - \frac{p^2}{u^2}\right) - 2\frac{p^2w'}{u} . \quad . \quad . \quad (18)$$

For a point on the X_1 axis, the right-hand side is equal to $\mathfrak{A}_1$. If we consider a point not on the X_1 axis, the same expression holds for the radial component $\mathfrak{A}_r$.

Formula (18) holds for each system of coordinates in which the spherical symmetry is taken into consideration, if only the time-coordinate is chosen so that $g_{r_4} = 0$. If we specialize the system of coordinates by putting $p = 1$, then the expressions (11) and (12) are valid outside the body and for u and w we obtain

$$\mathfrak{A}_r = \frac{2c}{r}\left(\frac{\alpha}{r} - \frac{\varepsilon^2}{r^2}\right) - c\left(\frac{\alpha}{r^2} - 2\frac{\varepsilon^2}{r^3}\right),$$

$$\mathfrak{A}_r = \frac{c\alpha}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Because of the spherical symmetry we have, of course, for the components of $\mathfrak{A}$ in the directions of the axes of coordinates in space

$$\mathfrak{A}_\tau = \frac{x_\tau}{r}\frac{c\alpha}{r^2}, \qquad \tau = 1, 2, 3. \quad . \quad . \quad . \quad . \quad . \quad (19a)$$

Making up the divergency we obtain

$$\sum_\tau \frac{\partial \mathfrak{A}_\tau}{\partial x_\tau} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and formula (16) gives

$$\mathfrak{t}_4^4 = -\tfrac{1}{2}\sum_\alpha \mathfrak{T}_\alpha^\alpha .$$

Outside the body the right-hand side is zero and therefore also

$$\mathfrak{t}_4^4 = 0 \qquad (r > R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the system of coordinates is chosen so that $p = 1$ [1]*, the gravitation field has everywhere outside the body the density of energy zero.* This is also true when the body is electrically charged, because for the electro-magnetic field the sum of the diagonal components of the stress-energy-tensor is equal to zero (see the note on p. 1087).

Now we may ask whether the other stress-energy-components t_μ^ν for the gravitation field are also equal to zero. This is suggested by formula (17) I, which gives

$$\sum_\alpha \mathfrak{t}_\alpha^\alpha = 0.$$

That really all $\mathfrak{t}_\mu^\nu$ are zero in systems of coordinates for which $p = 1$ is easily proved with the aid of formula (52) of EINSTEIN, Grundlage. The details will be left here aside.[2])

[1]) Then we have also $\sqrt{-g} = c$ as has been proved in § 1

[2]) For a non-electric centre E. SCHRÖDINGER has meanwhile proved this too: E. SCHRÖDINGER, Die Energiekomponenten des Gravitationsfeldes. Phys. Zeitschr. **19**, 1918 p. **4**. (Remark in the proof.)

If a system of coordinates is chosen for which $p \neq 1$, both t_μ^ν and the other t_μ^ν are different from zero. EINSTEIN deduced approximated values for the t_μ^ν in a system of coordinates in which the velocity of the light is independent of the direction of propagation [1]), and these expressions show that the t_μ^ν are not zero. In this case t_μ^ν can also be calculated by means of formulae (16) and (18). For that system of coordinates in which the velocity of light is independent of the direction of propagation, we have according to DROSTE [2]) for u, p, w outside the body

$$u = p = \left(1 + \frac{\alpha}{4r}\right)^2, \qquad w^2 = 1 - \frac{\alpha}{r\left(1 + \frac{\alpha}{4r}\right)} \quad . \quad . \quad (22)$$

These expressions can be introduced into (18). The further calculation will not be given here.

The property of the components t_μ^ν that they can be made to vanish by a transformation of coordinates when the field possesses spherical symmetry, is evidently connected with the circumstance, that not for every transformation of coordinates these quantities behave like tensor-components. It is remarkable, that the change of the system of coordinates, necessary to make the components t_μ^ν vanish is undetectably small for really existing gravitation fields. This circumstance renders the conception of the quantities t_μ^ν as stress-energy-components somewhat less sympathic, and supports the conception of the gravitation energy enunciated by LORENTZ. Whatever may be however our opinion on this subject we have no ground to banish the quantities t_μ^ν from EINSTEIN's gravitation theory. As is evident from our considerations in I § 2, several general theorems may be derived in a simple way with the aid of these quantities.

1) A. EINSTEIN, Näherungsweise Integration der Feldgleichungen der Gravitation, Berl. Ber. 1916, p. 688.

2) DROSTE, Het zwaartekrachtsveld, p. 20, equation (31).